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# THE INFINITE HORIZON OPEN-LOOP NASH LQ-GAME

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## Abstract

In this paper we consider the linear-quadratic differential game with an infinite planning horizon. We derive both necessary and sufficient conditions for existence of open-loop Nash equilibria for this game. Furthermore we show how all equilibria can be easily obtained from the eigenspace structure of a Hamiltonian matrix that is associated with the game.

## 1 Introduction

The last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Måler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Douven et al (1995)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with each other. One of these scenarios is the so-called open-loop strategy. This scenario can be interpreted as that the parties simultaneously determine their strategy, next submit their strategies to some authority who then enforces these plans as binding commitments. So, this strategy is based on the assumption that the parties act non-cooperatively and that the only information they have on the model is its present state

and the model structure. Obviously, since according to this scenario the participating parties can not react to each other's policies, its economic relevance is mostly rather limited. However, as a benchmark to see how much parties can gain by playing other strategies, it plays a fundamental role. Due to its analytic tractability the open-loop Nash equilibrium strategy is in particular very popular for problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives the parties are striving for can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first modeled and solved in a mathematically rigorous way by Starr and Ho in (1969) (see also Lukes et al (1971), Eisele (1982) and Engwerda (1996) for extensions and more precise formulations). In Abou-Kandil et al. (1993), Weeren (1995) and Engwerda (1996) also convergence of this equilibrium strategy was studied if the planning horizon expands. Like in the optimal linear quadratic regulator theory it turns out that under some conditions it can be shown that this strategy converges. Furthermore, this converged solution is rather easy to calculate and much easier to implement than the finite planning horizon equilibrium solution. So, the question arises whether this (converged) solution also solves the game if the parties consider an infinite-planning horizon. In Engwerda (1996) this problem was partly solved. That is, on the one hand a sufficient condition was given under which open-loop Nash equilibria exist and, on the other hand, for stable systems both a necessary and sufficient existence condition was derived. In this paper we will extend this approach. We will show that the condition derived in the above mentioned paper is also necessary and sufficient for the general case. We will conclude this paper by a discussion on the consequences of this result for numerical calculation of equilibrium solutions and by considering some special cases when (generically) a unique equilibrium exists.

The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper and present some preliminary properties. Section three contains the basic result, whereas section four contains the concluding remarks.

## 2 Preliminaries

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. As already mentioned in the introduction we assume that both players have to formulate their strategy already at the moment the system starts to evolve and this strategy can not be changed once the system runs. So, the players have to minimize their performance criterion based on the information that they only know the differential equation and its initial state. We are looking now for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = x_0, \quad (1)$$

where  $x$  is the  $n$ -dimensional state of the system,  $u_i$  is an  $m_i$ -dimensional (control) vector player  $i$  can manipulate,  $x_0$  is the initial state of the system,  $A$ ,  $B_1$ , and  $B_2$  are constant matrices of appropriate dimensions, and  $\dot{x}$  denotes the time derivative of  $x$ .

The performance criterium player  $i = 1, 2$  aims to minimize is:

$$\lim_{t_f \rightarrow \infty} J_i(u_1, u_2),$$

where

$$J_1(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_1 x(t) + u_1(t)^T R_{11} u_1(t)\} dt,$$

and

$$J_2(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_2 x(t) + u_2(t)^T R_{22} u_2(t)\} dt,$$

in which matrix  $R_{ii}$  is positive definite,  $Q_i$  is semi-positive definite and additionally is positive definite w.r.t. the controllability subspace  $\langle A, B_i \rangle$ ,  $i = 1, 2$ .

Note that usually in literature each player's performance criterium also includes a cross term, penalizing the control efforts of the other player. Since, however, this cross

term does not play a role in the analysis of open-loop Nash equilibria, we dropped this term here (see any of the references quoted above).

The set of admissible control functions we consider, is given by  $U :=$

$$\left\{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, t \in [0, \infty) \mid \lim_{t_f \rightarrow \infty} J_i(u_1, u_2) < \infty, i = 1, 2. \right\}$$

Note that  $U$  depends on the initial state of the system. For simplicity of notation we omit, however, this dependency. Furthermore, it is clear that  $u_i(\cdot) \in L^2$ , the set of square integrable functions, but that  $U$  is not a linear subspace of  $L^2$ . First, since the zero-function will in general not belong to  $U$  and, second, in general with  $v, w \in U$ ,  $v + w \notin U$ . However,  $U$  does satisfy the following important property:

### Lemma 1:

Assume that both  $v$  and  $v + w$  are an element of  $U$ . Then for any real  $\epsilon$  also  $v + \epsilon w \in U$ .

