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INTERSECTION THEOREMS ON THE UNIT SIMPLEX AND THE SIMPLOTOPE

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ABSTRACT

In this paper we give a survey and present several new results of intersection theorems on the unit simplex and the simplotope. The most familiar intersection theorem on the unit simplex is the KKM lemma which states that under some boundary condition the intersection of $n+1$ closed subsets covering the n-dimensional unit simplex S^n is nonempty. Other intersection theorems with $n+1$ subsets are Scarf's lemma and its generalization without boundary conditions. We also consider Shapley's and Ichiishi's lemma, where the unit simplex S^n is covered by subsets C^T with T being a subset of the set $\{1, \ldots, n+1\}$ instead of an integer. We generalize these results to one without boundary condition and introduce for that purpose the concept of T-balancedness. Next we generalize the theorems stated on the unit simplex to the simplotope, this being the cartesian product of unit simplices. This leads to some known intersection results on the simplotope but also to several new theorems, thereby generalizing the concept of balancedness and T-balancedness to the simplotope.

1 INTRODUCTION

Intersection theorems are used to prove the existence of solutions to mathematical programming problems. The most well-known intersection theorem is probably the Knaster-Kuratowski-Mazurkiewicz lemma. This lemma (see [5]) states that $n+1$ closed subsets covering the n-dimensional unit simplex S^n and satisfying some boundary condition have a nonempty intersection. The unit simplex S^n is the subset of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ where all the components are nonnegative and sum up to one. If we label the

$n+1$ subsets of S^n by $C^1, \ldots, C^{n+1}$ then the boundary condition is that for every x in the boundary of the unit simplex there is some index i for which $x \in C^i$ and $x_i > 0$. The dual of the KKM lemma was introduced in Scarf [12]. This lemma says that $n+1$ closed sets $C^1, \ldots, C^{n+1}$ covering S^n have a nonempty intersection if for x in the boundary it holds that $x \in C^i$ for some index i for which $x_i = 0$. The Scarf lemma was generalized in [10] as follows. If $C^1, \ldots, C^{n+1}$ are closed subsets covering S^n then there exists an $x^* \in S^n$ such that $x^*_j = 0$ whenever $x^* \notin C^j$.

The KKM lemma has been generalized in Shapley [13] to an intersection theorem of sets C^T with T arbitrary subsets of the index set $\{1, \ldots, n+1\}$ instead of consisting out of just one index. Shapley's lemma states that a collection of closed subsets $C^T, T \subset \{1, \ldots, n+1\}$, covering S^n contains a balanced collection of sets $C^{T_1}, \ldots, C^{T_k}$ having a nonempty intersection if for every x in the boundary of S^n there exists a subset C^T containing x such that $x_j > 0$ for all $j \in T$. The lemma of Shapley was introduced to prove the nonemptiness of the core of a balanced cooperative game with nontransferable utilities of the players. Its dual counterpart was proved in Ichiishi [3] and gives the same result when for every point x in the boundary of S^n there exists a subset C^T containing x such that $i \in T$ for all indices i for which $x_i = 0$. In this paper we generalize the lemma of Ichiishi to an intersection theorem in which the subsets C^T do not satisfy any boundary condition. For that purpose we need to generalize balancedness to T-balancedness with respect to some subset T of the index set $\{1, \ldots, n+1\}$. Then it can be proved that for a collection of closed subsets $C^T, T \subset \{1, \ldots, n+1\}$, covering S^n there exists an $x^* \in S^n$ such that x^* lies in the intersection of some T^*-balanced collection of subsets, where T^* is the set of indices j for which $x^*_j > 0$.

In the paper we also generalize the intersection theorems on the unit simplex mentioned above to such theorems on the simplotope, this being the product space of several unit simplices. We first give intersection theorems where the number of subsets covering the simplotope is equal to the number of variables and these sets therefore can be labelled by the indices corresponding to the variables. These theorems were stated in Freund [2], van der Laan, Talman, and Van der Heyden [9], and van der Laan and Talman [7]. Next we consider intersection theorems where the subsets covering the simplotope are labelled by a set of indices. Generalizations of the KKM lemma and the Scarf lemma are obtained when these sets are labelled by exactly N indices where N is the number of unit simplices out of which the simplotope exists. These theorems were developed recently in van der Laan and Talman [8] and can be used to prove the existence of a Nash equilibrium in a noncooperative game. Finally, we generalize the results of Shapley and Ichiishi to intersection theorems on the simplotope. For that purpose we generalize the concept of balancedness to the simplotope. Also T-balancedness for an arbitrary index set T is introduced and an intersection theorem is given for an arbitrary collection of closed subsets covering the simplotope. This results in an nonempty intersection of some T-balanced subcollection of subsets where the set T is the set of indices for which the corresponding components of the intersection point is positive.

This paper is organized as follows. Section 2 consists of the mathematical preliminaries. Several concepts and some notations are introduced concerning the simplotope and intersection properties. Also a unifying result for the existence of a stationary point is given. This result is used in almost all proofs for showing the intersection theorems. Section 3 discusses the intersection theorems on the unit simplex whereas Section 4 treats the theorems on the simplotope.

2 PRELIMINARIES

For some positive integer k, let S^k be the k-dimensional unit simplex, i.e.,

$$S^k = \{x \in \mathbb{R}^{k+1} | \sum_{j=1}^{k+1} x_j = 1, x_i \geq 0 \text{ for } i = 1, \ldots, k+1\}.$$

Let $n_1, \ldots, n_N$ be N positive integers for some given positive integer N and let n be equal to $\sum_{j=1}^{N} n_j$. We call the cartesian product of $S^{n_1}, \ldots, S^{n_N}$, denoted S, a simplotope, so

$$S = S^{n_1} \times \cdots \times S^{n_N}.$$

The dimension of S is equal to n. An element in S is denoted by

$$x = (x_1, \ldots, x_N) \text{ with } x_j \in S^{n_j} \text{ for all } j.$$

The k-th component of the vector x_j in S^{n_j} is denoted by x_{jk} or $x_{j,k}$ and is also called the (j, k)-th component of an element x in S, for $k \in \{1, \ldots, n_j + 1\}$. For $j = 1, \ldots, N$, the set $I(j)$ will be equal to the index set $\{(j, 1), \ldots, (j, n_j + 1)\}$, and I will denote the union of $I(j)$ over all j. The set I_k will denote the index set $\{1, \ldots, k\}$.

Let F be an upper-hemicontinuous mapping from a nonempty, convex, compact subset X in $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ to the collection of (nonempty) subsets of X such that for each x in X the set $F(x)$ is convex and compact, then according to

Kakutani's fixed point theorem there exists an x^* in X such that $x^* \in F(x^*)$, see [4].

