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Investment under Uncertainty and Policy Change*

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Abstract

In this paper the impact of policy change on the investment behavior of the firm is studied. The change occurs when a stochastic process describing the state of the economic environment reaches a certain trigger. In our setting both the firm's conjecture concerning the trigger as well as the precision of this conjecture serve as input parameters. We derive the optimal investment rule maximizing the value of the firm. We show that the impact of trigger value uncertainty is non-monotonic: the investment threshold decreases with the trigger value uncertainty for low levels of uncertainty, while the reverse is true for high uncertainty levels. Furthermore, it is shown that the uncertainty concerning the magnitude of the change delays investment. Finally, based on the firm's value-maximizing behavior, policy implications for the authority are presented.

Keywords: investment under uncertainty, real options, policy change

JEL classification: C61, D81, G31

1 Introduction

Corporate investment opportunities may be represented as a set of (real) options to acquire productive assets. In the literature it is widely assumed that the present values of cash flows generated by these assets are uncertain and that their evolution can be described by a stochastic process. Consequently, an appropriate identification of the optimal exercise strategies for real options plays a crucial role in capital budgeting and in the maximization of a firm's value.

So far, the real options literature provides relatively little insight into the impact of structural changes of the economic environment on the investment decisions of a firm. The existing papers (excellent surveys of those are provided by Dixit and Pindyck [5] and by Lander and Pinches [11]) mainly consider continuous changes in the value of relevant variables. This, most of the time, results in the assumption that the entire uncertainty in the economy can be described by a geometric Brownian motion process.

It is often more realistic to model an economic variable as a process that makes infrequent but discrete jumps.[1] In such cases use is made of a Poisson (jump) process. An interesting application is provided by Hassett and Metcalf [9] who analyze the impact of an expected reduction in the investment tax credit.[2] In their setting a Poisson process describes the changes in the tax regime that affect the value of the investment opportunity. Within such a framework an implicit assumption is made that the firm has very limited information about the mechanisms governing the shocks in the economy.

When a change in the economic environment reflects a new policy implemented by the authority, it may be more realistic to assume that the firm has some conjecture about the expected moment of the change. Referring to the example of the investment tax credit, the firm typically expects the reduction to be imposed when the economy is booming and an active pro-investment policy is no longer needed or desired.[3] Conversely, applying the Poisson based methodology is equivalent to assuming that it is time itself and not the state of economic environment that governs the change.

[1] For instance, recent tax debates across Europe are a significant source of uncertainty associated with discountinuous changes in the economic environment.

[2] Another, albeit less closely related, reference is Pennings [15] that provides the example of optimal fiscal policy associated with investment when the cash flows received after completing the project are uncertain.

[3] Although some of the changes of tax rates in Europe result from the need to unify the EU tax systems, in many cases the policy change can be attributed to the pace of economic growth. After a period of fast economic growth, in 1999 Ireland announced an end to its 10% corporate tax rate for new foreign manufacturing and financial investors as one of the means to avoid "overheating" of the economy. Other EU proposals include abandoning corporate tax exemptions in Germany and withdrawing approximately seventy tax reliefs used so far by European governments to draw investment. The tax reliefs subject to change range from Belgian exemptions on multinational haedquarters to incentives made by Spain for investors in the Basque region. Cf. "Hey, Let's All Get Together and Raise Taxes!", *Businessweek*, 25 Nov., 1998 and "Ireland: Burning Too Bright; Can Ireland control its rapid growth?", *Businessweek*, 10 Apr., 2000.

Moreover, the firm (at least to some extent) can assess the precision of its conjecture concerning the moment of change, i.e. the variance of the estimate of the timing of the future event. A Poisson based approach does not allow for including this type of uncertainty in the analysis since it entails a single parameter characterizing the arrival rate of the jump. Consequently, such a modelling approach lacks degrees of freedom to capture both the expectation and the precision of this expectation.[4]

In this paper, we propose a method to model the impact of a policy change on the investment strategy of the firm that takes into account the type of information possessed by the firm while making the investment decision. In our approach the subjective expectation concerning the moment of the change as well as the level of imprecision of such a conjecture serve as input parameters. We model the policy change as being triggered by a sufficiently high realization of a stochastic process related to the value of the investment opportunity. This, for instance, reflects the fact that - as we already argued - a tax credit reduction is more likely to occur when the economy is booming. Hence, the moment of the reduction depends on the state of the economy. This is in contrast with the models based on the Poisson process where the probability of the change is constant over time.[5]

There are other economic situations in which it is realistic to impose a certain relationship between the occurrence of the shock and the state of the economy. A foreign direct investment decision to purchase a privatized enterprise where the local government may increase the offering price after the performance of the enterprise improves, can also be perceived as an option with an embedded risk of an increase in the strike price. A non-exclusive investment opportunity for which a competitive bid can be expected can serve as another example.[6,7]

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[4] It may be more realistic to assume that a firm expects Mr. Greenspan to increase the interest rate when the DJIA reaches the barrier of 12,000 pts with a precision +/- 500 pts than to assume that the firm's conjecture about such an event can be expressed with a single Poisson arrival rate. The latter would mean that the occurrence of an interest rate increase solely depends on time and the level of uncertainty concerning this event is predetermined by the arrival rate.

[5] Hassett and Metcalf [9] try to correct this by letting the arrival rate depend on the output price. But still it is then possible that an investment subsidy is reduced for low output prices, while the subsidy was maintained under high output prices. This kind of inconsistency in the authority's behavior is no longer possible under our approach.

[6] See Smets [19] and Cherian and Perotti [3] for a discussion of the effects of strategic interactions and political risk.

[7] The same idea can also be applied within the topic of technology adoption. In Farzin et al. [6] the arrival of a more efficient technology satisfies a Poisson process (this assumption is also adopted by Baudry [1] where the new technology has the advantage of being less polluting, and by Mauer and Ott [12] where maintenance and operation cost are lowered after the technological breakthrough). This way of modeling is satisfactory only when the firm has no insight at all in the innovation process of new technologies. If, instead, the firm could observe progress (but has no perfect information), a way to model it is to introduce a variable that stands for the state of technological progress. The firm is able to observe perfectly the realizations of this variable. As soon as the state of technological progress hits a certain barrier, which is ex ante unknown to the firm, the new technology is invented. This approach is similar to the one in Grenadier and Weiss [7] but there it was assumed that the value of

We consider the possibility of an upward jump in the (net) investment cost. This jump is caused, for instance, by the reduction of an investment tax credit. It occurs at the moment that an underlying variable reaches a certain trigger. Here, the underlying variable is the value of the investment project. The firm is not aware of the exact value of the trigger but it knows the probability distribution underlying the trigger. Taking into account consistent authority's behavior, the firm knows that a jump will not occur as long as the current value of the variable remains below the maximum that this variable has attained in the past. When the underlying variable reaches a new maximum and the jump does still not occur, the firm updates its conjecture about the value of the barrier.

Consequently, our objective is to determine the optimal timing of an irreversible investment when the investment cost is subject to change and the firm has incomplete information about the moment of the change. It is clear that the value of the investment opportunity drops to zero at the moment that the investment cost jumps to infinity. However, we mainly consider scenarios where the cost of investment is still finite after the upward jump occurred. In this way this work generalizes Lambrecht and Perraudin [10] , Schwartz and Moon [17], and Berrada [2], where the value of the project drops to zero at the unknown point of time.

Our main results are the following. An equation is derived that implicitly determines the value of the project at which the firm is indifferent between investing and refraining from the investment. This value is the optimal investment threshold and it is shown that this threshold is decreasing in the hazard rate of the cost-increase trigger. For the most frequently used density functions it holds that, for a given value of the project, the hazard rate first increases and then decreases with trigger value uncertainty. This leads to the conclusion that the investment threshold decreases with the *trigger value* uncertainty when the uncertainty is low, while it increases with uncertainty for high uncertainty levels. Hence, for a policy maker interested in accelerating investment, an optimal (strictly positive) level of the trigger value uncertainty can be identified which is the level corresponding to the minimal investment threshold. Furthermore, it is shown that the uncertainty concerning the *magnitude* of the change delays investment. This implies that an effective policy stimulating early investment is associated with minimizing the investors' uncertainty about the size of the expected change.

