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On perfectness concepts for bimatrix games

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Summary. It is shown that the perfect and the proper equilibria for $2 \times n$ bimatrix games can be determined systematically by means of the geometric-combinatorial approach of Borm et al. (1988). Moreover, for these games, stable sets and persistent retracts can be characterized. In particular, it is found that each stable set consists of either one or two perfect equilibria, that each stable component contains a proper equilibrium and that each persistent equilibrium is perfect.

Zusammenfassung. Ein Prozedur für die Bestimmung aller perfekten, properen und persistenten Gleichgewichte als aller stabilen Mengen eines $2 \times n$ Bimatrix Spieles wird angegeben. Besonders folgt, daß jede stabile Menge entweder ein oder zwei Elemente hat, daß jede stabile Komponente ein properes Gleichgewicht enthält und daß jedes persistente Gleichgewicht perfekt ist.

Key words: $2 \times n$ bimatrix game, perfect equilibrium, proper equilibrium, stable set, persistent retract

Schlüsselwörter: $2 \times n$ Bimatrix Spiele, perfekte, propere und persistente Gleichgewichte, stabile Mengen

1. Introduction

This paper considers $2 \times n$ (or $n \times 2$) bimatrix games and concentrates on the geometric-combinatorial approach (GC-approach) of Borm et al. (1988) to characterize various subclasses of perfect equilibria. Although the class of $2 \times n$ bimatrix games is a "small" class of games, it is a rather important class in the sense that many examples on non-cooperative game theory lie within this class. In this context the study of $2 \times n$ bimatrix games is worthwile and, as we will show, the GC-approach offers a useful tool to determine in a systematic way *all* perfect equilibria, proper equilibria, stable sets and persistent retracts for any game within this class. This leads to a better understanding of the special features of various examples encountered in the literature. Moreover, "full" knowledge of the class of $2 \times n$ bimatrix games provides a valuable "play-ground" for conjectures on general bimatrix games.

The organization of this paper is as follows. In Sect. 2 we establish notation and illustrate how the GC-approach enables us to determine perfect equilibria (Selten 1975).

The set of proper equilibria (Myerson 1978) is characterized in Sect. 3. This result allows for a better understanding of the rather special features of a 2×3 bimatrix game considered by van Damme (1987), regarding the "position" of the set of proper equilibria within the set of perfect equilibria. This game is discussed in detail.

Section 4 concentrates on stable sets in the sense of Kohlberg and Mertens (1986). It is shown that each stable consists of either one or two perfect equilibria. Moreover, it is found that each connected component of the set of Nash equilibria that contains a stable set, also contains a proper equilibrium. This is conjectured by van Damme (1987) for general bimatrix games.

Section 5 provides a characterization of persistent retracts (Kalai and Samet 1984). As a corrollary, it is found that each persistent equilibrium is perfect. This result does not hold for general bimatrix games.

Notation. $\mathbb{N} := \{1, 2, 3, \ldots\}$. Let $S \subset \mathbb{R}^t$, $t \in \mathbb{N}$. Then $\text{Conv}(S)$ denotes the convex hull of S and, if S is convex, $\text{Ext}(S)$ denotes the set of extreme points of S and $\text{Relint}(S)$ the relative interior of S.

The unit vectors of $\mathbb{R}^t$ are denoted by $e_1, e_2, \ldots, e_t$, $\Delta_t := \text{Conv}\{e_1, e_2, \ldots, e_t\} \subset \mathbb{R}^t$ and $\mathring{\Delta}_t := \text{Relint}(\Delta_t)$.

Let $x, y \in \mathbb{R}^t$. Then $x \geq (>) y$ if $x_k \geq (>) y_k$ for all $k \in \{1, \ldots, t\}$,

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$[x,y] := \{\lambda x + (1-\lambda)y \mid 0 \le \lambda \le 1\}$,

$(x,y) := \{\lambda x + (1-\lambda)y \mid 0 < \lambda < 1\}$

and $(x,y]$, $[x,y)$ are defined in the obvious way.