### Proof:

First we introduce some notation.

Let  $x_u$  denote the state trajectory obtained by using the control function  $u$ , that is,  $x_u(t) := e^{At} x_0 + \int_0^t e^{A(t-\tau)} (B_1 \ B_2) u(\tau) d\tau$ .

Since by assumption both  $v$  and  $v + w$  belong to  $U$ ,  $x_v(t)$  and  $x_{v+w}(t)$  converge to zero if  $t \rightarrow \infty$ . So,  $x_v(t) - x_{v+w}(t) = \int_0^t e^{A(t-\tau)} (B_1 \ B_2) w(\tau) d\tau \rightarrow 0$ , if  $t \rightarrow \infty$ . Moreover, since both  $x_v$  and  $x_{v+w}$  are square integrable, also the righthandside of the above equation is square integrable. Now, consider  $x_{v+\epsilon w}(t)$ . Elementary calculation shows that  $x_{v+\epsilon w}(t) = x_{v+w}(t) - (1 - \epsilon) \int_0^t e^{A(t-\tau)} (B_1 \ B_2) w(\tau) d\tau$ . So, using the above result, it is clear that  $x_{v+\epsilon w}(t)$  is square integrable. Moreover, since both  $v$  and  $v + w$  are square integrable it follows that  $w$  has to be square integrable too. From this follows then immediately that also  $v + \epsilon w$  is square integrable. Combining both results gives then that  $\lim_{t_f \rightarrow \infty} J_i(v + \epsilon w) < \infty$ ,  $i = 1, 2$ . Which implies that  $v + \epsilon w \in U$ .  $\square$

Next, we introduce the set of coupled algebraic asymmetric Riccati-type equations associated with this problem:

$$0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \quad (2)$$

$$0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1, \quad (3)$$

where  $S_i = B_i R_{ii}^{-1} B_i^T$ ,  $i = 1, 2$ .

We will see in the next section that the solutions  $K_1, K_2$  to this set of equations, that satisfy an additional stability property, play a similar role like the stabilizing solution of the algebraic Riccati equation in the standard LQ regulator problem.

### 3 The equilibrium strategies

The basic result of this paper is summarized in the next theorem:

**Theorem 2:**

The two-player linear quadratic differential game (1) has for every initial state an open-loop Nash equilibrium  $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  if and only if there exist  $K_1$  and  $K_2$  that are solutions of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of  $A_{cl} := A - S_1 K_1 - S_2 K_2$  are all situated in the left half complex plane.

In that case, the strategy

$$u_i(t) = -R_{ii}^{-1} B_i^T K_i x(t), i = 1, 2$$

is an open-loop Nash equilibrium strategy.

Moreover, the costs obtained by using this strategy for the players are

$$\int_0^\infty \{(\epsilon^{A_{cl}t} x_0)^T (Q_i + K_i^T S_i K_i) \epsilon^{A_{cl}t} x_0 dt, i = 1, 2.$$

□

**Proof:**

" $\Leftarrow$ " This part was proved by Engwerda (1996), theorem 12.

" $\Rightarrow$ " To prove this part we use the variational approach (see e.g. Friedman (1971), Lukes and Russell (1971) and Engwerda (1996)).

Suppose that  $\bar{u}_1, \bar{u}_2$  are a Nash solution. That is,

$$J_1(u_1, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2) \text{ and } J_2(\bar{u}_1, u_2) \geq J_2(\bar{u}_1, \bar{u}_2). \quad (4)$$

Then, for any control function  $\begin{pmatrix} w \\ 0 \end{pmatrix}$  such that  $\begin{pmatrix} \bar{u}_1 + w \\ \bar{u}_2 \end{pmatrix} \in U$  we have, according lemma 1, that for any real number  $\epsilon$

$$J_1(\epsilon) := J_1(\bar{u}_1 + \epsilon w, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2). \quad (5)$$

Let  $x_{\bar{u}}(t)$  and  $x_{\bar{u} + \epsilon w}(t)$  be the solutions to (1) corresponding to the controls  $\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}$  and  $\begin{pmatrix} \bar{u}_1 + \epsilon w \\ \bar{u}_2 \end{pmatrix}$ , respectively. Then it is easily verified that (see also proof of lemma 1)

$$x_{\bar{u} + \epsilon w}(t) = x_{\bar{u}}(t) + \epsilon g(t), \quad (6)$$

where  $g(t) := \int_0^t e^{A(t-s)} B_1 w(s) ds$  is a square integrable function.