In Section 2 we recall the basic model of investment under uncertainty, while in Section 3 the investment cost jump resulting from a policy change is introduced. Section 4 provides the major results and Section 5 contains a numerical analysis including some comparisons with Poisson based models. Section 6 extends the model to allow for a stochastic size of the jump in the cost. In Section 7 we present the implications of our model for the authority that considers an investment tax credit policy change, and Section 8 concludes.

the barrier is known beforehand.

2 The Basic Model

We start by considering the basic model of investment under uncertainty developed by McDonald and Siegel [13], and extensively analyzed in Dixit and Pindyck [5]. The general problem is to find the optimal timing of an irreversible investment, I, given that the value of the investment project, $\{V_t : t \geq 0\}$, follows a geometric Brownian motion process

$$dV_t = \alpha V_t dt + \sigma V_t dw_t, \tag{1}$$

which is defined on the complete filtered probability space (- ,$\mathcal{F}, P, \mathbb{F}$) satisfying the usual hypotheses.[8] The parameter α denotes the deterministic drift parameter, σ is the instantaneous standard deviation, and dw is the increment of a Wiener process.[9] The firm is assumed to be risk-neutral and maximizes the expected present value of cash flow by choosing the optimal V at which the project is undertaken.[10] A well-described procedure (see Dixit and Pindyck [5]), involving the use of Itô's lemma and solving a differential equation under the corresponding value-matching and smooth-pasting conditions, yields the value of the optimal investment threshold, V_m:

$$V_m = \frac{\beta}{\beta - 1} I, \tag{2}$$

where

$$\beta = -\frac{\alpha}{\sigma^2} + \frac{1}{2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{3}$$

and r is the instantaneous interest rate.[11] For the value of the investment opportunity , $W(V)$, evaluated at $V \in (0, V_m)$, it holds that

$$W(V) = E\left[\int_{T_m}^{\infty} V\,(r-\alpha)\,e^{-rt}dt - Ie^{-rT_m}\right] = (V_m - I)\left(\frac{V}{V_m}\right)^{\beta}, \tag{4}$$

where T_m is the first passage time corresponding to the threshold V_m. By rearranging 4 we obtain that[12]

$$E\left[T_m\right] = -\frac{1}{\alpha - \frac{1}{2}\sigma^2} \ln\left(\frac{V}{V_m}\right). \tag{5}$$

According to (4), there are two factors determining the value of the investment opportunity. The first factor, $V_m - I$, corresponds to the net payoff realized

[8] Shiryaev [18] provides a detailed exposition of the probabilistic concepts applied to finance.

[9] To simplify notation, from now on we skip the time subscripts.

[10] Risk-neutrality assumption can be replaced by applying the replicating portfolio methodology. In order to capture the discountinuous changes of volatility and/or in the value of investment opportunity resulting from the policy change, we repeat the argument of Naik [14].

[11] The problem has a finite solution for $\alpha < r$.

[12] The expectation exists if and only if $\alpha > \frac{1}{2}\sigma^2$.

at the time of the optimal exercise. The second one, often referred to as a probability-weighted discount factor, $(V/V_m)^{\beta}$, allows for translating the future payoff from the investment opportunity into its present value.

The value of the optimal investment threshold is positively related both to the volatility of the project's value as well as to its growth rate (the higher they are, the higher V must be reached for the project to be undertaken). $W(V)$ increases in the volatility of the value of the project (β is a decreasing function of σ and W is decreasing in β) which results from the convex payoff of the investment opportunity. Moreover, W is increasing in the growth rate, α, since the effective discount rate of future cash flows decreases linearly in α.

3 The Model with a Jump in the Investment Cost

In this section we develop the model that allows for incorporating the impact of the expected policy change on the firm's investment strategy. If the value of the investment project reaches a critical level, a certain policy instrument is imposed and, as a result, an effective increase in the investment cost occurs.[13] This instrument can be interpreted, among others, as a reduction in the investment tax credit, an increase in the cost of capital via lending rates or an increase in the offering price for a privatized enterprise. Allowing for a broader interpretation, an arrival of a competitive firm offering a higher bid for a particular project belongs to the set of potential sources of the investment cost shock as well.

We denote by V^* such a realization of the process for which the new policy is imposed and the investment cost changes from I_l to I_h, where $I_h > I_l$. At this stage we assume that I_h is deterministic. Later we consider I_h to be stochastic and discuss implications of such an extension. The firm does not know the value of V^* but knows only its cumulative density function, $F(V^*)$. $F(\cdot)$ is continuous and twice differentiable everywhere in the interior of its domain. To provide a simple interpretation, we assume that $F(\cdot)$ is completely defined by its first two moments. Moreover, $F(\cdot)$ is stationary over time. Consequently, if the investment cost has not increased by time τ, while $\bar{V}$ is the highest realization of the process so far, the cost will not increase at any $u > \tau$ as long as $V_t \leq \bar{V}$ for all $t \leq u$. Hence, the probability of the jump in investment cost is a function of V alone.

In order to restrict our analysis to the most interesting case, we impose

[13] If, instead, a downward change in investment cost is considered, the same solution methodology can be applied as in the remainder of the paper. Consequently, a unique realization of the underlying process has to be found for which the marginal cost of waiting beyond the optimal investment threshold equals the benefit of waiting associated with the expected decrease in the investment cost.

the following assumptions on the values of the variables used in the model:

$$\begin{cases} V^*_{\max} > V_0 & (i) \\ V^*_{\min} < \frac{\beta}{\beta-1} I_l & (ii) \\ 1_{\{V^*<V_h\}} \left(V_h - I_h\right) \left(\frac{V^*}{V_h}\right)^{\beta} < V^* - I_l, & (iii) \end{cases} \tag{6}$$

where $V^*_{\min}$ and $V^*_{\max}$ are the lower and higher bound of the domain of $F(\cdot)$. V_0 denotes the initial value of the project and V_h $(\equiv \beta I_h / (\beta - 1))$ is the unconditional optimal investment threshold corresponding to the cost I_h.[14] Assumptions (i) and (ii) ensure that the problem is relevant, i.e. that the policy change has not occurred yet and that there is a positive probability that the change will take place before the optimal threshold corresponding to I_l is reached. Assumption (iii) states that ex post it is never optimal to wait with investing until the upward change in cost occurs.

3.1 Value of the Investment Opportunity

Since the value of the project that triggers the increase in the investment cost is not known beforehand, two scenarios are possible. In the first scenario the investment occurs *before* the change in the investment cost, and in the second scenario the investment takes place *after* the upward change. Consequently, the value of the investment opportunity reflects the structure of the expected payoff:

$$\begin{aligned} W_s(V, \bar{V} | I &= I_l) = p_s(\bar{V}) E\left[\int_{T_s}^{\infty} V\left(r-\alpha\right) e^{-rt} dt - I_l e^{-rT_s}\right] + \\ &+ \left(1 - p_s(\bar{V})\right) E\left[\int_{T_h}^{\infty} V\left(r-\alpha\right) e^{-rt} dt - I_h e^{-rT_h}\right], \end{aligned} \tag{7}$$

where $p_s(\bar{V})$ is the conditional (on the highest realization of V, $\bar{V}$) probability that the investment cost will not increase *before* the investment is made optimally, and T_s and T_h denote the first passage time corresponding to the optimal investment threshold at the low and at the high cost, respectively. Expectations of T_s and T_h can be calculated in a similar way as (5). After rearranging and including these expectations, we obtain the following maximization problem that allows for finding the optimal investment threshold:

$$\begin{aligned} W_s(V, \bar{V} | I &= I_l) = \max_{V_s} \left[(V_s - I_l) \left(\frac{V}{V_s}\right)^{\beta} \frac{1 - F(V_s)}{1 - F(\bar{V})} + \right. \\ &\left. + (V_h - I_h) \left(\frac{V}{V_h}\right)^{\beta} \left(1 - \frac{1 - F(V_s)}{1 - F(\bar{V})}\right) \right]. \end{aligned} \tag{8}$$

[14] 1_B denotes an indicator function of B such that $1_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$.