Next, let F be an upper-hemicontinuous mapping from a nonempty, convex, compact subset $X = \prod_{j=1}^N X^j$ in $\prod_{j=1}^N \mathbb{R}^{n_j+1}$ to (the set of subsets of) $\prod_{j=1}^N \mathbb{R}^{n_j+1}$ such that for every $x \in X$ the set $F(x)$ is nonempty, convex, compact and such that $\bigcup_{x\in X} F(x)$ is a bounded set. We call an element x^* in X a stationary point of F on X if for some y^* in $F(x^*)$ it holds that for all $j \in I_N$

$$x_j^\top y_j^* \leq (x_j^*)^\top y_j^* \text{ for all } x_j \in X^j.$$

Lemma 2.1 *The point-to-set mapping F has a stationary point on X.*

Proof Let Y be a compact, convex set in $\prod_{j=1}^N \mathbb{R}^{n_j+1}$, containing the set $\bigcup_{x\in X} F(x)$. Then we define the mapping H from Y to (the set of subsets of) X by

$$H(y) = \{x^* \in X | x_j^\top y_j \leq (x_j^*)^\top y_j \text{ for all } x_j \in X^j \text{ and } j \in I_N\}.$$

Clearly, H is upper-hemicontinuous and for every y in Y the set $H(y)$ is nonempty, convex, compact, whereas $\bigcup_{y\in Y} H(y)$ as a subset of X is bounded. For $(x,y) \in X \times U$, let $G(x,y)$ be defined by $G(x,y) = H(y) \times F(x)$, then G is an upper-hemicontinuous mapping from $X \times Y$ into itself satisfying that for every (x,y) in $X \times Y$ the set $G(x,y)$ is nonempty, convex, compact. Therefore, according to Kakutani's fixed point theorem, there exists an $(x^*, y^*) \in X \times Y$ such that $x^* \in H(y^*)$ and $y^* \in F(x^*)$, which proves the lemma. □

Given the unit simplex S^n, for a nonempty subset T of I_{n+1}, m^T denotes the barycentre of the face $S^n(T)$ of S^n, where $S^n(T) = \{x \in S^n | x_j = 0 \text{ for every } j \notin T\}$. So, $m_j^T = 1/|T|$ for $j \in T$ and $m_j^T = 0$ for $j \notin T$, with $|T|$ denoting the number of elements of T. Similarly, given the simplotope S, let T be a subset of the index set I such that $T \cap I(j) \neq \emptyset$ for every $j \in I_N$. Then m^T will denote the barycentre of the face $S(T)$ of S where $S(T) = \{x \in S | x_{jk} = 0$ for every $(j,k) \notin T\}$, i.e., $m_{jk}^T = 1/|T \cap I(j)|$ for $(j,k) \in T$ and $m_{j,k}^T = 0$ for $(j,k) \notin T$. When T is equal to the index set I (or I_{n+1} on the unit simplex) we often write m instead of m^I (or $m^{I_{n+1}}$). When for $T \subset I_{n+1}$ the set T consists of only one index, say j, we also write $e(j)$ instead of $m^{\{j\}}$. Similarly, when for $T \subset I$ the set $T \cap I(j)$ consists of only one index for every $j \in I_N$, we also write $e(T)$ instead of m^T. Notice that for such a T the vector m^T is a vertex of S and that $e_j(T)$ is an (n_j+1)-dimensional unit vector for every $j \in I_N$.

3 INTERSECTION THEOREMS ON THE UNIT SIMPLEX

The most well-known intersection theorems on the unit simplex are the lemma's of Knaster–Kuratowski–Mazurciewicz (KKM) [5], Scarf [12], Ichiishi [3], and Shapley [13]. All these theorems give sufficient conditions under which a certain subset of closed sets covering the unit simplex has a nonempty intersection. In this section we state and prove these theorems by using either Kakutani's fixed point theorem or Lemma 2.1 and we generalize the result of Ichiishi to an intersection theorem without boundary condition.

Theorem 3.1 (KKM lemma) *Let $C^1, C^2, \ldots, C^{n+1}$ be closed subsets covering S^n such that if x lies in the boundary of the unit simplex then $x \in C^i$ for some index i for which $x_i > 0$. Then*

$$\bigcap_{i=1}^{n+1} C^i \neq 0.$$

<u>Proof</u> Let W^n be the set defined by

$$W^n = \{w \in \mathbb{R}^{n+1} | \sum_{j=1}^{n+1} w_j = 1, \ \ w_j \leq 1 + (n+1)^{-1} \text{ for all } j\}.$$

Then W^n is the convex hull of the $n+1$ points $w(1), \ldots, w(n+1)$, given by

$$\begin{aligned} w_i(j) \ &= -(n^2+n-1)/(n+1) && \text{if } i = j \\ &= 1 + (n+1)^{-1} && \text{if } i \neq j. \end{aligned}$$

Clearly, W^n contains the n-dimensional unit simplex S^n in its interior. For $w \in W^n$, let $p(w)$ be the relative projection of w on S^n, i.e., $p(w) = w$ when $w \in S^n$ and

$$\begin{aligned} p_j(w) \ &= 0 && \text{if } w_j \leq 0 \\ &= w_j / \textstyle\sum_{\{i|w_i>0\}} w_i > 0 && \text{if } w_j > 0 \end{aligned}$$

when $w \in W^n \backslash S^n$.

Next let the point-to-set mapping F from W^n to the set of subsets of W^n be defined by

$$F(w) = \text{Conv}(\{w(j) | p(w) \in C^j \text{ and } w_j \geq 0\}).$$

Then F is upper-hemicontinuous and for every w in W^n the set $F(w)$ is a nonempty, convex, compact subset of W^n. According to Kakutani's fixed point theorem, there exists an w^* in W^n such that $w^* \in F(w^*)$. We will show that w^* lies in S^n and that $w^* \in C^i$ for every $i \in \{1,\ldots,n\}$. Let T^* be the subset of $\{1,\ldots n+1\}$ such that $j \in T^*$ if and only if $p(w^*) \in C^j$ and $w_j^* \geq 0$. By assumption, the set T^* is nonempty.
Suppose that w^* lies in $W^n \backslash S^n$. Then there exists an index h such that $w_h^* < 0$. Consequently, h is not an element of T^*. Since $w^* \in F(w^*)$, there exist nonnegative numbers $\lambda_j^*, j \in T^*$, summing up to one such that

$$w^* = \sum_{j \in T^*} \lambda_j^* w(j).$$

Since $w_h^* < 0, h \notin T^*$, and $w_h(j) = 1 + (n+1)^{-1}$ when $j \neq h$, we obtain that

$$0 > w_h^* = \sum_{j \in T^*} \lambda_j^* [1 + (n+1)^{-1}] = 1 + (n+1)^{-1} > 0,$$

yielding a contradiction. Therefore w^* lies in S^n. We now show that $T^* = \{1,\ldots,n+1\}$. Suppose that $h \notin T^*$ for some $h \in \{1,\ldots,n+1\}$. Then since $w_h^* \leq 1, h \notin T^*$, and $w_h(j) = 1 + (n+1)^{-1}$ when $j \neq h$,

$$1 \geq w_h^* = \sum_{j \in T^*} \lambda_j^* [1 + (n+1)^{-1}] = 1 + (n+1)^{-1} > 1,$$

yielding again a contradiction. Consequently, $T^* = \{1,\ldots,n+1\}$ and so $w^* \in \bigcap_{i=1}^{n+1} C^i$. □

Theorem 3.2 (Generalized Scarf lemma [10]) *Let $C^1, C^2, \ldots, C^{n+1}$ be closed subsets covering S^n. Then there exists an $x^* \in S^n$ such that $x_j^* = 0$ whenever $x^* \notin C^j$, $j = 1,\ldots,n+1$.*

<u>Proof</u> Let the point-to-set mapping F from S^n to the set of subsets of S^n be defined by

$$F(x) = \text{Conv}(\{e(j) | x \in C^j\}).$$

Then F is upper-hemicontinuous and for every $x \in S^n$ the set $F(x)$ is nonempty, convex and compact. According to Kakutani's fixed point theorem there exists an x^* in S^n such that $x^* \in F(x^*)$. Let T^* be such that $T^* = \{j | x^* \in C^j\}$.