2. The GC-approach: perfect equilibria

Let A and B be two real $m\times n$ matrices. The $m\times n$ *bimatrix game* (A,B) is defined to be the two-person game $(\Delta_m, \Delta_n, K, L)$ in strategic form with mixed strategy spaces Δ_m and Δ_n, and payoff functions $K:\Delta_m\times\Delta_n\to\mathbb{R}$ and $L:\Delta_m\times\Delta_n\to\mathbb{R}$, with $K(p,q)=pAq$ and $L(p,q)=pBq$ for all $(p,q)\in\Delta_m\times\Delta_n$. The strategies $e_r\in\Delta_m(\Delta_n)$ are called *pure* and correspond to choosing rows (columns) in the payoff matrices A and B. Strategies $p\in\mathring{\Delta}_m(q\in\mathring{\Delta}_n)$ are called *completely mixed.*

For $p\in\Delta_m$, we define the *carrier* $C(p):=\{r\in\{1,2,\ldots,m\}\mid p_r>0\}$, the set $PB_2(p):=\{l\in\{1,2,\ldots,n\}\mid pBe_l\ge pBe_s$ for all $s\in\{1,2,\ldots,n\}\}$ of *pure best replies* of player 2 to p, and the set $B_2(p):=\mathrm{Conv}(\{e_l\in\Delta_n\mid l\in PB_2(p)\})$ of *best replies to* p. For $q\in\Delta_n$, the sets $C(q)$, $PB_1(q)$ and $B_1(q)$ are defined analogously.

A *Nash equilibrium* $(p,q)\in\Delta_m\times\Delta_n$ for (A,B) is such that $p\in B_1(q)$ and $q\in B_2(p)$ or, equivalently, such that $C(p)\subset PB_1(q)$ and $C(q)\subset PB_2(p)$. By $E(A,B)$ we denote the non-empty (Nash 1951) set of Nash equilibria for (A,B). Let $BG(m,n)$ denote the set of all $m\times n$ bimatrix games.

Let $(A,B)\in BG(2,n)$. Consider the function $g:\Delta_2\to\mathbb{R}$, defined by $g(p):=\max_{l\in\{1,\ldots,n\}} pBe_l$ for all $p\in\Delta_2$.

Obviously, g is a piecewise linear function: there exists a minimal number $\nu+1(\le n+1)$ of strategies

$$e_2=p(0),\, p(1),\, p(2),\,\ldots,\, p(\nu)=e_1$$

such that g is affine on $[p(k-1),\, p(k)]$ for all $k\in\{1,2,\ldots,\nu\}$. Note that g exactly describes the best reply structure for player 2 and that, for all $p^1,p^2\in\mathring{\Delta}_2$ with $p^1\ne p^2$, we have $PB_2(p^1)=PB_2(p^2)$ if and only if there is a $k\in\{1,2,\ldots,\nu\}$ such that $p^1,p^2\in(p(k-1),p(k))$.

In geometric representations, a pure strategy $e_l\in\Delta_n$ will be provided with a label [1] if $PB_1(e_l)=\{1\}$, with a label [2] if $PB_1(e_l)=\{2\}$ and with a label [12] if $PB_1(e_l)=\{1,2\}$. Let $I([1]):=\{l\in\{1,\ldots,n\}\mid PB_1(e_l)=\{1\}\}$ represent the set of pure strategies with label [1]. The sets $I([2])$ and $I([12])$ are defined analogously. Further, for $k\in\{1,2,\ldots,\nu\}$, let $I_k:=PB_2\big(\frac{1}{2}p(k-1)+\frac{1}{2}p(k)\big)$ denote the set of pure best replies to any strategy in $(p(k-1),p(k))$.

For $p\in\Delta_2$, let the set $S(p)$ of *solutions* to p be defined by

$$S(p):=\{q\in\Delta_n\mid(p,q)\in E(A,B)\} \tag{2.1}$$

It is straightforward to verify that $S(p)$ is a bounded polyhedral set and hence a polytope. So, using the Krein-Milman theorem, $S(p)$ is the convex hull of its finitely many extreme points.