V_s is the optimal investment threshold in case the investment takes place before the change in cost, and $\bar{V}$ is the highest realization of the process so far. Hence, $(1-F(V_s))/(1-F(\bar{V}))$ is the probability that the jump in the investment cost will not occur by the moment V is equal to V_s, given that the shock has not occurred for V smaller than $\bar{V}$. Equation (8) is therefore interpreted as follows: the value of the investment opportunity is equal to the weighted average of the values of two investment opportunities. They correspond to the investment cost I_l and I_h, respectively, given that the investment is made optimally (at V_s if the cost is still equal to I_l and at V_h if the upward change has already occurred).[15]

The value of the investment opportunity depends on the highest realization of the process, $\bar{V}$. A higher $\bar{V}$ (thus a one closer to V_s) implies a lower probability of the trigger falling into the interval $(\bar{V}, V_s)$ and, as a consequence, a higher probability of making the investment at the lower cost, I_l. In order to calculate the value of the investment opportunity, we first need to establish the value of V_s by solving the maximization problem.

3.2 Optimal Investment Threshold

The optimal investment threshold, V_s, is determined by maximizing the value of the investment opportunity or the RHS of the Equation (8).

Proposition 1 *Under the sufficient condition that*

$$h'(V_s)V_s + h(V_s) \geq 0, \tag{11}$$

the investment is made optimally at V_s which is the solution to the following equation:

$$h(V_s)V_s^2 + (\beta-1)V_s - (V_s h(V_s) + \beta)I_l - h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}} = 0, \tag{12}$$

where $h(x) = \frac{F'(x)}{1-F(x)}$ denotes the hazard rate.[16]

[15] It is worth pointing out that for $I_h \to \infty$ the value of the investment opportunity boils down to:

$$W_s(V,\bar{V}|I=I_l) = \max_{V_s}(V_s - I_l)\left(\frac{V}{V_s}\right)^\beta \frac{1-F(V_s)}{1-F(\bar{V})}, \tag{9}$$

which directly corresponds to the result of Lambrecht and Perraudin [10]. In the other limiting case, i.e. for $I_h \to I_l$, the value of investment opportunity converges to

$$W_s(V,\bar{V}|I=I_l) = (V_l - I_l)\left(\frac{V}{V_l}\right)^\beta \tag{10}$$

which is the formula obtained by McDonald and Siegel [13].

[16] In our case, the hazard rate has the following interpretation. The probability of the upward change in the investment cost during the nearest increment of the value of the project, dV, (given that the cost-increase has not occurred by now) is equal to the appropriate hazard rate multiplied by the size of the value increment, i.e. to $h(x;\cdot)dV$.

Proof. See Appendix. ■

A sufficient condition for (11) to hold is that the hazard rate has to be non-decreasing.[17] Condition (11) is satisfied for most of the common density functions as, e.g., exponential, uniform and Pareto.[18]

4 Solution Characteristics

In this section we analyze the sensitivity of the optimal threshold with respect to changes in the parameters characterizing the dynamics of the project value. Moreover, we determine the direction of the impact of the changes in the investment costs under both policy regimes. Subsequently, we examine how the uncertainty concerning the moment of imposing the change influences the firm's optimal investment rule.

4.1 Changing the Parameters of the Investment Opportunity

We are interested in how potential changes in the characteristics of the investment opportunity influence the optimal investment rule. For this purpose we formulate the following proposition.

Proposition 2 *The effects on the investment threshold level of the changes in the different parameters are as follows:*

$$\frac{dV_s}{dI_l} > 0, \tag{13}$$

$$\frac{dV_s}{dI_h} < 0, \tag{14}$$

$$\frac{dV_s}{d\beta} < 0, \tag{15}$$

$\forall I_l, I_h$ *satisfying* $0 < I_l < I_h$, $\forall \beta \in (1, r/\alpha)$ *if* $\alpha > 0$ *and* $\forall \beta \in (1, \infty)$ *if* $\alpha \leq 0$.

Proof. See Appendix. ■

Consequently, the optimal threshold (*ceteris paribus*) increases in the initial investment cost and decreases in the size of the potential cost-increase as well as in the parameter β. The latter implies that the threshold increases with uncertainty of the value of the project and decreases with the wedge between interest rate and the project's growth rate. All these results are intuitively plausible.

[17] More precisely, the elasticity of the hazard rate with respect to the value of the process evaluated at the optimal investment threshold has to be larger than -1.

[18] In fact, the hazard rate based on the Pareto function is decreasing at the order of $1/x$ and the property (11) is still met.

4.2 Impact of Policy Change

The optimal investment rule depends not only on the characteristics of the project itself but also on the firm's conjecture about the probability distribution underlying the expected policy change. The parameters of this distribution can be influenced by actions of the authority. For instance, an information campaign about the expected changes in the investment tax credit leads to a reduction of the variance (often to zero) of the distribution underlying the value triggering the change. Therefore, it is important to know how changes in the uncertainty related to the project value triggering the jump in the investment cost influence the firm's optimal investment rule. Knowing that the firms are going to act optimally, the authority can implement a desired policy, which is, for instance, accelerating the investment expenditure, by changing the level of the firms' uncertainty about the tax strategy. We come back to this point in Section 7, where policy implications for the authority are considered.

4.2.1 Hazard Rate

The hazard rate of the arrival of the cost-increase trigger is one of the basic inputs for calculating the optimal investment threshold. Although it is exogenous to the firm, it may well be controlled by another party such as the authority. Here, we determine the impact of its change on the firm's investment rule. Later, we discuss some of the policy implications of the obtained result.

After applying the envelope theorem to the LHS of (12), we can formulate the following proposition.

Proposition 3 *The optimal investment threshold is decreasing in the corresponding hazard rate, i.e. the following inequality holds:*

$$\left.\frac{dV_s}{dh\left(V\right)}\right|_{V=V_s} < 0. \tag{16}$$

Proof. See Appendix. ■

This result implies that an increasing risk of the jump leads to an earlier optimal exercise. The intuition is quite simple: an increasing probability of a partial deterioration of the investment opportunity after a small appreciation in the project value decreases the value of waiting. Consequently, (16) implies that for any parameter of the density function underlying the jump, denoted by a, the following condition holds:

$$\forall a \quad \operatorname{sgn} \left.\frac{\partial h\left(V\right)}{\partial a}\right|_{V=V_s} = -\operatorname{sgn} \frac{dV_s}{da}. \tag{17}$$

Using (17) we can establish how the investment threshold is affected by changes in the parameters of the distribution function underlying the occurrence of the jump.

4.2.2 Trigger Value Uncertainty

Now the aim is to analyze how the optimal investment threshold is affected by uncertainty related to the value of the cost-increase trigger. To do so, we only need to establish the sign of the relationship between the hazard rate and the uncertainty related to the value of the trigger. We measure the trigger-value uncertainty by applying a mean-preserving spread (see Rotschild and Stiglitz [16])

If the cost-increase trigger is known with certainty, the investment is made optimally at an infinitesimal instant before V^* is reached. At this point, the hazard rate is zero (there is no risk that the cost increases before the optimal threshold is reached). As the uncertainty marginally increases, the hazard rate is affected by: 1) the value of the density function underlying the trigger, denoted by $f(V^*)$, and 2) a change in the value of the survival function, $1 - F(V^*)$. It is easy to verify that, for the most frequently used density functions, such as normal, uniform, exponential and Pareto, the value of the hazard rate, for any $V \in [V_0, E[V^*])$, first increases and then decreases in the mean-preserving spread. An example for the normal density function is shown in Figure 1.[19]

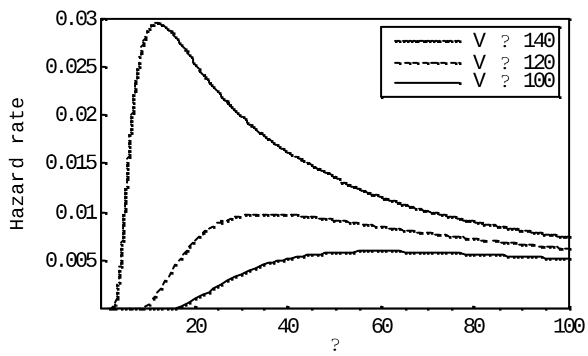

Figure 1

Figure 1. The relationship between the hazard rate and standard deviation of a normal density function $N\left(150, \omega^2\right)$. Hazard rates are plotted for $V = 100$, 120 and 140.