Then there exist nonnegative numbers λ_j^*, $j \in T^*$, summing up to one such that

$$x^* = \sum_{j \in T^*} \lambda_j^* e(j).$$

Consequently, $x_j^* = 0$ when $j \notin T^*$, and $x^* \in C^j$ when $j \in T^*$, which proves the theorem. □

Corollary 3.3 (Scarf lemma) *Let $C^1, C^2, \ldots, C^{n+1}$ be closed subsets covering S^n such that if x lies in the boundary of S^n then $x \in C^i$ for some index i for which $x_i = 0$. Then $\bigcap_{i=1}^{n+1} C^i \neq \emptyset$.*

Proof Since $C^1, C^2, \ldots, C^{n+1}$ are closed subsets covering S^n there exists according to Theorem 3.2 an $x^* \in S^n$ such that $x^* \notin C^j$ implies $x_j^* = 0$ for $j = 1, \ldots, n+1$. Suppose $x^* \notin C^h$ for some $h \in I_{n+1}$. Then $x_h^* = 0$. Let $\{x^\ell | \ell = 1, 2, \ldots\}$ be a sequence of points in S^n converging to x^* such that $x_h^\ell = 0$ and $x_j^\ell > 0$ for all $j \neq h$. By the statement of the corollary, x^ℓ must lie in C^h for every ℓ. Since C^h is closed and x^ℓ converges to x^* for ℓ going to infinity, x^* must lie in C^h, yielding a contradiction. Therefore, $x^* \in C^j$ for every $j \in I_{n+1}$. □

Other intersection theorems on the unit simplex are obtained when the unit simplex is covered by subsets C^T with T being arbitrary subsets of I_{n+1}. A well known result from Shapley [13] is that under some boundary condition similar to the one in the KKM lemma the intersection of at least one balanced collection of sets C^T is nonempty.

Definition 3.4 *Let B be a collection of a finite number of subsets of I_{n+1}, say $B = \{T_1, \ldots, T_k\}$. The collection B is* **balanced** *if there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ such that $\sum_{j=1}^k \lambda_j^* = 1$ and $\sum_{j=1}^k \lambda_j^* m^{T_j} = m$.*
The collection B is **balanced with respect to some subset** *T^* of I_{n+1} (or T^*-balanced) if there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ such that $\sum_{j=1}^k \lambda_j^* =$ 1 and for some $\alpha^* > 0$*

(i) $\sum_{j=1}^k \lambda_j^* m_i^{T_j} = \alpha^*$ *if* $i \in T^*$.

(ii) $\sum_{j=1}^k \lambda_j^* m_i^{T_j} \leq \alpha^*$ *if* $i \notin T^*$.

Balancedness of B means that every set T^j in B can be given a weight λ_j^* such that for every index $i \in I_{n+1}$ it holds that

$$\sum_{i \in T_j} \frac{\lambda_j^*}{|T_j|} = \frac{1}{n+1}.$$

So for every $i \in I_{n+1}$ the total weight of index i in the sets T_j containing i, being λ_j^* divided by the number of indices in T_j, aggregated over all $j = 1, \ldots, k$ is the same and therefore equal to $(n+1)^{-1}$. In case of T^*-balanced the same holds for every $i \in T^*$ whereas the total weight for $i \notin T^*$ might be less. Notice that a balanced collection is T^*-balanced for every nonempty $T^* \subset I_{n+1}$.

The next theorem states that when S^n is covered by closed subsets C^T such that if x lies in the boundary of S^n there is some index set T such that $x \in C^T$ and $x_j > 0$ for every $j \in T$, then there exists a balanced collection of subsets having a nonempty intersection. This result is called the Shapley lemma and is a generalization of the KKM-lemma, see also [14] for a similar proof.

Theorem 3.5 (Shapley lemma) *Let $C^T, T \subset I_{n+1}$, be a collection of closed subsets covering S^n such that for every x in the boundary of S^n there is a subset C^T containing x for which $x_j > 0$ for all $j \in T$. Then there exists a balanced collection of index sets $T_1, \ldots T_k$ such that $\bigcap_{j=1}^k C^{T_j} \neq \emptyset$.*

Proof Let W^n be as defined in the proof of Theorem 3.1, let $p(w)$ be again the relative projection of $w \in W^n$ on S^n, and for $j = 1, \ldots, n+1$ let $w(j)$ be the vertex of W^n defined by

$$\begin{aligned} w_i(j) &= -(n^2+n-1)/(n+1) && \text{if } i = j \\ &= 1 + (n+1)^{-1} && \text{if } i \neq j. \end{aligned}$$

Next, let the point-to-set mapping F from W^n to $\mathbb{R}^{n+1}$ be defined by

$$F(w) = \text{Conv}(\{m - m^T | p(w) \in C^T \text{ and } w_j \geq 0 \text{ for all } j \in T\})$$

Notice that $\sum_{i=1}^{n+1} y_i = 0$ for all $y \in F(w)$. Clearly, F is an upper-hemicontinuous mapping and for every $w \in W^n$ the set $F(w)$ is nonempty, convex and compact. Moreover, the set $\bigcup_{w \in W^n} F(w)$ is compact. Hence, according to Lemma 2.1 there exist $x^* \in W^n$ and $y^* \in F(x^*)$ such that

$$x^\top y^* \leq (x^*)^\top y^* \quad \text{for all } x \in W^n.$$

We will show that x^* lies in S^n and that x^* lies in the intersection of a balanced set of C^T's. Let $(x^*)^\top y^*$ be equal to α^* then $\alpha^* \geq 0$ if we take x equal to m.

If we take x equal to $w(j) = (n+2)m - (n+1)e(j)$, we obtain that $y_j^* \geq -\alpha^*/(n+1)$ for all $j = 1, \ldots, n+1$. When $x_j^* < 1 + (n+1)^{-1}$ we obtain for x equal to $(1+\varepsilon)x^* - \varepsilon w(j)$ for small enough $\varepsilon > 0$ that $y_j^* \leq -\alpha^*/(n+1)$. Therefore,

$$\begin{aligned} y_j^* &= -\alpha^*/(n+1) \quad \text{if } x_j^* < 1 + (n+1)^{-1} \\ &\geq -\alpha^*/(n+1) \quad \text{if } x_j^* = 1 + (n+1)^{-1}. \end{aligned}$$