Defining the set $PS(p)$ of *pure solutions* to p by

$$PS(p):=\{e_l\in\Delta_n\mid(p,e_l)\in E(A,B)\} \tag{2.2}$$

it follows that $PS(p)\subset\mathrm{Ext}(S(p))$ and

$$PS(p)=\begin{cases}\{e_l\in\Delta_n\mid l\in PB_2(p,[12])\} & \text{if } p\in\mathring{\Delta}_2\\ \{e_l\in\Delta_n\mid l\in PB_2(p,[12])\cup PB_2(p,[1])\} & \text{if } p=e_1\\ \{e_l\in\Delta_n\mid l\in PB_2(p,[12])\cup PB_2(p,[2])\} & \text{if } p=e_2\end{cases}$$

where $PB_2(p,[12]):=PB_2(p)\cap I([12])$ denotes the set of pure best replies to p having label [12]; $PB_2(p,[1])$ and $PB_2(p,[2])$ are defined in a similar way.

However, pure solutions need not be the only extreme points of the set of solutions.

Lemma 2.1. *Let $(A,B)\in BG(2,n)$ and $A=[a_{rs}]_{r=1}^{2}{}_{s=1}^{n}$. Let $i\in I([1])$ and $j\in I([2])$. Then there exists a unique (i,j)-coordination strategy $q(i,j)\in\Delta_n$ such that $C(q(i,j))=\{i,j\}$ and $e_1Aq(i,j)=e_2Aq(i,j)$. In particular,*

$$q_i(i,j)=\frac{a_{2j}-a_{1j}}{(a_{1i}-a_{2i})+(a_{2j}-a_{1j})}$$

Having this lemma, we can define the set $CS(p)$ of *coordination solutions to* $p\in\Delta_2$ by

$$CS(p):=\{q(i,j)\in\Delta_n\mid i\in PB_2(p,[1]) \text{ and } j\in PB_2(p,[2])\}. \tag{2.3}$$

Obviously, $CS(p)\subset\mathrm{Ext}(S(p))$. Moreover, we have

Theorem 2.2 (cf. Borm and Gijsberts 1990).
Let $(A,B)\in BG(2,n)$. Then, for all $p\in\Delta_2$,

$$\mathrm{Ext}(S(p)=PS(p)\cup CS(p).$$

Since $S(p^1)=S(p^2)$ for all $p^1,p^2\in(p(k-1),\,p(k))$ and $k\in\{1,\ldots,\nu\}$, Theorem 2.2 implies

Corollary 2.3. *Let $(A,B)\in BG(2,n)$. Then $E(A,B)$ is the union of finitely many polytopes T of the following form: either*

(a) there is a $k\in\{1,2,\ldots,\nu\}$ such that $T=[p(k-1),p(k)]\times S\big(\frac{1}{2}p(k-1)+\frac{1}{2}p(k)\big)$ or

(b) there is a $k\in\{0,1,\ldots,\nu\}$ such that $T=\{p(k)\}\times S(p(k))$.

Example 1. Let $(A,B)\in BG(2,6)$ be given by

$$(A,B)=\begin{bmatrix}(2,0) & (0,6) & (0,8) & (0,6) & (3,8) & (-1,3)\\ (0,8) & (0,6) & (1,0) & (2,6) & (1,-8) & (0,7)\end{bmatrix}$$

Identifying a strategy $(p,1-p)\in\Delta_2$ with $p\in[0,1]$, the GC-approach uses the geometric representation in Fig. 1 (the subindices of the labels indicate the corresponding pure strategies).

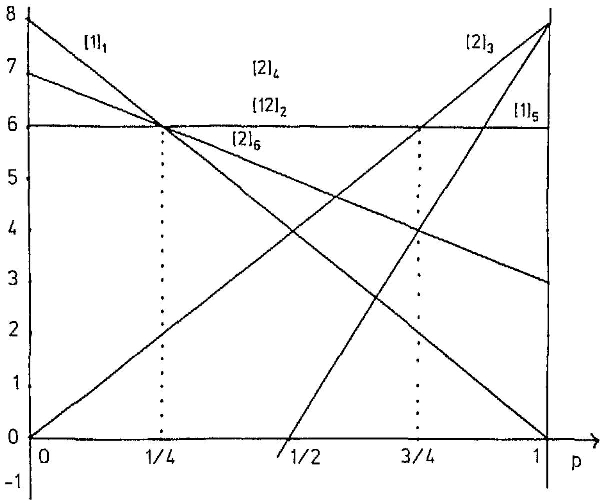

Fig. 1 (see text)