Moreover, for each degree of the trigger value uncertainty, there exists such a value of $V < E[V^*]$, say $\widetilde{V}$, that for $V \in [V_0, \widetilde{V})$ the hazard rate increases, and for $V \in (\widetilde{V}, E[V^*])$ decreases, in this uncertainty. This form of the relationship between the hazard rate and the uncertainty implies (via Proposition 3) that V_s decreases in the uncertainty if it falls into the interval $[V_0, \widetilde{V})$ and increases otherwise, as depicted in Figure 2.

[19] Although the concepts of the mean-preserving spread and increased standard deviation are, in general, not equivalent, they may be treated as such for the types of density functions referred to in this paper.

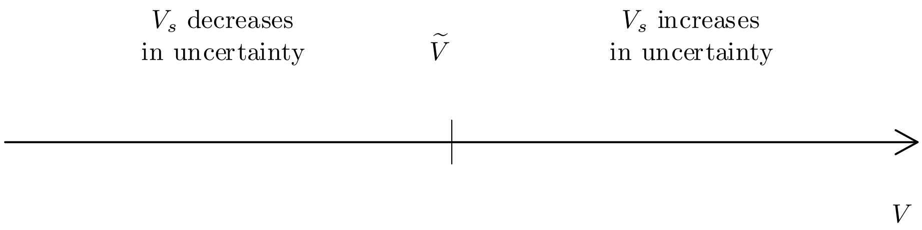

Figure 2

Figure 2. The relationship between trigger value uncertainty and the optimal investment threshold.

Consequently, in order to determine the sign of the effect of uncertainty on V_s, we need to establish the relative position of V_s with respect to $\widetilde{V}$. Let us denote the standard deviation of the density function underlying the cost-increase trigger by ω. Since the expression for V_s is already known (see (12)), all we have to calculate is $\widetilde{V}$ as a function of ω, such that, for each pair (V, ω), the following condition holds:[20]

$$\left.\frac{\partial h(V)}{\partial \omega}\right|_{V=\widetilde{V}} = 0. \tag{18}$$

For the most frequently used density functions it can be shown that $\widetilde{V}$ decreases with uncertainty. For a relatively low degree of uncertainty, it holds that $V_s < \widetilde{V}$ $(< E[V^*])$. Since for $V < \widetilde{V}$ the hazard rate increases in ω, V_s falls when the uncertainty rises. After the uncertainty reaches a critical level, say ω^e, at which $V_s = \widetilde{V}$, the hazard rate at V_s decreases in ω and the optimal threshold begins to increase. This implies that optimal investment threshold attains the minimum for $\omega = \omega^e$. Now, we are able to formulate the following proposition.

Proposition 4 *Consider the following unrestrictive conditions*

$$\begin{array}{l} \lim_{\omega \to \infty} f(V, \cdot) = 0, \ \forall V \quad \text{and} \\ f(V, \cdot) \text{ is unimodal,} \end{array} \tag{19}$$

Then, there exists a non-monotonic relationship between the optimal investment threshold and the trigger value uncertainty. At a low degree of uncertainty, the marginal increase in uncertainty leads to an earlier optimal investment. The reverse is true for a high degree of uncertainty. There exists a unique ω^e*, such that* $V_s(\omega^e) = \widetilde{V}(\omega^e)$*, which separates the areas of low and high uncertainty levels.*

Proof. Proposition 4 directly follows from the analysis performed so far. ■

[20] Although $\widetilde{V}(\omega)$ cannot be written explicitly in a general form, its values corresponding to a given density function may be easily found numerically.

The interpretation of the proposition is relatively simple. At low levels of uncertainty concerning the policy change the firm responds to an increase of this uncertainty by investing earlier (i.e. at a lower V). This is because the chance of earlier implementation of a new instrument increases. However, when this uncertainty becomes sufficiently high, the firm is more willing to ignore the information about the expected change since the quality of this information has deteriorated too much. The marginal impacts of a higher probability of an early change and of the increased "noisiness" of the firm's conjecture offset exactly at the level of uncertainty equal to ω^e.

Figures 3 and 4 show the relationship between the uncertainty, ω, and the optimal investment threshold.

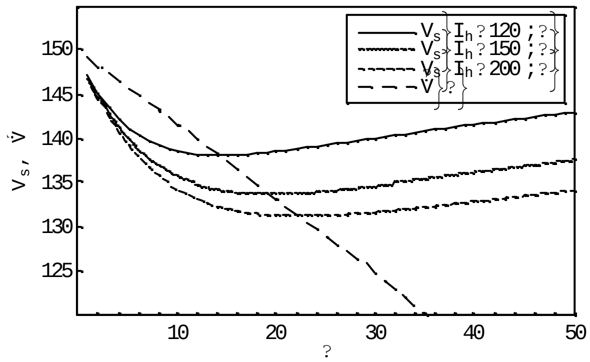

Figure 3

Figure 3. The relationship between the uncertainty, ω, and the optimal investment threshold, V_s, for different sizes of the high investment cost ($I_h = 120, 150$ and 200). The values are calculated for a normal density function with mean 150. The original investment cost, I_l equals 100. An intersection of V_s and $\widetilde{V}$ corresponds to the minimal investment threshold, $V_s(\omega^e)$. The parameters of the underlying process are: $\alpha = 0$, $r = 0.025$ and $\sigma = 0.1$.[21]

In Figure 3 it can be seen that the optimal investment threshold is first decreasing and then increasing in the uncertainty concerning the value of the trigger. The minimum is always reached when $V_s(\omega)$ intersects $\widetilde{V}(\omega)$. The hazard rate increases in ω in the area located to the south-west from $\widetilde{V}(\omega)$ and decreases in the north-eastern region. The opposite holds for V_s. Moreover, the optimal threshold is higher if the expected change in the investment cost is smaller (cf. Proposition 2).

[21] This set of parameters is used in Dixit [4], and Lambrecht and Perraudin [10] (we rescale the investment cost with the factor 100).

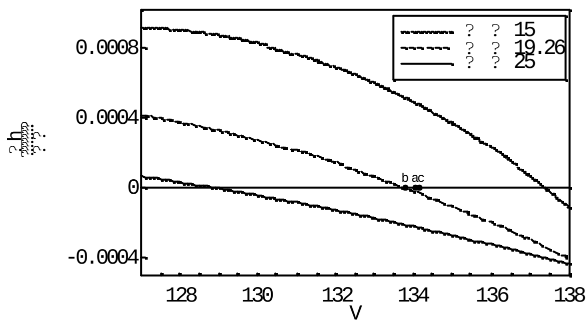

Figure 4

Figure 4. The relationship between V and the derivative of the hazard rate with respect to the trigger value uncertainty. The optimal investment thresholds for $I_h = 150$ and different uncertainty levels are shown on the horizontal axis (Point a corresponds to $V_s(15)$, b to $V_s(\omega^e = 19.26)$ and c to $V_s(25)$. The values are calculated for a normal density function with mean 150. The parameters of the underlying process are: $\alpha = 0$, $r = 0.025$ and $\sigma = 0.1$.

In Figure 4 it can be noticed that the point, $\widetilde{V}$, at which the derivative of the hazard rate is equal to zero moves to the left when the trigger uncertainty increases. As long as $V_s < \widetilde{V}$, the optimal threshold also moves to the left (cf. the location of $V_s(15)$). When the standard deviation is equal to $\omega^e = 19.26$, V_s equals $\widetilde{V}$. After a further increase in the uncertainty, $\widetilde{V}$ continues moving to the left and V_s starts moving to the right (cf. $V_s(25)$). For a sufficiently high degree of uncertainty V_s tends to the unconditional threshold, denoted by V_l $(\equiv \beta I_h / (\beta - 1))$.[22]

5 Comparative Statics

In this section we provide a numerical illustration of the results of our model. In Table 1 the relationship between the uncertainty about the timing of the jump in the investment cost and the optimal investment threshold is shown for different levels of the after-shock investment cost. The results are grouped in three panels corresponding to the different combinations of the rate of growth and volatility of the project's value.