So,  $J_1(\epsilon)$  can be rewritten as:

$$\frac{1}{2} \int_0^\infty f(t, \epsilon) dt,$$

where

$$f(t, \epsilon) := (x_{\bar{u}}(t) + \epsilon g(t))^T Q_1 (x_{\bar{u}}(t) + \epsilon g(t)) +$$

$$(\bar{u}_1(t) + \epsilon w(t))^T R_{11} (\bar{u}_1(t) + \epsilon w(t)).$$

Note that  $f(t, \epsilon)$  is differentiable w.r.t.  $\epsilon$  for every  $t \in (0, \infty)$ . Simple calculations show that

$$\begin{aligned} \frac{\partial f}{\partial \epsilon} &= 2\epsilon(g^T(t)Q_1g(t) + w^T(t)R_{11}w(t)) + \\ &2(g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t)) \end{aligned}$$

Using the facts that  $g(t), w(t)$  and  $\bar{u}_1(t)$  are square integrable, it is obvious now that  $\frac{\partial f}{\partial \epsilon}$  is integrable for, e.g., all  $\epsilon \in [-1, 1]$ . Using standard arguments we have then that  $J_1(\epsilon)$  is differentiable on  $(-1, 1)$  and that

$$\begin{aligned} \frac{dJ_1(\epsilon)}{d\epsilon} &= \int_0^\infty \{ \epsilon(g^T(t)Q_1g(t) + w^T(t)R_{11}w(t)) + \\ &(g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t)) \} dt \end{aligned}$$

From (5) we get

$$\left. \frac{dJ_1(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0. \quad (7)$$

So, we get:

$$\int_0^\infty \{g^T(t)Q_1x_{\bar{u}}(t) + w^T(t)R_{11}\bar{u}_1(t)\} dt = 0.$$

Substitution of the expression for  $g(t)$  into this equation and then interchanging the order of integration yields:

$$\begin{aligned} \int_0^\infty \{ \int_s^\infty (e^{A(t-s)} B_1 w(s))^T Q_1 x_{\bar{u}}(t) dt \} ds + \\ \int_0^\infty w^T(t) R_{11} \bar{u}_1(t) dt = 0. \end{aligned}$$

Which can be restated as:

$$\begin{aligned} \int_0^\infty \{ w^T(t) B_1^T e^{-A^T t} \int_t^\infty e^{A^T s} Q_1 x_{\bar{u}}(s) ds \} dt + \\ \int_0^\infty w^T(t) R_{11} \bar{u}_1(t) dt = 0. \end{aligned}$$

Now, choose in the above expression consecutively  $w(t) := \text{sgn}[e_i^T (B_1^T \int_t^\infty e^{A^T(s-t)} Q_1 x_{\bar{u}}(s) ds + R_{11} \bar{u}_1(t))] e^{-\lambda t} e_i$ ,  $i=1, \dots, n$ , where  $\lambda$  is an arbitrary real number larger than the spectral radius of matrix  $A$  and  $e_i$  is the  $i$ -th standard basis vector in  $\mathbb{R}^n$ . Then it is clear that for every choice of  $w(t)$ ,  $\bar{u}_1 + w(t) \in U$ . Consequently it follows that

$$\bar{u}_1(t) = -R_{11}^{-1} B_1^T \int_t^\infty e^{A^T(s-t)} Q_1 x_{\bar{u}}(s) ds. \quad (8)$$

Similarly, it can be shown that  $\bar{u}_2(t)$  necessarily satisfies:

$$\bar{u}_2(t) = -R_{22}^{-1} B_2^T \int_t^\infty e^{A^T(s-t)} Q_2 x_{\bar{u}}(s) ds \quad (9)$$

Next, we introduce the vector  $v := (v_1^T \ v_2^T \ v_3^T)^T$  as follows:  $v_1(t) := x_{\bar{u}}(t)$ ,  $v_2(t) := \int_t^\infty e^{A^T(s-t)} Q_1 x_{\bar{u}}(s) ds$

and  $v_3(t) := \int_t^\infty e^{A^T(s-t)} Q_2 x_{\bar{u}}(s) ds$ . Then  $\bar{u}_1 = -R_{11}^{-1} B_1^T v_2(t)$  and  $\bar{u}_2 = -R_{22}^{-1} B_2^T v_3(t)$ . Moreover, it is easily verified by differentiation of  $v_2(t)$  and  $v_3(t)$ , that  $v(t)$  satisfies

$$\dot{v}(t) = - \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} v(t), \text{ with } v_1(0) = x_0. \quad (10)$$