It is found that $v=3$,

$$p(0)=e_2,\quad p(1)=\tfrac{1}{4}e_1+\tfrac{3}{4}e_2,$$

$$p(2)=\tfrac{3}{4}e_1+\tfrac{1}{4}e_2,\quad p(3)=e_1$$

and $E(A,B)=T^1\cup T^2\cup T^3$, with

$$T^1=\{\tfrac{1}{4}e_1+\tfrac{3}{4}e_1\}\times \operatorname{Conv}\{e_2,q(1,4),q(1,6)\}$$

$$=\{\tfrac{1}{4}e_1+\tfrac{3}{4}e_1\}\times \operatorname{Conv}\{e_2,(\tfrac{1}{2},0,0,\tfrac{1}{2},0,0),$$

$$(\tfrac{1}{3},0,0,0,0,\tfrac{2}{3})\}$$

$$T^2=[\tfrac{1}{4}e_1+\tfrac{3}{4}e_2,\tfrac{3}{4}e_1+\tfrac{1}{4}e_2]\times\{e_2\}$$

and

$$T^3=\{e_1\}\times\operatorname{Conv}\{e_5,q(5,3)\}$$

$$=\{e_1\}\times\operatorname{Conv}\{e_5,(0,0,\tfrac{2}{3},0,\tfrac{1}{3},0)\}.$$

Perfect equilibria were introduced by Selten (1975). Van Damme (1983) proved that perfect equilibria for bimatrix games exactly correspond to undominated Nash equilibria, i.e. to equilibrium pairs that consist of two undominated strategies.

Definition. Let $(A,B)\in BG(m,n)$. A strategy $\bar{p}\in\Delta_m$ is said to dominate $p\in\Delta_m$ if

$$\bar{p}A\geq pA \quad\text{and}\quad \bar{p}A\neq pA.$$

Similarly, a strategy $\bar{q}\in\Delta_n$ dominates $q\in\Delta_n$ if $B\bar{q}\geq Bq$ and $B\bar{q}\neq Bq$. A strategy $p\in\Delta_m$ ($q\in\Delta_n$) is called *undominated* if there is no $\bar{p}\in\Delta_m$ ($\bar{q}\in\Delta_n$) which dominates $p(q)$. By $U_i(A,B)$ we denote the set of undominated strategies of player $i\in\{1,2\}$.

A straightforward characterization of the undominated strategies of both players in $2\times n$ bimatrix games, is provided in

Lemma 2.4. *Let $(A,B)\in BG(2,n)$. Then*

$$U_1(A,B)=\begin{cases}\{e_1\} & \text{if } I([1])\neq\emptyset \text{ and } I([2])=\emptyset\\ \{e_2\} & \text{if } I([1])=\emptyset \text{ and } I([2])\neq\emptyset\\ \Delta_2 & \text{otherwise}\end{cases}$$

and

$$U_2(A,B)=\bigcup_{p\in\mathring{\Delta}_2}B_2(p)$$

Let $PE(A,B):=E(A,B)\cap(U_1(A,B)\times U_2(A,B))$ denote the set of perfect equilibria for (A,B). Lemma 2.4 immediately implies

Theorem 2.5. *Let $(A,B)\in BG(2,n)$ and $(p,q)\in\Delta_2\times\Delta_n$. Then $(p,q)\in PE(A,B)$ if and only if the following three assertions hold:*

(i) *If $I([1])\neq\emptyset$ and $I([2])=\emptyset$, then $p=e_1$.*
If $I([2])\neq\emptyset$ and $I([1])=\emptyset$, then $p=e_2$
(ii) $q\in S(p)$
(iii) *If $p=e_1$, then $C(q)\subset I_\nu$. If $p=e_2$, then $C(q)\subset I_1$.*

Example 2. For the 2×6 bimatrix game (A,B) of example 1, it is found that

$$U_1(A,B)=\Delta_2,$$

$$U_2(A,B)=\operatorname{Conv}(\{e_1,e_2,e_4,e_6\})\cup\operatorname{Conv}(\{e_2,e_3,e_4\})$$

and $PE(A,B)=T^1\cup T^2$, with T^1 and T^2 as in Example 1.

3. Proper equilibria

Proper equilibria were introduced by Myerson (1978) as a modification of the perfectness concept of Selten (1975).