[22] The necessary and sufficient condition for $\lim_{\omega \to \infty} V_s = V_l$ is $\lim_{\omega \to \infty} h(V_s, \cdot) = 0$.

$E[V^*]=160$	V_s				
I_h	ω 100	50	25	10	5
110	186.48	177.91	169.62	162.32	159.57
125	176.96	166.88	158.90	153.95	154.10
150	169.02	158.64	151.65	149.62	151.99
200	161.85	151.68	145.98	146.76	150.71
500	152.76	143.32	139.64	143.93	149.47
∞	148.22	NOW	NOW	142.74	148.94
$V_l=200$	$\alpha=0.02$		$\sigma=0.1$		$r=0.05$
110	153.41	150.18	147.36	147.40	150.42
125	149.08	144.74	142.11	144.45	149.04
150	145.39	140.59	138.53	142.69	148.25
200	142.21	NOW	NOW	141.45	147.69
500	NOW	NOW	NOW	140.35	147.20
∞	NOW	NOW	NOW	140.11	147.10
$V_l=158.77$	$\alpha=0.01$		$\sigma=0.1$		$r=0.05$
110	302.09	281.54	271.10	302.07	302.07
125	270.82	248.50	236.21	230.52	201.37
150	246.79	223.99	210.45	203.01	201.22
200	225.19	202.79	188.74	179.47	176.70
500	194.54	174.42	162.24	155.32	154.80
∞	160.54	145.73	140.46	144.60	149.97
$V_l=371.85$	$\alpha=0.02$		$\sigma=0.3$		$r=0.05$

Table 1. The optimal investment thresholds calculated for three different combinations of the rate of growth and volatility of the project's value. NOW means that investment takes place immediately. The results are presented for the following parameter values: investment cost before the jump $I_l = 100$, investment cost after the jump ranging from 110 to infinity, uncertainty concerning the occurrence of the shock, ω, ranging from 5 to 100. The initial value of the process equals $V_0 = 140$.

The results indicate a clear non-monotonic dependence of the optimal investment threshold on the uncertainty related to the occurrence of the shock. For example, consider the case where $\alpha = 0.02$ and $\sigma = 0.1$. When the firm's conjecture about the expected occurrence of the shock is relatively precise ($\omega = 5$), the possibility of doubling the effective investment cost results in the expected timing of undertaking the project being equal to 4.91 years.[23] When the uncertainty concerning the occurrence of the jump becomes moderately higher ($\omega = 25$), the firms is expected to invest within 2.78 years. Finally, when the firm's conjecture about the moment of the shock is highly imprecise ($\omega = 100$),

[23]The result is obtained by substituting appropriate values into (5), i.e.

$$-\frac{1}{0.02-\frac{1}{2}0.01}\ln\frac{140}{150.71}\approx 4.91.$$

the expected time to invest equals 9.67 years. If the project is about to deteriorate completely after the shock in the economy, the expected timing of investment shortens significantly, especially if the uncertainty concerning the occurrence of the shock is high. For $\omega = 5$ it is equal to 4.13 years, and for $\omega = 25$ it is optimal to invest immediately. In the case corresponding to a very high imprecision of the conjecture ($\omega = 100$) the expected time to invest equals 3.80 years. The direction of the impact of change in the growth rate and/or volatility of the project's value is consistent with the conclusions in the existing real options literature.

In Table 2 we show the values corresponding to the investment opportunity and probabilities that the investment is made before the increase in the investment cost (provided that the cost still equals I_l at V_0).

$E[V^*] = 160$	$W(V), P(V_s < V^* \| V^* > V_0)$					
I_h	ω	100	50	25	10	5
110		61.54	66.65	70.58	71.24	66.00
		0.68	0.55	0.44	0.42	0.53
125		55.82	57.11	56.94	53.25	48.66
		0.75	0.68	0.66	0.74	0.88
150		50.93	50.01	48.28	46.28	45.27
		0.80	0.78	0.80	0.87	0.95
200		46.69	44.70	42.93	43.01	43.92
		0.85	0.86	0.90	0.93	0.97
500		42.16	40.51	40.00	40.86	42.98
		0.91	0.96	1.00	0.97	0.98
∞		40.62	40.00	40.00	40.30	42.66
		0.94	1.00	1.00	0.98	0.97
$V_l = 200$	$\alpha = 0.02$		$\sigma = 0.1$		$r = 0.05$	

Table 2. The values of the investment opportunity and probabilities of investing at I_l for the following parameter values: investment cost before the jump $I_l = 100$, investment cost after the jump ranging from 110 to infinity, uncertainty concerning the occurrence of the shock ranging from 5 to 100. The initial value of the process equals $V_0 = \overline{V} = 140$.

From Table 2 it can be concluded that increasing the magnitude of the change in the investment cost results in i) deteriorating the value of the investment opportunity, and ii) an increased probability of investing before the shock occurs (which is a direct consequence of the lower optimal threshold).

An interesting observation can be made upon analyzing the relationship between the trigger-value uncertainty and the value of the investment opportunity. The non-monotonicity of this relationship results from the interaction of two opposite effects. First, increasing the variance, ω, implies lower quality of the firm's information about the moment of the policy change. This factor affects the value of the investment opportunity negatively. On the other hand, higher uncertainty makes the firm update its beliefs about the distribution of V^* in a way that enhances the value of the investment opportunity (the probability of survival on the interval $[V_0, V_s]$ becomes higher).[24] It appears that in situations where the magnitude of the change in the investment cost is small, the value of the project is the highest for a moderate precision of the conjecture about the timing of the change. Conversely, if the investment opportunity is to deteriorate completely upon the occurrence of the shock, the value of the project is most likely to be equal to its static NPV, i.e. the value of the project minus investment cost, for a moderate precision of the conjecture.

To provide some intuition of how the results of our model correspond to the outcome of Poisson based models, in which the whole information about the shock is aggregated in a single arrival parameter, we present some comparative statics comparing both approaches in Table 3.[25]

$V_l = 200.00$		$I_h = 150$			
λ	$E[V^*]$	V_P	$\frac{1}{\lambda}$ $(years)$	$\omega(\lambda)$	$V_s(\omega(\lambda))$
0.01	627.44	191.64	100	1031.90	196.61
0.05	188.98	172.25	20	56.91	166.49
0.10	162.66	161.11	10	24.42	152.51
0.25	148.66	148.48	4	8.92	142.49
0.33	146.51	145.47	3	6.66	140.98
0.50	144.26	141.67	2	4.33	NOW
$\alpha = 0.02$		$\sigma = 0.1$		$r = 0.05$	

Table 3. The optimal investment thresholds based on the model with the policy change triggered by V^* compared with the outcomes of the Poisson based model with the arrival rate λ ranging from 0.01 to 0.50 where the initial value of the process equals $V_0 = 140$. $\omega(\lambda)$ is a geometric average of an upward and downward deviation from $E[V^*]$, that are associated with the expected first passage time $\frac{1}{\lambda}$.[26]

In Table 3 $E[V^*]$ is selected in such a way that its expected first passage time is equal to the expected time of a Poisson jump of a given arrival rate. Moreover, the level of uncertainty concerning the cost-increase trigger corresponds to the standard deviation of the trigger implied by the Poisson process.

[24] The positive impact of updating on the value of the investment opportunity results from the fact that conditional on $V^* > V_0$ the cumulative density function of V^* is decreasing in ω for sufficiently large ω. This is equivalent, by definition, to the increase of the value of the conditional survival function.

[25] In order to calculate the optimal thresholds based on the Poisson arrivals, we apply a similar methodology as Dixit and Pindyck [5], pp. 305-306.

[26] Consequently, $\omega(\lambda)$ is defined as

$$\omega(\lambda) \equiv E\sqrt{\left(E[V^*] - V^{sd-}\right)\left(V^{sd+} - E[V^*]\right)},$$

where V^{sd+} (V^{sd-}) is the upward (downward) deviation from $E[V^*]$ such that the expected first-passage time of reaching V^{sd+} ($E[V^*]$) when the process originates at $E[V^*]$ (V^{sd-}) equals $\frac{1}{\lambda}$.