Since by assumption for arbitrary  $x_0$ ,  $v(t)$  converges to zero, it follows that there exist  $K_1, K_2$  and a stable matrix  $\Lambda$  such that  $-\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} \Lambda$ . Writing out these equations yields then the advertised result.  $\square$

Remarks:

Parts of the above proof can be substituted by using the results of Haurie et al. (1984, lemma 5.1). This requires, however, the introduction of the concept of weak overtaking optimality. In this framework it is not required that the state or the performance criterium converge (see Halkin (1974)). Since we like to stay in the framework of bounded performance criteria, we choose to give an elementary selfcontained proof of the theorem.

Note that in case the system is not stabilizable, the problem has no solution.

In the above theorem the costs for the individual players are expressed as an integral. In fact, analogously to the optimal LQ regulator theory, we have that the costs can be obtained indirectly by solving the following associated Lyapunov equations<sup>1</sup>

$$A_{cl}^T M_i + M_i A_{cl} + Q_i + K_i^T S_i K_i = 0, \quad (11)$$

where  $A_{cl} := A - S_1 K_1 - S_2 K_2$ ,  $i=1,2$ .

Note that if all eigenvalues of  $A_{cl}$  are in the left half complex plane and  $Q_i + K_i^T S_i K_i \geq 0$ , this equation has a unique positive semi-definite solution  $M_i$ . The equilibrium costs can then also be obtained as

Proposition 3:

Assume that  $K_1, K_2$  solve (ARE) and satisfy the stabilization property mentioned in theorem 2. Let  $M_i$  be the unique positive semi-definite solution of the above Lyapunov equation (11): Then,

$$J_i(\bar{u}_1, \bar{u}_2) = x_0^T M_i x_0, \quad i = 1, 2. \quad (12)$$

Proof:

Let  $M_i \geq 0$  be the unique solution of (11). Then, using

<sup>1</sup>I like to thank Arie Weeren for pointing out this to me

the notation  $x(t) := e^{A_{cl}t} x_0$ , we have

$$\begin{aligned} J_i(\bar{u}_1, \bar{u}_2) &= \int_0^\infty x(t)^T (Q_i + K_i^T S_i K_i) x(t) dt \\ &= - \int_0^\infty x(t)^T (A_{cl}^T M_i + M_i A_{cl}) x(t) dt \\ &= - \int_0^\infty \frac{d}{dt} [x(t)^T M_i x(t)] dt \\ &= x_0^T M_i x_0 - \lim_{t \rightarrow \infty} x(t)^T M_i x(t) \\ &= x_0^T M_i x_0, \end{aligned}$$

which proves the claim.  $\square$

## 4 The solutions for the algebraic Riccati equation

In the previous section we saw that we can find all equilibrium solutions by determining all solutions  $K_1, K_2$  of the set of algebraic Riccati equations (ARE), which satisfy the additional property that all eigenvalues of the corresponding matrix  $A - S_1 K_1 - S_2 K_2$  lie in the left half complex plane.

MacFarlane (1963) and Potter (1966) independently discovered that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix in linear quadratic regulator problems. Abou-Kandil et al. (1993) already pointed out the existence of a similar relationship for our problem. One of their results was that if the planning horizon  $t_f$  in (1) tends to infinity, under some technical conditions, the solution of the finite planning horizon problem converges to a solution which requires the calculation of a solution  $K_1, K_2$  of (ARE) which can be calculated from the eigenspaces of the matrix

$$M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}. \quad (13)$$

In Engwerda et al. (1995) this relationship between solutions of (ARE) and eigenspaces of matrix  $M$  was elaborated. Using the notation  $\mathcal{M}^{inv}$  for the set of all  $M$ -invariant subspaces, it was shown that all solutions for the set of algebraic Riccati equations can be calculated from the following subset of  $M$ -invariant subspaces:

$$\mathcal{K}^{pos} := \left\{ \mathcal{K} \in \mathcal{M}^{inv} \mid \mathcal{K} \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

Note that elements in this set  $\mathcal{K}^{pos}$  can be calculated using the set of matrices

$$\mathcal{K}^{pos} := \left\{ K \in \mathbb{R}^{3n \times n} \mid \text{Im}K \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

The following result was proved:

Proposition 4:

(ARE) has a real solution  $(K_1, K_2)$  if and only if  $K_1 = YX^{-1}$  and  $K_2 = ZX^{-1}$  for some  $\mathcal{K} =: \text{Im} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}$ .