Let B^* be the collection of subsets $T_1, \ldots, T_k$ such that, for $j = 1, \ldots, k$, $p(x^*) \in C^{T_j}$ and $x_i^* \geq 0$ for all $i \in T_j$. Since $y^* \in F(x^*)$ there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ summing up to one such that

$$y^* = \sum_{j=1}^{k} \lambda_j^*(m - m^{T_j}). \tag{1}$$

Suppose that $x_i^* < 0$ for some $i \in \{1, \ldots, n+1\}$. Then $i \notin T_j$ for $j = 1, \ldots, k$ and therefore $y_i^* = 1/(n+1) > 0$. On the other hand, $x_i^* < 0$ implies that $x_i^* < 1 + (n+1)^{-1}$ and hence that $y_i^* = -\alpha^*(n+1)^{-1} \leq 0$. Consequently, from this contradiction it follows that $x_i^* \geq 0$ for all i and so $x^* \in S^n$. The latter implies that $x_i^* < 1 + (n+1)^{-1}$ for all i and hence that $y_i^* = -\alpha^*/(n+1)$ for all i. Since $\sum_{i=1}^{n+1} y_i^* = 0$, this yields $\alpha^* = 0$ and so $y_i^* = 0$ for $i = 1, \ldots, n+1$. From (3.1) it then follows that

$$\sum_{j=1}^{k} \lambda_j^* m^{T_j} = m.$$

Consequently, the collection B^* is balanced. Moreover, since $x^* \in S^n$ and hence $p(x^*) = x^*$, we also have that $x^* \in \bigcap_{j=1}^{k} C^{T_j}$, which proves the theorem. □

The next result can be considered as the dual of Shapley lemma and is due to Ichiishi [3].

Theorem 3.6 (Ichiishi lemma) *Let C^T, $T \subset I_{n+1}$, be a collection of closed subsets covering S^n such that for every x in the boundary of S^n there is a subset C^T containing x for which $i \in T$ when $x_i = 0$. Then there exists a balanced collection of index sets $T_1, \ldots, T_k$ such that $\bigcap_{j=1}^{k} C^{T_j} \neq \emptyset$.*

Proof Let the set V^n be given by

$$V^n = \{v \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} v_i = 1, v_i \geq -(n+1)^{-1} \text{ for all } i \in I_{n+1}\}.$$

Clearly, V^n is the convex hull of the points $v(i) = 2e(i) - m$, $i = 1, \dots, n+1$. Let $p(v)$ be the relative projection of $v \in V^n$ on S^n as defined in the proof of Theorem 3.1. Next, let the mapping F from V^n to the set of subsets of $\mathbb{R}^n$ be defined by

$$F(v) = \text{Conv}(\{m^T - m | p(v) \in C^T \text{ and } i \in T \text{ if } v_i < 0\}.$$

Then F is upper-hemicontinuous and for every $v \in V^n$ the set $F(v)$ is nonempty, convex and compact. Moreover the set $\bigcup_{v \in V^n} F(v)$ is compact. According to Lemma 2.1, there exist $x^* \in V^n$ and $y^* \in F(x^*)$ such that for all $x \in V^n$

$$x^\top y^* \leq (x^*)^\top y^*.$$

Let B^* be the collection of index sets $T_1, \dots, T_j$ such that for all $j \in \{1, \dots, k\}$ it holds that $p(x^*) \in C^{T_j}$ and $i \in T_j$ if $x_i^* < 0$. We will show that B^* is balanced. Let α^* be equal to $(x^*)^\top y^*$, then $\alpha^* \geq 0$ when we take x equal to m. Moreover, when we take x equal to $v(i)$, $i \in \{1, \dots, n+1\}$, we obtain $y_i^* \leq \frac{1}{2}\alpha^*$. On the other hand, in case $x_i^* > -(n+1)^{-1}$ and we take x equal to $(1+\varepsilon)x^* - \varepsilon v(i)$ for small enough $\varepsilon > 0$ we obtain $y_i^* \geq \frac{1}{2}\alpha^*$. All of this together implies

$$\begin{aligned} y_i^* \; & = \tfrac{1}{2}\alpha^* \geq 0 && \text{when } x_i^* > -(n+1)^{-1} \\ & \leq \tfrac{1}{2}\alpha^* && \text{when } x_i^* = -(n+1)^{-1}. \end{aligned}$$

Moreover, for $j = 1, \dots, k$ we have that $i \in T_j$ when $x_i^* = -(n+1)^{-1}$, by construction of B^*. Since $y^* \in F(x^*)$, there exist nonnegative numbers $\lambda_1^*, \dots, \lambda_k^*$ summing up to one such that

$$y^* = \sum_{j=1}^{k} \lambda_j^* (m^{T_j} - m).$$

Therefore, when $x_i^* = -(n+1)^{-1}$ and so $i \in T_j$ for all j, we have that

$$y_i^* = \sum_{j=1}^{k} \lambda_j^* (|T_j|^{-1} - (n+1)^{-1}) \geq 0.$$

Consequently, $y_i^* \geq 0$ for all i. Since $\sum_{i=1}^{n+1} y_i^* = 0$, this implies $y^* = 0$ and so $\alpha^* = 0$. Hence,

$$\sum_{j=1}^{k} \lambda_j^* m^{T_j} = m.$$

So, B^* is a balanced collection of index sets $T_1, \ldots, T_k$ such that $p(x^*) \in \bigcap_{j=1}^k C^{T_j}$, which completes the proof. □

The next theorem is new and states that when S^n is covered by an arbitrary collection of closed subsets C^T with $T \subset \{1, \ldots, n+1\}$, then there exists an $x^* \in S^n$ such that x^* is an intersection point of some T^*-balanced collection of sets, with $T^* = \{j | x_j^* > 0\}$.

Theorem 3.7 *Let C^T, $T \subset I_{n+1}$, be a collection of closed subsets covering S^n. Then there exists an $x^* \in S^n$ such that for $T^* = \{i | x_i^* > 0\}$ there is a T^*-balanced collection of index sets $T_1, \ldots, T_k$ for which $x^* \in \bigcap_{j=1}^k C^{T_j}$.*

Proof Let the point-to-set mapping F from S^n to the set of subsets of S^n be defined by

$$F(x) = \mathrm{Conv}(\{m^T | x \in C^T\})$$

The mapping F is upper–hemicontinuous and for every x in S^n the set $F(x)$ is nonempty, convex and compact. Moreover, the union of $F(x)$ over all $x \in S^n$ is bounded. Hence, according to Lemma 2.1 there exist $x^* \in S^n$ and $y^* \in F(x^*)$ such that

$$x^\top y^* \leq (x^*)^\top y^* \quad \text{for all } x \in S^n.$$

When we take x equal to m, we obtain that $\alpha^* = (x^*)^\top y^* > 0$. Moreover, when we take x equal to $e(i)$ for $i \in \{1, \ldots, n+1\}$ we get

$$\begin{aligned} y_i^* &= \alpha^* \quad \text{if } x_i^* > 0 \\ &\leq \alpha^* \quad \text{if } x_i^* = 0. \end{aligned}$$

Next, let B^* be the collection of index sets $T_1, \ldots, T_k$ such that $x^* \in C^{T_j}$ for $j = 1, \ldots, k$. We will show that B^* is T^*-balanced where $T^* = \{i | x_i^* > 0\}$. Since $y^* \in F(x^*)$ there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ summing up to one such that

$$y^* = \sum_{j=1}^{k} \lambda_j^* m^{T_j}.$$

Consequently, since $\sum_{j=1}^k \lambda_j^* = 1, y_i^* = \alpha^*$ when $x_i^* > 0$, and $y_i^* \leq \alpha^*$ when $x_i^* = 0$, we have that

$$\begin{aligned} \textstyle\sum_{i \in T^j} \lambda_j^* / |T_j| &= \alpha^* \quad \text{for } i \in T^* \\ &\leq \alpha^* \quad \text{for } i \notin T^*. \end{aligned}$$