Definition. Let $(A,B)\in BG(m,n)$. A (perfect equilibrium) $(p,q)\in\Delta_m\times\Delta_n$ is called *proper* if there are sequences $\{p^k,q^k\}_{k\in\mathbb{N}}\subset\mathring{\Delta}_m\times\mathring{\Delta}_n$ converging to (p,q) and $\{\varepsilon_k\}_{k\in\mathbb{N}}$

of positive reals converging to zero, such that (p^k, q^k) is ε_k-*proper* for all $k \in \mathbb{N}$, i.e.

$$p^k Be_j < p^k Be_s \Rightarrow q_j^k \leq \varepsilon_k q_s^k \quad (j, s \in \{1, 2, \ldots, n\}) \quad (3.1)$$

$$e_i Aq^k < e_r Aq^k \Rightarrow p_i^k \leq \varepsilon_k p_r^k \quad (i, r \in \{1, 2, \ldots, m\}) \quad (3.2)$$

By $PR(A,B)$ we denote the non-empty set of proper equilibria for (A,B).

Note that, in order to determine proper equilibria, one has to distinguish between (first) best replies, second-best replies, third-best replies etc. For this aim we introduce the following notations.

Let $(A,B) \in BG(2,n)$ and $p \in \Delta_n$. For $t \in \{1, 2, \ldots, n\}$, $PB_2^t(p)$ denotes the set of *t-th pure best replies* to p, i.e. $PB_2^1(p) := PB_2(p)$ and, for $t \in \{2, \ldots, n\}$,

$$PB_2^t(p) := \left\{ j \in \{1, 2, \ldots, n\} | pBe_j = \max \left\{ pBe_l | l \in \{1, 2, \ldots, n\} \setminus \bigcup_{s=1}^{t-1} PB_2^s(p) \right\} \right\}$$

Further, $PB_2^t(p,[1]) := PB_2^t(p) \cap I([1])$; $PB_2^t(p,[2])$ and $PB_2^t(p,[12])$ are defined in a similar way.

Theorem 3.1 describes how proper equilibria for a $2 \times n$ bimatrix game (A,B) depend on perfect equilibria for a "smaller" game $(\bar{A}, \bar{B})$.

Definition. Let $(A,B) \in BG(2,n)$ and $I([12]) \neq \{1, 2, \ldots, n\}$. Then the $2 \times \bar{n}$ bimatrix game $(\bar{A}, \bar{B})$ is obtained form (A,B) by deleting all pure strategies $e_j \in \Delta_n$ with $j \in I([12])$.

Theorem 3.1. *Let $(A,B) \in BG(2,n)$ and $(p,q) \in \Delta_2 \times \Delta_n$.*

If $I([12]) = \{1, 2, \ldots, n\}$, then $PR(A,B) = PE(A,B)$. Otherwise, we have that $(p,q) \in PR(A,B)$ if and only if the following three assertions hold:

(i) *There is a $\bar{q} \in \Delta_{\bar{n}}$ such that $(p, \bar{q})$ is perfect for $(\bar{A}, \bar{B})$.*
(ii) *$q \in S(p)$.*
(iii) *If $p = e_1$, then $C(q) \subset I_v$. If $p = e_2$, then $C(q) \subset I_1$.*

Proof. The proof consists of two parts: in the first part we assume p to be completely mixed, in the second part to be pure.

First, let $p \in \mathring{\Delta}_2$. Define $\alpha_t := |PB_2^t(p)|$ for all $t \in \{1, 2, \ldots, n\}$ and $\tau := \max\{t \in \{1, 2, \ldots, n\} | \alpha_t > 0\}$.

(a) Assume $I([12]) = \{1, 2, \ldots, n\}$. Let $(p,q) \in PE(A,B)$. We show that (p,q) is proper. Let, for $k \in \mathbb{N}$,

$$\varepsilon_k := \frac{1}{k+1}, \quad p^k := p \quad (3.3)$$

and

$$q^k := (k(k+1)^{-1} + \delta(k))q + \sum_{t=1}^{\tau} \alpha_t^{-1}(k+1)^{-2t} \sum_{l \in PB_2^t(p)} e_l, \quad (3.4)$$

where

$$\delta(k) := (k+1)^{-1} - \sum_{t=1}^{\tau} (k+1)^{-2t} \quad (3.5)$$

is such that $\sum_{l=1}^n q_l^k = 1$. Obviously, $(p^k, q^k) \in \mathring{\Delta}_2 \times \mathring{\Delta}_n$. Further, the coefficients are chosen in such a way that (3.1) is satisfied for large k. Condition (3.2) trivially holds, because $e_1 Aq' = e_2 Aq'$ for all $q' \in \Delta_n$. So (p^k, q^k) is ε_k-proper for large k. Hence, $(p,q) \in PR(A,B)$.