It appears that the slope of the relationship between the cost-increase trigger uncertainty and the optimal investment threshold is higher when our model is used than in the Poisson based approach. In other words, the resulting investment thresholds will be more responsive to the changes in ω. Consequently, for high levels of cost-increase trigger uncertainty, the optimal investment threshold under our approach will be higher than for Poisson based models (a cost increase trigger combined with very noisy information will not have a substantial effect on the firm's investment behavior). Conversely, if the prediction of the policy change is more reliable, the firm will invest more carefully (therefore earlier).

Finally, in Table 4 we show the outcomes of the Poisson based model in which the arrival rate is positively related to the value of the project.

$V_l = 200.00$			$\lambda_{Var}\|_{V=V_{PVar}} = \lambda$		$\lambda_{Var}\|_{V=V_0} = \lambda$	
λ	$E[V^*]$	V_P	d	V_{PVar}	d	V_{PVar}
0.01	627.44	191.64	5.195×10^{-5}	192.52	7.143×10^{-5}	190.20
0.05	188.98	172.75	2.875×10^{-4}	173.71	3.511×10^{-4}	170.69
0.10	162.66	161.11	6.160×10^{-4}	162.49	7.143×10^{-4}	160.33
0.25	148.66	148.48	1.675×10^{-3}	149.24	1.178×10^{-3}	148.53
0.33	146.51	145.47	2.284×10^{-3}	145.96	2.357×10^{-3}	145.64
0.50	144.26	141.67	3.520×10^{-3}	142.07	3.571×10^{-3}	141.95
			$\alpha = 0.02$		$\sigma = 0.1$	$r = 0.05$

Table 4. The optimal investment threshold calculated according to the Poisson based model with a variable arrival rate. The initial value of the process equals $V_0 = 140$, and the investment cost after the jump is given by $I_h = 150$. Parameter d corresponding to the variable arrival rate $\lambda = Vd$ is a solution to $\lambda = V_{PVar}d$ in column 4 and $\lambda = V_0 d$ in column 6, while the relevant λ is presented in column 1.

Table 4 illustrates the impact on the optimal investment threshold of introducing a variable arrival rate. The arrival rate increases with the value of the project. For the first set of solutions (columns 4-5) the variable $\lambda(V)$ equals λ in column 1 exactly at the level of V triggering the investment, i.e. $\lambda(V_{PVar}) = \lambda$. Analogously, the second set of solutions (columns 5-6) correspond to such a normalization upon which the variable rate $\lambda(V)$ equals to a constant λ in column 1 at V_0. Despite the fact that the variable λ has been normalized in two extreme ways, the differences in outcomes are relatively small. Therefore, we conclude that an attempt to introduce a variable arrival rate in the Poisson-based model does not alter the firm's investment behavior significantly.

6 Extension: Stochastic Jump Size

In this section we relax the assumption that the magnitude of the change in the investment cost is known beforehand. The firm is assumed to know only the density function of the size of the jump. Consequently, the random variable I_h

is distributed according to the cumulative density function $G(I_h)$ with a support $\left[\underline{I_h}, \overline{I_h}\right]$ such that I_h is nonnegative. Moreover, we impose a condition

$$\left(\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta} dG\left(I_h\right)\right)^{\frac{1}{1-\beta}} \geq I_l \tag{20}$$

that ensures that the firm prefers incurring the cost I_l to spending the stochastic amount I_h.[27]

Like in the deterministic case, the value of the investment opportunity, W_s, reflects the structure of the expected payoffs maximized with respect to the optimal investment threshold, V_s. For stochastic I_h, the value of investment opportunity becomes (cf. (8)):

$$\begin{aligned} W_s(V, \bar{V}|I &= I_l) = \max_{V_s}(V_s - I_l)\left(\frac{V}{V_s}\right)^{\beta} \frac{1 - F(V_s)}{1 - F(\bar{V})} + \\ &+ \int_{\underline{I_h}}^{\overline{I_h}} (V_h - I_h)\left(\frac{V}{V_h}\right)^{\beta}\left(1 - \frac{1 - F(V_s)}{1 - F(\bar{V})}\right) dG\left(I_h\right). \end{aligned} \tag{21}$$

Equation (21) is interpreted analogously to (8), and the second component is the expected value of the option to invest after the upward change in the investment cost occurs. We prove in the Appendix that the following proposition holds.

Proposition 5 *In case of a stochastic size of the jump in the investment cost, the optimal investment rule can be determined by replacing the deterministic counterpart I_h by*

$$I_h^* = \left(\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta} dG\left(I_h\right)\right)^{\frac{1}{1-\beta}} \tag{22}$$

in expression (12) for the optimal threshold.

Formula (22) can be interpreted as a certainty equivalent of the high investment cost.[28]. In other words, the investment policy of the firms is identical in the following two cases: *i*) investment cost I_h is stochastic and distributed according to $G\left(I_h\right)$, and *ii*) I_h is deterministic and equal to I_h^*. This allows for a relatively simple analysis of the impact on the optimal investment timing of the uncertainty concerning the magnitude of the jump.

The impact of the uncertainty concerning the magnitude of the jump can be analyzed by directly comparing (12) and (14). By Jensen's inequality it

[27] The LHS of (20) has a natural interpretation presented in the remainder of the section.

[28] Using the term *certainty equivalent* is a simplification since the firm is assumed to be risk-neutral. In our sense, (22) corresponds to such a value of a certain investment cost (within the high regime) that yields an identical optimal investment rule as when uncertain costs are distributed according to $G(I_h)$.

holds that

$$\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta} dG\left(I_h\right) > \left(\int_{\underline{I_h}}^{\overline{I_h}} I_h dG\left(I_h\right)\right)^{1-\beta}, \tag{23}$$

since the function $f(x) = x^a, a < 0$, is convex for all $x > 0$. The RHS of Equation (23) is an inversely monotonic transformation of the expected value of I_h. Since, by (14), $\frac{\partial V_s}{\partial I_h} < 0$, the threshold increases in $I_h^{1-\beta}$. Consequently, the threshold is higher for LHS than for RHS.

This result can be explained in the following way. The optimal timing is a convex function of the new investment cost, I_h. Therefore, the gains from below average realizations of the jump are assigned a larger weight by the firm than the symmetric losses resulting from above-average realizations. Consequently, the firm is going to wait longer if the realizations are random than in the case when all of them are equal to the average.

Compared to the basic model where investment cost is constant, the threat of an upward change in the investment cost reduces the optimal investment threshold. Now, we can see that the uncertainty in the size of the jump mitigates this reduction of the threshold value. Again, it holds that increased uncertainty raises the option value of waiting.

Apart from the overall difference between the uncertain and deterministic outcome, we are interested in the marginal impact of uncertainty on the optimal investment strategy. In other words, we aim at establishing how the investment threshold behaves for different degrees of uncertainty concerning the size of the jump. Therefore, we compare the investment triggers corresponding to a relatively small and a high degree of uncertainty. For this purpose, we use the concept of mean preserving spread (Rotschild and Stiglitz [16]). In this setting, the effect of increasing uncertainty is examined by replacing the original random variable I_h ('low uncertainty' case) by a new random variable $I_h + \xi$ ('high uncertainty' case), where $E[\xi] = 0$ and $\sigma_\xi \in (0, \infty)$. By applying Jensen's inequality it can be proven that the expected value of a convex function (in our case $f(I_h) = I_h^{1-\beta}$) increases as its argument undergoes a mean preserving spread (cf. Hartman [8]). Consequently, an increase in the uncertainty leads to the higher expected value of $I_h^{1-\beta}$ which corresponds to the lower I_h^*. This observation results in the following corollary.

Corollary 6 *Increasing the uncertainty concerning the magnitude of the jump of the investment cost (in a mean-preserving spread sense) leads to a higher optimal investment threshold and is equivalent to decreasing the expected magnitude of the jump.*

The impact on the optimal investment rule of uncertainty related to the magnitude of the change in the cost is monotonic. Furthermore, (13) implies that a lower potential increase in the investment cost is associated with a higher optimal investment threshold. In Table 5 we present the numerical results illustrating the impact of the uncertainty related to the magnitude of the change on the optimal investment threshold.