Therefore, the collection B^* is T^*-balanced. Furthermore,

$$x^* \in \bigcap_{j=1}^{k} C^{T_j}$$

and $x_i^* = 0$ whenever $i \notin T^*$, which concludes the theorem. □

In the theorems above we gave all kinds of intersection theorems on the unit simplex in case this set is covered by a finite number of sets labelled by subsets of the index set I_{n+1}, where n is the dimension of the simplex. First we discussed theorems in which the sets are labelled by just one index out of the set I_{n+1}. Then we gave theorems where the sets are labelled by arbitrary sets of indices out of I_{n+1}. In the next section we generalize these theorems to intersection theorems on the simplotope. We remark that there are also intersection theorems on S^n in which the sets covering S^n are labelled by index sets different from being elements or subsets from the set $\{1, \ldots, n+1\}$, e.g. see [1].

4 INTERSECTION THEOREMS ON THE SIMPLOTOPE

In this section we consider intersection theorems on the simplotope S where each of the sets is labelled by an index or by a set of indices in I. Labelling by sets of indices on S is new but labelling by just one index on S is well-known. One of these theorems, see Kuhn [6], concerns a collection of sets equal to the dimension of the simplotope plus one and is a trivial generalization of the KKM lemma on the unit simplex. In Freund [2] and in van der Laan, Talman, and Van der Heyden [9] an intersection theorem is given on S for a collection of closed sets covering S equal to the number of variables in S. This theorem can be considered as a generalization of the generalized Scarf lemma on the unit simplex to the simplotope.

Theorem 4.1 ([2] and [9]) *Let $C^{j,k}$, $(j,k) \in I$, be a collection of closed subsets covering S. Then there exists an $x^* \in S$ such that for some $j^* \in \{1, \ldots, N\}$ it holds that $x^*_{j^*,k} = 0$ if $x^* \notin C^{j^*,k}$, $k = 1, \ldots, n_{j^*} + 1$.*

Proof Let the point-to-set mapping φ from S to the set of subsets of $\mathbb{R}^{n+1}$ be defined by

$$\varphi(x) = \text{Conv}(\{e(j,k) | x \in C^{j,k}\}),$$

where $e_{j,k}(j,k) = 1$ and $e_{i,h}(j,k) = 0$ for $(i,h) \neq (j,k)$.
Then φ is upper hemi-continuous and for every $x \in S$ the set $\varphi(x)$ is a compact, convex, nonempty set. Moreover, the set $\bigcup_{x \in S} \varphi(x)$ is compact. Hence, according to Lemma 2.1 there exist $x^* \in S$ and $y^* \in \varphi(x^*)$ such that for all j

$$x_j^\top y_j^* \leq (x_j^*)^\top y_j^* \quad \text{for all } x \in S.$$

Letting $(x_j^*)^\top y_j^*$ be equal to α_j^* we obtain for all $(j,k) \in I$ that

$$\begin{array}{lll} y_{j,k}^* & = \alpha_j^* & \text{if } x_{jk}^* > 0 \\ & \leq \alpha_j^* & \text{if } x_{jk}^* = 0. \end{array}$$

On the other hand, let T^* be the set of indices (j,k) such that $x^* \in C^{j,k}$. Then, since $y^* \in \varphi(x^*)$, there exists nonnegative $\lambda_{j,k}^*$, $(j,k) \in T^*$, with $\sum_{(j,k) \in T^*} \lambda_{j,k}^* = 1$, such that

$$y^* = \sum_{(j,k) \in T^*} \lambda_{j,k}^* e(j,k).$$

Since $\sum_{(j,k) \in T^*} \lambda_{j,k}^* = 1$, there exists an $j^* \in I_N$ such that $\sum_{(j^*,k) \in T^*} \lambda_{j^*,k}^* > 0$. Clearly, $y_{j^*,k}^* = \lambda_{j^*,k}^*$ for all $(j^*,k) \in T^*$. Hence, $y_{j^*,k}^* > 0$ for at least one index (j^*,k), and so $\alpha_{j^*}^* > 0$. Consequently, $y_{j^*,k}^* = \lambda_{j^*,k}^* > 0$ if $x_{j^*,k}^* > 0$, which can only be the case when $(j^*,k) \in T^*$. Therefore, $(j^*,k) \in T^*$ whenever $x_{j^*,k}^* > 0$. So, $(j^*,k) \notin T^*$ and hence $x^* \notin C^{j^*,k}$ implies $x_{j^*,k}^* = 0$. □

Notice that in the theorem it is not guaranteed that for all $j = 1, \ldots, N$ it holds $x_{jk}^* = 0$ whenever $x^* \notin C^{j,k}$. This is caused by the fact that $T^* \cap I(j)$ might be empty for some $j \in I_N$. From Theorem 4.1 the next corollary immediately follows.

Corollary 4.2 *Let $C^{j,k}$, $(j,k) \in I$, be a collection of closed subsets covering S such that $C^{j,k}$ contains x whenever $x_{jk} = 0$. Then for some $j^* \in I_N$, the set $\bigcap_{(j^*,k) \in I(j^*)} C^{j^*,k} \neq \emptyset$.*

Theorem 4.1 can be considered as a generalization of Theorem 3.2. The next two theorems generalize in the same way the KKM lemma and Scarf lemma on S^n to S, respectively.

Theorem 4.3 (KKM lemma on S [7]) *Let $C^{j,k}$, $(j,k) \in I$, be a collection of closed subsets covering S such that if x lies in the boundary of S then $x \in C^{j,k}$ for some $(j,k) \in I$ for which $x_{j,k} > 0$. Then there is an index $j^* \in I_N$ such that*

$$\bigcap_{k=1}^{n_{j^*}+1} C^{j^*,k} \neq \emptyset.$$

Proof Let the set V in $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ be given by

$$V = \prod_{j=1}^{N} V^{n_j},$$

with $V^{n_j}, j \in \{1, \ldots, N\}$, defined as in the proof of Theorem 3.7, i.e.,

$$V^{n_j} = \{v_j \in \mathbb{R}^{n_j+1} | \sum_{k=1}^{n_j+1} v_{jk} = 1, v_{jk} \geq -(n_j+1)^{-1} \text{ for all } k\}.$$