(b) Assume $I([12]) \neq \{1, 2, \ldots, n\}$. Let $(p,q) \in PR(A,B)$. We show that (i), (ii) and (iii) are satisfied. The assertions (ii) and (iii) are obvious. Suppose (i) does not hold. Then, without loss of generality, there exists a $t^* \in \{1, 2, \ldots, n\}$ such that $PB_2^{t^*}(p,[1]) \neq \emptyset$, $PB_2^{t^*}(p,[2]) = \emptyset$ and $PB_2^t(p) \subset I([12])$ for all $t \in \{1, 2, \ldots, t^* - 1\}$. In particular choosing $i \in PB_2^{t^*}(p,[1])$, we have that $pBe_i > pBe_l$ for all $l \in I([2])$.

Take sequences $\{(p^k, q^k)\}_{k \in \mathbb{N}} \subset \mathring{\Delta}_2 \times \mathring{\Delta}_n$ converging to (p,q) and $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive reals converging to zero, such that (p^k, q^k) is ε_k-proper for all $k \in \mathbb{N}$. Obviously, if k is large enough, then $p^k Be_i > p^k Be_l$ for all $l \in I([2])$. Let $A = [a_{rs}]_{r=1}^{2}{}_{s=1}^{n}$. Condition (3.1) implies that, for large k,

$$e_1 Aq^k - e_2 Aq^k = \sum_{l \in I([1])} (a_{1l} - a_{2l}) q_l^k + \sum_{l \in I([2])} (a_{1l} - a_{2l}) q_l^k \geq q_i^k \left((a_{1i} - a_{2i}) + \varepsilon_k \sum_{l \in I([2])} (a_{1l} - a_{2l}) \right) > 0.$$

Consequently, using (3.2), $p_2^k \leq \varepsilon_k p_1^k$ for large k. However, this would imply that $p = e_1$. So we may conclude that (i) is satisfied.

(c) Assume $I([12]) \neq \{1, 2, \ldots, n\}$. Let (i), (ii) and (iii) be satisfied. We show that (p,q) is proper.

Because of (i), there are $t^* \in \{1, 2, \ldots, n\}$, $i \in PB_2^{t^*}(p,[1])$ and $j \in PB_2^{t^*}(p,[2])$ with $PB_2^t(p) \subset I([12])$ for all $t \in \{1, \ldots, t^* - 1\}$. For $k \in \mathbb{N}$, let ε_k and p^k be as in (3.3), and, with $\delta(k)$ as in (3.5),

$$q^k := (k(k+1)^{-1} + \delta(k))q + \sum_{t=1}^{\tau} \alpha_t^{-1}(k+1)^{-2t} \left(\sum_{l \in PB_2^t(p,[12])} e_l + \sum_{l \in PB_2^t(p,[1])} q(l,j) + \sum_{l \in PB_2^t(p,[2])} q(i,l) \right) \quad (3.6)$$

By construction, (3.1) is satisfied for large k. Since $e_1 Aq^k = e_2 Aq^k$ for all $k \in \mathbb{N}$, also (3.2) holds. So (p^k, q^k) is ε_k-proper for large k. Hence, $(p,q) \in PR(A,B)$.

Secondly, let $p = e_2$ (the case $p = e_1$ can be treated in a similar way).

Choose $p^* \in \mathring{\Delta}_2$ such that $(PB_2^t(p^*))_{t\in\{1,2,\dots,n\}} = (PB_2^t(p'))_{t\in\{1,2,\dots,n\}}$ for all $p' \in [p^*, e_2)$. Define $\alpha_t := |PB_2^t(p^*)|$ for all $t \in \{1,2,\dots,n\}$, and let $\tau := \max\{t \in \{1,2,\dots,n\} | \alpha_t > 0\}$.