Moreover, if the control functions  $u_i^*(t) = -R_{ii}^{-1}B_i^T K_i \Phi(t)x_0$  are used to control the system (1), the spectrum of the closed-loop matrix  $A - S_1 K_1 - S_2 K_2$  coincides with  $\sigma(-M|_{\mathcal{K}})$ .  $\square$

From this result we first of all observe that every element of  $\mathcal{K}^{pos}$  defines exactly one solution of (ARE). Furthermore, this set contains only a finite number of elements if and only if the geometric multiplicities of all eigenvalues of  $M$  is one (see e.g. Lancaster and Tismenetsky (1985)). So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions. Furthermore, we see that

Corollary 5:

(ARE) will have a set of solutions  $(K_1, K_2)$  stabilizing the closed-loop system matrix  $A - S_1 K_1 - S_2 K_2$  if and only if there exists an  $M$  invariant subspace  $\mathcal{K}$  in  $\mathcal{K}^{pos}$  such that  $\text{Re } \lambda > 0$  for all  $\lambda \in \sigma(M|_{\mathcal{K}})$ .  $\square$

So, the study of the equilibria of our LQ game boils down to the study of all  $M$  invariant subspaces  $\mathcal{K}$  in  $\mathcal{K}^{pos}$  for which  $\text{Re } \lambda > 0$  for all  $\lambda \in \sigma(M|_{\mathcal{K}})$ .

The next example illustrates that in general there may be more than one equilibrium:

Example 6:

In Engwerda et al. (1995) the following example was considered:

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = I_4, \quad Q_1 =$$

$\text{diag}(3.5, 2, 4, 5)$ ,  $Q_2 = \text{diag}(1.5, 6, 3, 1)$ ,  $R_{11} = I_4$ , and  $R_{22} = 2I_4$ .

By considering the eigenvalues and corresponding eigenspaces of matrix  $M$ , it was shown that (ARE) has 7 solutions  $K_1, K_2$  satisfying the stabilization property. So, according to theorem 2, for this choice of matrices, the infinite planning horizon game has 7 open-loop Nash equilibria.  $\square$

## 5 Concluding remarks

In this paper we considered the existence of open-loop Nash equilibrium solutions in the two-player infinite-planning horizon linear quadratic game. We derived both necessary and sufficient conditions for existence of such equilibria. Furthermore we showed how these equilibria can be calculated from the invariant subspaces of the Hamiltonian matrix associated with this game  $M =$

$$\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}.$$

It turns out that the eigenvalues of the closed-loop system, if the open-loop control strategies are implemented in (1), can be obtained from the eigenvalues of this matrix.

As was illustrated in an example, in general the game will have more than one equilibrium. An important open problem remains the study of the eigenstructure of this matrix  $M$ .

One property, which was already noted in Engwerda (1996), is that if a discounting factor is included in the performance function that is large enough, matrix  $M$  will have  $n$  stable eigenvalues. So, in that case there exists at most one equilibrium. Since generically the corresponding eigenvectors will be such that they together form an element of  $\mathcal{K}^{pos}$ , we get generically a unique equilibrium in the discounted case. Moreover, combining the results on scalar systems from Engwerda (1996) and the results of theorem 2, we have that for scalar systems there exists always a unique equilibrium.

Finally we note that the obtained results can be straightforwardly generalized to the  $N$  player game.

Given the results presented here, one might wonder whether we can exploit some of them to get results for the linear quadratic feedback Nash game. At a first glance this seems, however, not the case. This, since the corresponding Hamiltonian matrix  $M$  depends on the solutions of the corresponding algebraic Riccati equations (see e.g. Weeren (1995)). In Weeren et al. (1994) a sufficient condition can be found for existence of a feedback Nash equilibrium in linear stationary strategies for this differential game over an infinite planning horizon. Moreover, this paper contains a thorough dynamical analysis for the scalar case. For more references and results on this subject we refer to Başar and Olsder (1995), Weeren (1995) and the quoted references in both these references.

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