Clearly, for $j \in I_N$, V^{n_j} is the convex hull of the points $v_j(j,k) = 2e_j(j,k) - m_j$, $k = 1, \ldots, n_j + 1$. For $v \in V$, the point $p(v)$ in S will denote the relative projection of v on S, i.e., $p(v) = (p_1(v_1), \ldots, p_N(v_N))$, with $p_j(v_j)$ the relative projection of v_j in V^{n_j} on S^{n_j} as defined in the proof of Theorem 3.1. Next, let the point-to-set mapping F from V to the set of subsets of $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ be defined by

$$F(v) = \mathrm{Conv}(\{m(j) - e(j,k) | p(v) \in C^{j,k} \text{ and } v_{jk} \geq 0\},$$

where $m_j(j) = m_j$ and $m_i(j) = 0$ for $i \neq j$. Then F is upper-hemicontinuous and for every $v \in V$ the set $F(v)$ is nonempty, convex and compact. Moreover, the set $\cup_{v \in V} F(v)$ is compact. According to Lemma 2.1 there exist $x^* \in V$ and $y^* \in F(x^*)$ such that for all $x \in V$ and $j \in I_N$,

$$x_j^\top y_j^* \leq (x_j^*)^\top y_j^*.$$

Let α_j^* be equal to $(x_j^*)^\top y_j^*$, then $\alpha_j^* \geq 0$ by taking $x_j = m_j$, since $m_j^\top y_j^* = 0$, $j = 1, \ldots, N$. When we take x_j equal to the vertex $v_j(j,k)$ of V^{n_j+1} we obtain $2y_{jk}^* \leq \alpha_j^*$, for $(j,k) \in I$. On the other hand, if $x_{jk}^* > -(n_j+1)^{-1}$ and when taking x_j equal to $-\varepsilon v_j(j,k) + (1+\varepsilon)x_j^*$ for arbitrarily small $\varepsilon > 0$, we obtain that $2y_{jk}^* \geq \alpha_j^*$. Therefore,

$$y_{jk}^* \begin{array}{ll} = \frac{1}{2}\alpha_j^* \geq 0 & \text{if } x_{jk}^* > -(n_j+1)^{-1} \\ \leq \frac{1}{2}\alpha_j^* & \text{if } x_{jk}^* = -(n_j+1)^{-1}. \end{array}$$

Let $T^* = \{(j,k) \in I | p(x^*) \in C^{j,k}$ and $x^*_{jk} \geq 0\}$, and for $j = 1, \ldots, N$ let $T^*_j = T^* \cap I(j)$. Since $y^* \in F(x^*)$ there exist nonnegative numbers $\lambda^*_{j,k}$ for $(j,k) \in T^*$ such that

$$y^* = \sum_{(j,k) \in T^*} \lambda^*_{j,k}(m(j) - e(j,k))$$

and $\sum_{(j,k)} \lambda^*_{j,k} = 1$. Suppose that $x^*_{jk} = -(n_j+1)^{-1}$ for some $(j,k) \in I$. Then $(j,k) \notin T^*$ and hence $y^*_{jk} \geq 0$. Therefore,

$$0 \leq y^*_{jk} = \frac{1}{2}\alpha^*_j \quad \text{if } x^*_{jk} > -(n_j+1)^{-1}$$
$$0 \leq y^*_{jk} \leq \frac{1}{2}\alpha^*_j \quad \text{if } x^*_{jk} = -(n_j+1)^{-1}.$$

Since $\sum_{k=1}^{n_j+1} y^*_{jk} = 0$, for every $j \in I_N$ we must have that $\alpha^*_j = 0$ and $y^*_{jk} = 0$ for $(j,k) \in I(j)$, so $y^*_j = 0$. Since $\sum_{(j,k)\in T^*} \lambda^*_{j,k} = 1$, there is a $j^* \in I_N$ such that

$$\lambda^* := \sum_{(j^*,k) \in T^*_{j^*}} \lambda_{j^*k} > 0.$$

Together this implies $(j^*,k) \in T^*$ and $\lambda^*_{j^*,k} = \lambda^*(n_{j^*}+1)^{-1} > 0$ for every $(j^*,k) \in I(j^*)$. Hence, $x^* \in C^{j^*,k}$ for all $(j^*,k) \in I(j^*)$. □

Corollary 4.4 (Scarf lemma on S) *Let $C^{j,k}$, $(j,k) \in I$, be a collection of closed subsets covering S such that if x lies in the boundary of S then $x \in C^{j,k}$ for some index (j,k) for which $x_{jk} = 0$. Then there is a $j^* \in I_N$ such that*

$$\bigcap_{k=1}^{n_{j^*}+1} C^{j^*,k} \neq \emptyset.$$

<u>Proof</u> From Theorem 4.1 it follows that there exists an $x^* \in S$ such that for some $j^* \in I_N$ it holds that $x^*_{j^*,k} = 0$ when $x^* \notin C^{j^*,k}$. This implies that $x^* \in C^{j^*,k}$ when $x^*_{j^*,k} > 0$. Suppose now that $x^*_{j^*,k} = 0$ for some $(j^*,k) \in I(j^*)$. Then we will show that also $x^* \in C^{j^*,k}$. Let $\{x^\ell, \ell = 1,2,\ldots\}$ be a sequence of points in S converging to x^* such that $x^\ell_{j^*,k} = 0$ and $x^\ell_{jk} > 0$ for all $(j,k) \neq (j^*,k)$, $\ell = 1,2,\ldots$. Then $x^\ell \in C^{j^*,k}$ for all ℓ and therefore also $x^* \in C^{j^*,k}$ since $C^{j^*,k}$ is closed. □

In the previous theorems which can be considered as direct generalizations of the KKM and Scarf lemmas, the sets covering S were labelled by just one index. It is also possible to cover S by sets labelled by a vector or set of indices. First we give theorems where each set of the collection is labelled by a set of N indices, one index for each $j \in I_N$. Let $\mathcal{T}$ be the collection of sets T of indices $(j,k) \in I$ such that the set $T \cap I(j)$ consists of one element for each $j \in I_N$. When C^T, $T \in \mathcal{T}$, is a collection of sets covering S then for $T^* \subset I$ a point x lies in the set C^{T^*} if there exist $T_1, \ldots, T_k$ in $\mathcal{T}$ such that $T^* = \bigcup_{j=1}^{k} T_j$ and

$$x \in \bigcap_{j=1}^{k} C^{T_j}.$$

Theorem 4.5 (van der Laan and Talman [8]). *Let C^T, $T \in \mathcal{T}$, be a collection of closed subsets covering S. Then there is an $x^* \in S$ such that for some $T^* \subset I$ it holds that $x^* \in C^{T^*}$ and $x^*_{jk} = 0$ if $(j,k) \notin T^*$.*

Proof Let the point-to-set mapping F from S into the set of subsets of S be given by

$$F(x) = \text{Conv}(\{e(T) | x \in C^T\}).$$

Clearly, for every $x \in S$ the set $F(x)$ is convex, nonempty and compact, and F is upper-hemicontinuous. According to Kakutani's fixed point theorem on S, there exists an x^* such that $x^* \in F(x^*)$, i.e., there exist nonnegative numbers λ_i^*, $i = 1, \ldots, k$, such that $x^* = \sum_{i=1}^{k} \lambda_i^* e(T_i^*)$, where $T_1^*, \ldots, T_k^*$ are such that $x^* \in C^{T_i^*}$ for all i. Let T^* be the union of T_i^* over all i. Then $x^* \in C^{T^*}$ and, since $e_{jh}(T_i^*) = 0$ when $(j,h) \notin T_i^*$, $x^*_{jh} = 0$ if $(j,h) \notin T^*$. □

Also this theorem can be considered as a generalization of the Generalized Scarf lemma on the unit simplex, since it coincides with Theorem 3.2 in case N is equal to 1. The next result generalizes Scarf lemma in the same way.