(a) Assume $I([12]) = \{1,2,\dots,n\}$. Let $(e_2,q) \in PE(A,B)$. To show that (e_2,q) is proper, the sequences $\{(p^k,q^k)\}_{k\in\mathbb{N}}$ and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of (3.3) and (3.4) are appropriate, if we change the definition of p^k into $p^k := \frac{k}{k+1} e_2 + \frac{1}{k+1} p^*$ for all $k \in \mathbb{N}$.

(b) Assume $I([12]) \neq \{1,2,\dots,n\}$. Let $(e_2,q) \in PR(A,B)$. Since (e_2,q) is perfect, (ii) and (iii) are satisfied (cf. Theorem 2.5). Suppose (i) does not hold.

Then there exists an $i \in I([1])$ such that $p^*Be_i > p^*Be_l$ for all $l \in I([2])$.

However, using a similar line of argument as in (b) in the first part of this proof, this would imply that $p = e_1$.

(c) Assume $I([12]) \neq \{1,2,\dots,n\}$. Let (i), (ii) and (iii) be satisfied. We show that $(e_2,q) \in PR(A,B)$.

Because of (i), there are $t^* \in \{1,2,\dots,n\}$ and $j \in PB_2^{t^*}(p^*,[2])$ with $PB_2^t(p^*) \subset I([12])$ for all $t \in \{1,2,\dots,t^*-1\}$.

Let, for $k \in \mathbb{N}$,

$$\varepsilon_k := \frac{1}{k+1}, \quad p^k := \frac{k+1}{k+2} e_2 + \frac{1}{k+2} p^* \tag{3.7}$$

and, with $\delta(k)$ as in (3.5),

$$q^k := (k(k+1)^{-1} + \delta(k))q + \sum_{t=1}^{\tau} \alpha_t^{-1}(k+1)^{-2t} \left(\sum_{l \in PB_2^t(p,[12]) \cup PB_2^t(p,[2])} e_l + \sum_{l \in PB_2^t(p,[1])} q(l,j) \right) \tag{3.8}$$

Note that $C(q) \subset PB_2(p^*)$. By construction, (3.1) is satisfied for large k. With respect to (3.2) we have that $e_2Aq^k > e_1Aq^k$ for all $k \in \mathbb{N}$ and

$$p_1^k = \frac{1}{k+2} p_1^* \leq \frac{1}{k+2} \leq \frac{1}{k+1} \left(\frac{k+1}{k+2} + \frac{1}{k+2} p_2^* \right) = \varepsilon_k p_2^k.$$

So (p^k,q^k) is ε_k-proper for large k. Hence, $(e_2,q) \in PR(A,B)$. □

It may be noted that condition (i) of Theorem 3.1 provides a necessary and sufficient condition to decide whether, for any given strategy $p \in \Delta_2$, there exists a strategy q such that (p,q) is proper: if $I([12]) \neq \{1,2,\dots,n\}$ and (i) is satisfied for p, then there has to be a solution q to p satisfying (iii).

Example 3. For the 2×6 bimatrix game (A,B) of example 1, it is found that

$$PR(A,B) = \{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2\} \times \mathrm{Conv}\,\{e_2, (\tfrac{1}{2},0,0,\tfrac{1}{2},0,0), (\tfrac{1}{3},0,0,0,0,\tfrac{2}{3})\}.$$

Example 4 (cf. van Damme 1987). Let $(A,B) \in BG(2,3)$ be given by

$$(A,B) = \begin{bmatrix} (2,0) & (0,6) & (0,8) \\ (0,8) & (0,6) & (1,0) \end{bmatrix}$$

By deleting the second column, it is found that

$$(\bar{A},\bar{B}) = \begin{bmatrix} (2,0) & (0,8) \\ (0,8) & (1,0) \end{bmatrix}$$

A geometric representation is provided in Fig. 2. As before, $(p, 1-p) \in \Delta_2$ is identified with $p \in [0,1]$.