$V^* = 160$		V_s					
$\underline{I_h}$	$\overline{I_h}$	ω	100	50	25	10	5
150	150		169.02	158.64	151.65	149.62	151.99
125	175		169.34	158.96	151.92	149.76	152.06
100	200		170.39	160.02	152.83	150.26	152.29
50	250		177.29	167.25	159.23	154.18	154.21
25	275		192.81	186.54	179.08	171.64	168.60
$V_l = 200$		$\alpha = 0.02$		$\sigma = 0.1$			$r = 0.05$

Table 5. The impact of the uncertainty concerning the magnitude of the change in the investment cost on the optimal investment threshold.

The numerical results in Table 5 illustrate that a higher degree of uncertainty associated with the magnitude of the potential cost-increase results in a later investment (the first row of Table 5 corresponds to the third row of Table 1). Therefore, in the investment credit example, increasing this type of uncertainty has the same effect on the investment as the reduction of the magnitude of the change.

7 Implications for the Investment Credit Tax Policy Change

In our setting, the way in which the policy change is implemented by the authority can be expressed as a triple $\left\{\frac{I_h}{I_l}, V^*, \omega\right\}$. Consequently, as a result of the policy change, the investment cost is subject to increase by a proportion $\frac{I_h}{I_l}$. The increase is triggered by the project's value reaching the level V^* and ω corresponds to the precision of the firm's conjecture concerning the moment of change. To simplify the example we assume that the ratio $\frac{I_h}{I_l}$ is predetermined by the current amount of the tax credit (and is *a priori* common knowledge). The variables V^* and ω are the authority's decision variables.

As we already know, in case of a single firm whose investment opportunity satisfies (6), a decrease in a deterministic V^* results in a lower optimal threshold. Consequently, a reduction in the trigger value is going to accelerate this firm's investment. However, in case of multiple heterogenous firms, lowering the trigger has two opposite effects. First, as in the single-firm case, it leads to an earlier investment for those firms for which Assumption *iii* (6) is still satisfied. On the other hand, it results in the other firms waiting longer and investing at a high cost (i.e. those firms for which Assumption *iii* (6)) does not hold). Therefore, if the firms are sufficiently heterogenous, reducing V^* does not yield the desired effect of accelerating aggregate investment.

Therefore, the authority may prefer to resort to another instrument, such as ω. From Proposition 4 it can be concluded that there exists a U-shaped relationship between ω and the optimal threshold, V_s. An appropriately designed threat of abandoning the tax credit can accelerate investment by lowering the optimal threshold (see Dixit and Pindyck [5], Ch. 9). Since V_s reaches a minimum for a certain (strictly positive) degree of uncertainty, ω^e, the optimal strategy of the authority interested in accelerating the investment is to generate sufficiently (but not excessively) imprecise information about the conditions triggering the change. In purely analytical terms, this corresponds to setting the standard deviation of the density function associated with the conjecture about the policy change trigger, $F(V^*)$, to ω^e. Such a policy, while still accelerating investment, allows for Assumption *iii* (6) to be satisfied for a larger fraction of firms than in case of a reduction of a known V^*.

Since finding the true value of ω^e can be difficult in practice, we briefly discuss the impact on the investment behavior of misspecifying the optimal ω. A small deviation from ω^e results in a small relative delay in investment. Consequently, it is still desirable for the authority to create informational noise. However, if the misspecification of ω^e is large, it can happen that the resulting optimal investment threshold, V_s, is higher than the threshold corresponding to the case where V^* is known to the firm. Consequently, then the authority is better off by revealing the value of V^* to the investing firm. It is possible to find a critical level of ω, defined as $\overline{\omega}$, above which the optimal threshold is greater than the one corresponding to the known V^*. According to Proposition 1, $\overline{\omega}$ satisfies the following equation[29]

$$\begin{aligned} 0 &= h(V^*;\overline{\omega},\cdot)\,(V^*)^2 + (\beta-1)V^* - (V^*h(V^*;\overline{\omega},\cdot)+\beta)I_l - \\ &\quad - h(V^*;\overline{\omega},\cdot)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{(V^*)^{\beta+1}}{I_h^{\beta-1}}. \end{aligned} \tag{24}$$

If it is assumed that increasing the uncertainty by the authority is equivalent to applying a mean preserving spread, the change in the optimal investment threshold at $\overline{\omega}$ is discontinuous. Since the mean preserving spread implies that a policy change occurs at $V^* = E[V^*]$, imposing the level of uncertainty $\omega > \overline{\omega}$ results in the investment being made after the change in the cost occurs, i.e. at $V_h\,(\gg V_s)$. Therefore, increasing ω beyond $\overline{\omega}$ leads to a considerable delay of the investment.[30]

Consequently, the level of uncertainty concerning the value of the policy

[29] Equation (24) is also satisfied for $\omega = 0$, since the optimal threshold in the deterministic case is equal to V^*.

[30] The expected delay, ΔT, can be calculated from a direct application of the first passage time. In this case $\Delta T = -\frac{1}{\alpha-\frac{1}{2}\sigma^2}\ln\frac{V_h}{V_s}$.

change trigger can be characterized in the following way:

$$\omega \in \begin{cases} [0,\overline{\omega})\backslash\omega^e & : \text{ feasible (suboptimal) level of uncertainty,} \\ \omega^e & : \text{ optimal level of uncertainty,} \\ [\overline{\omega},\infty) & : \text{ excess uncertainty resulting in an investment delay.} \end{cases}$$

The threat of the policy change accelerates investment most significantly if the degree of uncertainty concerning the moment of the change is equal to ω^e. Therefore, from the point of view of the authority, this is the optimal level of the trigger value uncertainty. Revealing the value of V^* by the authority ($\omega = 0$) makes the firm invest an instant before V^* is reached. Excessive uncertainty (above $\overline{\omega}$) implies that information concerning the policy change is too unreliable to trigger investment before V^* is hit. As an effect, the optimal investment threshold exceeds the threshold corresponding to the known V^*. Consequently, there exists a set of feasible, though suboptimal, levels of uncertainty $\omega \in [0,\overline{\omega})\backslash\omega^e$ for which the optimal investment threshold is lower than V^*. For this set the threat of change remains high enough to trigger early investment.

The implications related to uncertainty in the *magnitude* of the policy change are straightforward, thus not requiring additional analysis. As shown in Section 7, an increase in the uncertainty concerning the *magnitude* of the change leads to a delay of the moment of investment. Consequently, ensuring that the magnitude of the policy change is known beforehand to potential investors lies in the interest of the authority interested in accelerating investment.

8 Conclusions

In the paper we consider an investment opportunity of a firm. The investment cost is irreversible and subject to an increase resulting from a policy change. The value of the cost-increase trigger is unknown to the firm and the firm knows the underlying density function instead. This corresponds to the situation where the firm has some information concerning the authority's future policy and this information is incomplete. Moreover, it is taken into account that a policy change mainly occurs under certain economic conditions.

We show that the threat of a policy change resulting in a higher investment cost leads to a reduction in the option value of waiting. Consequently, the firm invests earlier than in the case of a constant investment cost. The optimal investment threshold decreases in the magnitude of the change in investment cost and increases in market volatility (the latter result also hold for the Dixit and Pindyck [5] framework). One of our main results is that the impact of trigger value uncertainty on the optimal investment threshold is non-monotonic. If the uncertainty is sufficiently low, then the investment threshold is negatively related to the trigger value uncertainty. However, a rise in the uncertainty beyond a certain critical point reverses this relationship and leads to an increase of the optimal investment threshold.

Moreover, we extend the analysis by considering the case where the magnitude of the change is stochastic. This additional source of uncertainty results in a delay of investment. Increasing the uncertainty concerning the magnitude of the change leads to an outcome that is closer to the unconditional optimal threshold.

We apply our results to determine the optimal design of a change in the authority's policy, where the authority's aim is to accelerate investment undertaken by the firm. There exists a certain (strictly positive) level of the uncertainty concerning the policy change trigger that is associated with the earliest investment. Hence, a policy maker interested in accelerating investment should aim at achieving that particular level of uncertainty. In addition, in order to stimulate the firm to invest early, the authority should make sure that the magnitude of the policy change is known beforehand to potential investors.