Corollary 4.6 *Let C^T, $T \in \mathcal{T}$, be a collection of closed subsets covering S such that if x lies in the boundary of S then $x \in C^T$ for some $T \in \mathcal{T}$ containing an index $(j,k) \in I$ for which $x_{jk} = 0$. Then $C^I \neq \emptyset$.*

Proof According to Theorem 4.5 there exists an $x^* \in S$ such that $x^* \in C^{T^*}$ for some $T^* \subset I$ for which $x^*_{jk} = 0$ when $(j,k) \notin T^*$. We will show that $x^* \in C^I$. Clearly, $(j,k) \in T^*$ if $x^*_{jk} > 0$. So suppose that $x^*_{jk} = 0$. Let $\{x^\ell, \ell = 1, 2, \ldots\}$

be a sequence of points in S such that $x^{\ell}_{jk} = 0$ and $x^{\ell}_{ih} > 0$ for all $(i,h) \neq (j,k)$ whereas x^{ℓ} converges to x^* if ℓ goes to infinity. Then for every ℓ there exists a $T^{\ell} \in \mathcal{T}$ such that $(j,k) \in T^{\ell}$ and $x^{\ell} \in C^{T^{\ell}}$. Since there are only a finite numbers of index sets in $\mathcal{T}$, there is a T^o such that $T^{\ell} = T^o$ for infinitely many ℓ. Without loss of generality we can assume that $T^{\ell} = T^o$ for every ℓ. Consequently, $x^{\ell} \in C^{T^o}$ for every ℓ and $(j,k) \in T^o$. Since C^{T^o} is closed and x^{ℓ} converges to x^* we must have that $x^* \in C^{T^o}$. Hence, $x^* \in C^T$ with $T = T^* \cup T^o$. In this way we can extend for every $(j,k) \in I$ for which $x^*_{jk} = 0$ the set T^* with an index set $T^o \in \mathcal{T}$ such that $x^* \in C^{T^o}$ and $(j,k) \in T^o$. Consequently, $x^* \in C^I$. □

The next theorem can also be found in [8] and is a generalization of the KKM lemma.

Theorem 4.7 *Let C^T, $T \in \mathcal{T}$, be a collection of closed subsets covering S such that if x lies in the boundary of S then $x \in C^T$ for some $T \in \mathcal{T}$ for which $x_{jk} > 0$ when $(j,k) \in T$. Then $C^I \neq \emptyset$.*

Proof Let the set V be defined as in the proof of Theorem 4.3 and let the point-to-set mapping F from V to the set of subsets of $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ be given by

$$F(v) = \text{Conv}(\{m - e(T) | p(v) \in C^T \text{ and } (j,k) \in T \text{ implies } v_{jk} \geq 0\}).$$

Then F is upper-hemicontinuous, $\bigcup_{v \in V} F(v)$ is a compact set, and for every $v \in V$ the set $F(v)$ is nonempty, convex and closed. Following the proof of Theorem 4.3 there exist $x^* \in V$ and $y^* \in F(x^*)$ such that for all $j \in I_N$ for $\alpha^*_j = (x^*_j)^\top y^*_j$

$$\begin{array}{lll} y^*_{jh} & = \frac{1}{2}\alpha^*_j \geq 0 & \text{if } x^*_{jh} > -(n_j+1)^{-1} \\ & \leq \frac{1}{2}\alpha^*_j & \text{if } x^*_{jh} = -(n_j+1)^{-1}. \end{array}$$

Let $\mathcal{T}^* = \{T \in \mathcal{T} | p(x^*) \in C^T \text{ and } (j,k) \in T \text{ implies } x^*_{jk} \geq 0\}$ and let T^* be the union of $T \in \mathcal{T}^*$. We will show that T^* is equal to I. Let $\mathcal{T}^*$ be equal to the collection of sets $T_1, \ldots, T_k \in \mathcal{T}$. Since $y^* \in F(x^*)$, there exist nonnegative numbers $\lambda^*_1, \ldots, \lambda^*_k$ with sum equal to one such that

$$y^* = \sum_{i=1}^{k} \lambda^*_i (m - e(T_i)).$$

Suppose that $x^*_{jh} = -(n+1)^{-1}$ for some $(j,h) \in I$. Then $e_{jh}(T_i) = 0$ for all i and hence $y^*_{jh} \geq 0$. Therefore, $y^*_{jh} \geq 0$ for all $(j,k) \in I$ and since $\sum_k y^*_{jh} = 0$ we must have that $y^*_j = 0$ for $j = 1,\dots,N$. Hence, for every $(j,h) \in I$

$$(n_j+1)^{-1} = \sum_{i=1}^{k} \lambda^*_i e_{jh}(T_i) > 0.$$

This implies that for every $(j,h) \in I$ it must hold that $e_{jh}(T_i) > 0$ for at least one $i \in \{1,\dots,k\}$. Consequently, for every $(j,h) \in I$ there is an $i \in \{1,\dots,k\}$ such that $(j,h) \in T^i$, i.e., $T^* = \bigcup_{i=1}^{k} T^i$ is equal to I, and hence $x^* \in C^I$. □

In the previous theorems the sets C^T covering S were such that for every $j \in I_N$ the set $T \cap I(j)$ consists of just one element. In the next theorems we allow T to consist of more than one index out of each $I(j)$, $j \in I_N$. So, let $\mathcal{I}$ be the collection of subsets T of I such that for every $j \in I_N$ the set $T \cap I(j)$ consists of at least one element. Then balancedness of a collection of sets in $\mathcal{I}$ is defined as follows.

Definition 4.8 *Let $B = \{T_1,\dots,T_k\}$ be a collection of index sets such that $T_i \in \mathcal{I}$ for every $i = 1,\dots,k$. Then B is* **balanced** *if there exist nonnegative numbers $\lambda^*_1,\dots,\lambda^*_k$ such that it holds that $\sum_{j=1}^{k} \lambda^*_j = 1$ and*

$$\sum_{j=1}^{k} \lambda^*_j m^{T_j} = m.$$

*The collection B is T^*-***balanced with respect to some set** *$T^* \subset I$ if there exist nonnegative numbers $\lambda^*_1,\dots,\lambda^*_k$ summing up to one and positive numbers $\alpha^*_1,\dots,\alpha^*_N$ such that*

$$\begin{aligned} \textstyle\sum_{j=1}^{k} \lambda^*_j m^{T_j}_{ih} &= \alpha^*_i \quad \text{if } (i,h) \in T^* \\ &\leq \alpha^*_i \quad \text{if } (i,h) \notin T^*. \end{aligned}$$

Balancedness and T-balancedness have the same interpretation as these concepts have on the unit simplex, with the addition that for every index $(i,h) \in I$ (or T^*) the aggregated weight in every index set where (i,h) belongs to must be the same as for every other index $(i,\ell) \in I$ (or T^* and not less than for $(i,\ell) \notin T^*$). The next theorems are all new.

