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On S -Equivalence and Isomorphism of Games in Characteristic Function Form

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Abstract: In this note an example is given of two superadditive games which are isomorphic and not S -equivalent.

An n -person game (in characteristic function form) is an ordered pair $\langle N, \nu \rangle$, where $N = \{1, 2, \dots, n\}$, and where ν is a real-valued function on the family 2^N of subsets of N , with the property that $\nu(\emptyset) = 0$.

$\langle N, \nu \rangle$ is called a *superadditive game* if for each $S, T \in 2^N$ with $S \cap T = \emptyset$ we have

$$\nu(S \cup T) \geq \nu(S) + \nu(T).$$

For an n -person game $\langle N, \nu \rangle$ the *set of imputations*

$$\{x \in \mathbf{R}^n; x_i \geq \nu(\{i\}) \text{ for each } i \in N, \sum_{i \in N} x_i = \nu(N)\}$$

is denoted by $I(\nu)$.

Two n -person games $\langle N, \nu \rangle$ and $\langle N, w \rangle$ are called *S -equivalent* if there exist $a_1, a_2, \dots, a_n \in \mathbf{R}$ and a positive real number k such that

$$w(S) = k\nu(S) + \sum_{i \in S} a_i \text{ for each } S \in 2^N - \{\emptyset\}.$$

We say that $y \in I(\nu)$ is *dominated by* $x \in I(\nu)$ via $S \in 2^N - \{\emptyset\}$ (in the game $\langle N, \nu \rangle$) – and we write $x \text{ dom}_S y$ – if

$$y_i < x_i \text{ for each } i \in S, \tag{1}$$

$$\sum_{i \in S} x_i \leq \nu(S). \tag{2}$$

Two n -person games $\langle N, \nu \rangle$ and $\langle N, w \rangle$ are called *isomorphic*, if there exists a bijection $\psi : I(\nu) \rightarrow I(w)$ such that for each $x, y \in I(\nu)$ and each $S \in 2^N - \{\emptyset\}$ we have

$$x \text{ dom}_S y \text{ iff } \psi(x) \text{ dom}_S \psi(y).$$

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It is well-known (and easy to show) that two S -equivalent n -person games are also isomorphic.

McKinsey [1950] proved that two isomorphic superadditive constant-sum games are also S -equivalent. [$\langle N, \nu \rangle$ is a *constant-sum game* if there is a $c \in \mathbf{R}$ such that $\nu(S) + \nu(N - S) = c$ for each $S \in 2^N$]. The following theorem shows that the constant-sum condition in *McKinsey's* theorem is not superfluous.

Owen [1968, p. 161] and *Lucas* [1971, pp. 503, 504] erroneously neglected to mention this constant-sum condition in *McKinsey's* theorem. (See also *Rosenmüller* [1971, p. 10].)

Theorem: There exist two superadditive 3-person games which are isomorphic and not S -equivalent.

Proof: Let $\langle N, \nu \rangle$ be the superadditive 3-person game with $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{1,2\}) = \nu(\{1,3\}) = 0$, $\nu(\{2,3\}) = 1/2$ and $\nu(\{1,2,3\}) = 1$.

Let $\langle N, w \rangle$ be the superadditive 3-person game with $w(S) = \nu(S)$ for each $S \in 2^N - \{\{2,3\}\}$ and $w(\{2,3\}) = 1/4$.

It is obvious that $\langle N, \nu \rangle$ and $\langle N, w \rangle$ are not S -equivalent.

We will show that these two games are isomorphic. First we note that

$$I(\nu) = I(w) = \{x \in \mathbf{R}^3; x \geq 0, \sum_{i=1}^3 x_i = 1\}.$$

The map $\psi : I(\nu) \rightarrow I(w)$, defined by

$$\psi(x_1, x_2, x_3) = (1/2 + 1/2x_1, 1/2x_2, 1/2x_3) \text{ if } 1/2 \leq x_1 \leq 1, \text{ and}$$

$$\psi(x_1, x_2, x_3) = (3/2x_1, (1 - (3/2)x_1)(x_2 + x_3)^{-1}x_2,$$

$$(1 - (3/2)x_1)(x_2 + x_3)^{-1}x_3) \text{ if}$$

$$0 \leq x_1 < 1/2, \text{ is a bijection.}$$

In order to prove our theorem, it is sufficient to show that for each $x, y \in I(\nu) : x \text{ dom}_{\{2,3\}} y$ iff $\psi(x) \text{ dom}_{\{2,3\}} \psi(y)$, and this is straightforward.

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