9 Appendix

Proof of Proposition 1. The implicit solution for the optimal investment threshold is found by calculating the first order condition of (8). Consequently, by differentiating (8) with respect to V_s, and equalizing to zero, we obtain:

$$\begin{aligned} 0 \quad = \quad & -\frac{V^{\beta}}{V_s^{\beta+1}}\left(V_s-\beta V_s+\beta I_l\right)\frac{1-F(V_s)}{1-F(\bar{V})}+\left(V_s-I_l\right)\left(\frac{V}{V_s}\right)^{\beta}\left(\frac{f\left(V_s\right)}{1-F(\bar{V})}\right) \\ & -\left(V_h-I_h\right)\left(\frac{V}{V_h}\right)^{\beta}\frac{f\left(V_s\right)}{1-F(\bar{V})}, \end{aligned} \tag{25}$$

where $f\left(x\right)=\frac{\partial F\left(x\right)}{\partial x}$. Further simplification yields:

$$\begin{aligned} 0 \quad = \quad & \frac{1}{V_s^{\beta+1}}\left(V_s-\beta V_s+\beta I_l\right)\left(1-F(V_s)\right)-\left(V_s-I_l\right)\left(\frac{1}{V_s}\right)^{\beta}f\left(V_s\right) \\ & +\left(V_h-I_h\right)\left(\frac{1}{V_h}\right)^{\beta}f\left(V_s\right), \end{aligned}$$

thus

$$\left(\beta-1\right)V_s-\beta I_l+h\left(V_s\right)\left(V_s-I_l\right)V_s-\left(V_h-I_h\right)V_s^{\beta+1}\left(\frac{1}{V_h}\right)^{\beta}h\left(V_s\right)=0.$$

Since $V_h=\frac{\beta}{\beta-1}I_h$ (after the jump the McDonald-Siegel problem is left), this is equal to

$$h\left(V_s\right)V_s^2+\left(\beta-1\right)V_s-\left(h\left(V_s\right)V_s+\beta\right)I_l-V_s^{\beta+1}h\left(V_s\right)\frac{I_h}{\beta-1}\left(\frac{\beta-1}{\beta I_h}\right)^{\beta}=0,$$

what in a straightforward way leads to (12).

In order to prove that (12) is the expression for the maximal value of the project, we calculate the second order condition, which is equal to the following derivative:

$$\frac{\partial}{\partial V_s}\left(-h(V_s)V_s^2-(\beta-1)V_s+(V_sh(V_s)+\beta)I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}\right).$$

After differentiating, we obtain expression for the second order condition of (8):

$$\begin{aligned}\frac{\partial^2 W_s}{\partial V_s^2} &= -\left(h'\left(V_s\right)V_s+h\left(V_s\right)\right)\left(V_s-I_l-\frac{V_s}{\beta}\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1}\right)\\ &\quad -h\left(V_s\right)V_s\left(1-\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1}\right)-(\beta-1). \qquad (26)\end{aligned}$$

The sign of the second component can be proven to be negative by observing that:

$$1-\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1}>1-\left(\frac{\beta-1}{\beta}\frac{V_h}{I_h}\right)^{\beta-1}=0. \qquad (27)$$

The sign of the first component can be determined by notifying that the lower bound of V_s, denoted by $\underline{V_s}$, is a solution to the following equation:

$$\underline{V_s}-I_l=(V_h-I_h)\left(\frac{\underline{V_s}}{V_h}\right)^\beta. \qquad (28)$$

For $V_s=\underline{V_s}$ the second factor in the first component of (26) is equal to zero and for $V_s>\underline{V_s}$ it is positive. Therefore the whole expression is surely negative if (11) holds. ■

Proof of Proposition 2. Let us define the LHS of (12) as a function:

$$\begin{aligned}&H(V_s,I_l,I_h,\beta) \qquad (29)\\ =\ & h(V_s)V_s^2+(\beta-1)V_s-(V_sh(V_s)+\beta)I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}.\end{aligned}$$

Differentiating (29) with respect to I_l, I_h and β, respectively, yields:

$$\begin{aligned}\frac{\partial H}{\partial I_l} &= -(V_sh(V_s)+\beta)<0,\\ \frac{\partial H}{\partial I_h} &= (\beta-1)h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_H^\beta}>0, \qquad (30)\\ \frac{\partial H}{\partial \beta} &= V_s-I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}\ln\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)>0,\end{aligned}$$

$\forall I_l, I_h$ satisfying $0 < I_l < I_h$, $\forall\beta \in (1, r/\alpha)$ if $\alpha > 0$ and $\forall\beta \in (1, \infty)$ if $\alpha \leq 0$. Furthermore, differentiating (29) with respect to V_s gives:

$$\begin{aligned} \frac{\partial H}{\partial V_s} &= (h'(V_s)V_s + h(V_s)) \left(V_s - I_l - (V_h - I_h) \left(\frac{V_s}{V_h} \right)^\beta \right) + \\ &+ h(V_s)V_s \left(1 - \left(\frac{\beta - 1}{\beta} \frac{V_s}{I_h} \right)^{\beta - 1} \right) + (\beta - 1). \end{aligned} \tag{31}$$

From the proof of Proposition 1 it is known that under condition (11) $\frac{\partial H}{\partial V_s}$ is positive. Finally, by observing that

$$\frac{dV_s}{dz} = -\frac{\frac{\partial H}{\partial z}}{\frac{\partial H}{\partial V_s}}, \tag{32}$$

where z is an arbitrary parameter of our interest, we know that

$$\operatorname{sgn} \frac{dV_s}{dz} = -\operatorname{sgn} \frac{\partial H}{\partial z}, \tag{33}$$

what completes the proof. ◾

Proof of Proposition 3. By differentiating (29) with respect to the hazard rate, while taking into account that $V_h = \frac{\beta}{\beta - 1} I_h$, we obtain:

$$\frac{\partial H}{\partial h} = V_s \left(V_s - I_l - (V_h - I_h) \left(\frac{V_s}{V_h} \right)^\beta \right) > 0. \tag{34}$$

The inequality holds since the both factors are positive (cf. (28) and the proof of (31)). Since $\frac{\partial H}{\partial V_s}$ is also positive, from the envelope theorem we directly obtain the sign of (16). ◾

Proof of Proposition 5. Equation (22) requires the optimal investment threshold with a deterministic size of the jump be equal to the threshold with a jump with a stochastic size distributed according to $G(I_h)$. Since the maximization problem with a stochastic size of the jump can be expressed as follows:

$$\begin{aligned} W_s(V, \bar{V} | I &= I_l) = \max_{V_s} (V_s - I_l) \left(\frac{V}{V_s} \right)^\beta \frac{1 - F(V_s)}{1 - F(\bar{V})} + \\ &+ \left(1 - \frac{1 - F(V_s)}{1 - F(\bar{V})} \right) \frac{(\beta - 1)^{\beta - 1}}{\beta^\beta} \frac{V^\beta}{\int_{\underline{I_h}}^{\overline{I_h}} I_h^{\beta - 1} dG(I_h)}, \end{aligned} \tag{35}$$

the expression for the optimal investment threshold is a slight modification of (12):

$$\begin{aligned} 0 &= h(V_s)V_s^2 + (\beta - 1)V_s - (V_s h(V_s) + \beta) I_l - \\ &- V_s^{\beta + 1} h(V_s) \frac{(\beta - 1)^{\beta - 1}}{\beta^\beta} \int_{\underline{I_h}}^{\overline{I_h}} I_h^{1 - \beta} dG(I_h). \end{aligned} \tag{36}$$

Comparing (36) with (12) allows for observing that the threshold values are equal if:

$$h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^{\beta}}\frac{V_s^{\beta+1}}{I_h^{*\beta-1}}=\int_{\underline{I_h}}^{\overline{I_h}} h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^{\beta}}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}dG\left(I_h\right). \tag{37}$$

A simple algebraic manipulation yields:

$$I_h^{*1-\beta}=\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta}dG\left(I_h\right). \tag{38}$$

what in a straightforward way leads to (22). ■

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