Theorem 4.9 (Shapley lemma on S) *Let C^T, $T \subset \mathcal{I}$, be a collection of closed subsets covering S such that for every x in the boundary of S there exists* *a subset C^T containing x for which $x_{j,k} > 0$ for all $(j,k) \in T$. Then there exists a balanced collection of index sets $T_1, \ldots, T_k$ such that $\bigcap_{j=1}^{k} C^{T_j} \neq \emptyset$.*

Proof Let W be equal to the cartesian product of the sets $W^{n_1}, \ldots, W^{n_N}$ with W^{n_j}, $j \in \{1, \ldots, N\}$, defined as in the proof of Theorem 3.1. Let the point-to-set mapping F from W to the set of subsets of $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ be defined as in the proof of Theorem 3.5, i.e.,

$$F(w) = \text{Conv}(\{m - m^T | p(w) \in C^T \text{ and } w_{j,k} \geq 0 \text{ for all } (j,k) \in T\}),$$

where $p(w)$ is the relative projection of $w \in W$ on S. Following the proof of Theorem 3.5 there exist $x^* \in W, y^* \in F(x^*)$ and nonnegative numbers $\alpha_1^*, \ldots, \alpha_N^*$ such that

$$\begin{aligned} y_{jk}^* \quad &= -\alpha_j^*/(n_j+1) \quad \text{if } x_{jk}^* < 1 + (n_j+1)^{-1} \\ &\geq -\alpha_j^*/(n_j+1) \quad \text{if } x_{jk}^* = 1 + (n_j+1)^{-1}. \end{aligned}$$

Let B^* be the collection of subsets $T_1, \ldots, T_k$ such that for $j = 1, \ldots, k, p(x^*) \in C^{T_j}$ and $x_{ih}^* \geq 0$ for all $(i,h) \in T_j$.
Since $y^* \in F(x^*)$ there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ summing up to one such that

$$y_i^* = \sum_{j=1}^{k} \lambda_j^*(m_i - m_i^{T_j}) \text{ for all } i \in I_N.$$

Suppose that $x_{ih}^* < 0$ for some $(i,h) \notin I$. Then $(i,h) \in T_j$ for $j = 1, \ldots, k$ and therefore $y_{ih}^* = (n_i+1)^{-1} > 0$. On the other hand $x_{ih}^* < 0$ implies that $y_{ih}^* = -\alpha_i^*(n_i+1)^{-1} \leq 0$. Consequently, $x_{ih}^* \geq 0$ for all $(i,h) \in I$ and therefore $y_{ih}^* = -\alpha_i^*(n_i+1)^{-1} \leq 0$ for all $(i,h) \in I$. Since $\sum_{h=1}^{n_i+1} y_{ih}^* = 0$ this implies $\alpha_i^* = 0$ and so $y_i^* = 0$ for all $i \in I_N$. Hence, for every $i \in I_N$,

$$\sum_{j=1}^{k} \lambda_j^* m_i^{T_j} = m_i.$$

which proves the theorem. □

Similar to the proofs of Theorems 3.6 and 4.7 we can generalize the lemma of Ichiishi to the simplotope.

Theorem 4.10 (Ichiishi lemma on S) *Let C^T, $T \subset \mathcal{I}$, be a collection of closed subsets covering S such that if x lies in the boundary of S there exists* *a subset C^T containing x for which $(i,h) \in T$ for all indices (i,h) with $x_{ih} = 0$. Then there exists a balanced collection of index sets $T_1, \ldots, T_k$ such that $\bigcap_{j=1}^k C^{T_j} \neq \emptyset$.*

Proof Let V be the cartesian product of the sets $V^{n_1}, \ldots, V^{n_N}$, where for each j the set V^{n_j} is defined as in the proof of Theorem 3.6. Let F be the mapping from V to the set of subsets of $\prod_{j=1}^N \mathbb{R}^{n_j+1}$ defined by

$$F(v) = \text{Conv}(\{m^T - m | p(v) \in C^T \text{ and } i \in T \text{ if } v_i < 0\}).$$

Then there exist $x^* \in V$, $y^* \in F(x^*)$, and nonnegative numbers $\alpha_1^*, \ldots, \alpha_N^*$ such that

$$\begin{aligned} y_{ih}^* &= \tfrac{1}{2}\alpha_i^* \quad \text{if } x_{ih}^* > -(n_i+1)^{-1} \\ &\leq \tfrac{1}{2}\alpha_i^* \quad \text{if } x_{ih}^* = -(n_i+1)^{-1}. \end{aligned}$$

Let B^* be the collection of index sets $T_1, \ldots, T_k$ such that for all j it holds that $p(x^*) \in C^{T_j}$ and $(i,h) \in T_j$ if $x_{ih}^* = 0$. Then there exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ summing up to one such that

$$y_i^* = \sum_{j=1}^k \lambda_j^* (m_i^{T_j} - m_i) \text{ for all } i \in I_N.$$

Therefore, when $x_{ih}^* = -(n_i+1)^{-1}$ and so $(i,h) \in T_j$ for all $j, y_{ih}^* \geq 0$. Consequently, $\alpha_i^* = 0$ and $y_i^* = 0$ for all $i \in I_N$, and hence

$$\sum_{j=1}^k \lambda_j^* m^{T_j} = m,$$

which proves the theorem. □

The next theorem generalizes Theorem 3.7 to the simplotope. This theorem states that when S is covered by an arbitrary set of closed subsets C^T with $T \in \mathcal{I}$, then there exists an $x^* \in S$ such that x^* is an intersection point for some T^*-balanced collection of sets, with $T^* = \{(i,h) | x_{ih}^* > 0\}$.

Theorem 4.11 *Let C^T, $T \in \mathcal{I}$, be a collection of closed subsets covering S. Then there exists an $x^* \in S$ such that for $T^* = \{(i,h) \in I | x_{ih}^* > 0\}$ there is a T^*-balanced collection of index sets $T_1, \ldots, T_k$ for which $x^* \in \bigcap_{j=1}^k C^{T_j}$.*

Proof Let the point-to-set mapping F from S to the set of subsets of $\prod_{j=1}^{N} \mathbb{R}^{n_j+1}$ be defined by

$$F(x) = \text{Conv}(\{m^T | x \in C^T\}).$$

Following the proof of Theorem 3.7 there exist $x^* \in S$, $y^* \in F(x^*)$, and nonnegative numbers $\alpha_1^*, \ldots, \alpha_N^*$ such that

$$\begin{aligned} y_{ih}^* & = \alpha_i^* \quad \text{if } x_{ih}^* > 0 \\ & \leq \alpha_i^* \quad \text{if } x_{ih}^* = 0. \end{aligned}$$

Next, let B^* be again the collection of index sets $T_1, \ldots, T_k$ such that $x^* \in C^{T_j}$ for every j. We will show that B^* is T^*-balanced where $T^* = \{(i,h) \in I | x_{ih}^* > 0\}$. There exist nonnegative numbers $\lambda_1^*, \ldots, \lambda_k^*$ summing up to one such that

$$y_i^* = \sum_{j=1}^{k} \lambda_j^* m_i^{T_j} \text{ for all } i \in I_N.$$

Consequently,

$$\begin{aligned} \textstyle\sum_{(i,h)\in T_j} \lambda_j^* / |T_j^i| & = \alpha_i^* \quad \text{when}(i,h) \in T^* \\ & \leq \alpha_i^* \quad \text{when}(i,h) \notin T^*, \end{aligned}$$

where $T_j^i = T_j \cap I(i)$, which proves the theorem. □

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