On S-equivalence and isomorphism of games in characteristic function form

Tijs, S.H.

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On S-Equivalence and Isomorphism of Games in Characteristic Function Form

By *S.H. Tijs*, Nijmegen[1])

Abstract: In this note an example is given of two superadditive games which are isomorphic and not S-equivalent.

An *n-person game (in characteristic function form)* is an ordered pair $\langle N, v\rangle$, where $N = \{1, 2, \ldots, n\}$, and where v is a real-valued function on the family 2^N of subsets of N, with the property that $v(\phi) = 0$.

$\langle N, v\rangle$ is called a *superadditive game* if for each $S, T \in 2^N$ with $S \cap T = \phi$ we have

$$v(S \cup T) \geqslant v(S) + v(T).$$

For an n-person game $\langle N, v\rangle$ the *set of imputations*

$$\{x \in \mathbf{R}^n; x_i \geqslant v(\{i\}) \text{ for each } i \in N, \sum_{i \in N} x_i = v(N)\}$$

is denoted by $I(v)$.

Two n-person games $\langle N, v\rangle$ and $\langle N, w\rangle$ are called *S-equivalent* if there exist $a_1, a_2, \ldots, a_n \in \mathbf{R}$ and a positive real number k such that

$$w(S) = kv(S) + \sum_{i \in S} a_i \text{ for each } S \in 2^N - \{\phi\}.$$

We say that $y \in I(v)$ *is dominated by* $x \in I(v)$ via $S \in 2^N - \{\phi\}$ (in the game $\langle N, v\rangle$) – and we write $x \operatorname{dom}_S y$ – if

$$y_i < x_i \text{ for each } i \in S, \tag{1}$$

$$\sum_{i \in S} x_i \leqslant v(S). \tag{2}$$

Two n-person games $\langle N, v\rangle$ and $\langle N, w\rangle$ are called *isomorphic*, if there exists a bijection $\psi : I(v) \to I(w)$ such that for each $x, y \in I(v)$ and each $S \in 2^N - \{\phi\}$ we have

$$x \operatorname{dom}_S y \text{ iff } \psi(x) \operatorname{dom}_S \psi(y).$$

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[1]) Dr. *S.H. Tijs*, Department of Mathematics, Catholic University, Nijmegen, The Netherlands.

It is well-known (and easy to show) that two S-equivalent n-person games are also isomorphic.
McKinsey [1950] proved that two isomorphic superadditive constant-sum games are also S-equivalent. [$\langle N, v\rangle$ is a *constant-sum game* if there is a $c \in \mathbf{R}$ such that $v(S) + v(N - S) = c$ for each $S \in 2^N$]. The following theorem shows that the constant-sum condition in *McKinsey*'s theorem is not superfluous.

Owen [1968, p. 161] and *Lucas* [1971, pp. 503, 504] erroneously neglected to mention this constant-sum condition in *McKinsey*'s theorem. (See also *Rosenmüller* [1971, p. 10].)

Theorem: There exist two superadditive 3-person games which are isomorphic and not S-equivalent.

Proof: Let $\langle N, v\rangle$ be the superadditive 3-person game with
$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = v(\{1,3\}) = 0$, $v(\{2,3\}) = 1/2$ and $v(\{1,2,3\}) = 1$.
Let $\langle N, w\rangle$ be the superadditive 3-person game with
$w(S) = v(S)$ for each $S \in 2^N - \{\{2,3\}\}$ and $w(\{2,3\}) = 1/4$.
It is obvious that $\langle N, v\rangle$ and $\langle N, w\rangle$ are not S-equivalent.
We will show that these two games are isomorphic. First we note that

$$I(v) = I(w) = \{x \in \mathbf{R}^3; x \geqslant 0, \sum_{i=1}^{3} x_i = 1\}.$$

The map $\psi : I(v) \to I(w)$, defined by

$$\psi(x_1, x_2, x_3) = (1/2 + 1/2x_1, 1/2x_2, 1/2x_3) \text{ if } 1/2 \leqslant x_1 \leqslant 1, \text{ and}$$

$$\psi(x_1, x_2, x_3) = (3/2x_1, (1 - (3/2)x_1)(x_2 + x_3)^{-1}x_2,$$

$$(1 - (3/2)x_1)(x_2 + x_3)^{-1}x_3) \text{ if}$$

$0 \leqslant x_1 < 1/2$, is a bijection.

In order to prove our theorem, it is sufficient to show that for each $x, y \in I(v) : x \operatorname{dom}_{\{2,3\}} y$ iff $\psi(x) \operatorname{dom}_{\{2,3\}} \psi(y)$, and this is straightforward.

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