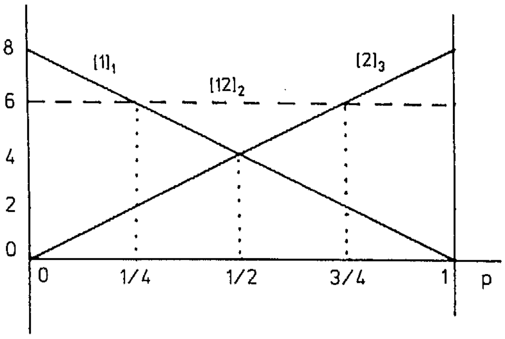

Fig. 2 (see text)

Clearly, if (p,q) is a (perfect) equilibrium for $(\bar{A},\bar{B})$, then $p = \frac{1}{2}e_1 + \frac{1}{2}e_2$. Hence,

$$PE(A,B) = [\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2] \times \{e_2\} \quad \text{and}$$

$$PR(A,B) = \{(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, e_2)\}$$

This approach enables us to understand the remarkable position of the unique proper equilibrium within the relative interior of the convex set of perfect equilibria.

There can be various isolated proper equilibria like this. Let $(A^1,B^1) \in BG(2,5)$ be given by

$$(A^1,B^1) = \begin{bmatrix} (2,0) & (0,6) & (0,8) & (0,4) & (1,6) \\ (0,8) & (0,6) & (1,0) & (1,6) & (0,4) \end{bmatrix}$$

Then,

$$PE(A^1,B^1) = [\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2] \times \{e_2\} \quad \text{and}$$

$$PR(A^1,B^1) = \{(\tfrac{1}{3}e_1 + \tfrac{2}{3}e_2, e_2),$$
$$(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, e_2)(\tfrac{2}{3}e_1 + \tfrac{1}{3}e_2, e_2)\}$$

4. Stable sets

In this section we concentrate on the set-valued stability concept of Kohlberg and Mertens (1986) that generalizes the concept of strictly perfect equilibria as introduced by Okada (1984). Although Mertens (1989) reformulated this stability concept, this section reserves the term stability for minimal strictly perfect sets. For $2\times n$ bimatrix games, it is shown that each stable set consists of either one or two perfect equilibria, and that each stable component contains a proper equilibrium.

Definition. Let $(A,B)\in BG(m,n)$. Then $\varepsilon^1\in\mathbb{R}^m$ and $\varepsilon^2\in\mathbb{R}^n$ are called *mistake vectors* for player 1 and player 2, respectively, if

$$\varepsilon^1>0,\ \varepsilon^2>0,\ \sum_{r=1}^{m}\varepsilon_r^1\le 1 \text{ and } \sum_{s=1}^{n}\varepsilon_s^2\le 1.$$

For mistake vectors ε^1 and ε^2, the $(\varepsilon^1,\varepsilon^2)$-*perturbed game* $(A,B;\varepsilon^1,\varepsilon^2)$ is defined as the strategic game that differs from (A,B) only in the fact that the strategy spaces of the players are restricted to $\Delta_m^{\varepsilon^1}:=\{p\in\Delta_m|p\ge\varepsilon^1\}$ and $\Delta_n^{\varepsilon^2}:=\{p\in\Delta_n|p\ge\varepsilon^2\}$.

The following lemma will be used frequently. The proof is straightforward and therefore omitted.

Lemma 4.1. *Let* $(A,B)\in BG(m,n)$.

(i) Let $\varepsilon^1\in\mathbb{R}^m$ *and* $\varepsilon^2\in\mathbb{R}^n$ *be mistake vectors and let* $(p,q)\in\Delta_m^{\varepsilon^1}\times\Delta_n^{\varepsilon^2}$. *Then* $(p,q)\in E(A,B;\varepsilon^1,\varepsilon^2)$ *if and only if the following two assertions hold:*

(i.1) $p_r=\varepsilon_r^1$ *for all* $r\in\{1,2,\dots,m\}$ *with* $r\notin PB_1(q)$.
(i.2) $q_s=\varepsilon_s^2$ *for all* $s\in\{1,2,\dots,n\}$ *with* $s\notin PB_2(p)$.

(ii) Let $(p,q)\in\Delta_m\times\Delta_n$ *and let* $\{(p^k,q^k)\}_{k\in\mathbb{N}}\subset\Delta_m\times\Delta_n$ *be a sequence that converges to* (p,q). *Then, for large* k,

$$C(p)\subset C(p^k),\quad C(q)\subset C(q^k),$$
$$PB_2(p)\supset PB_2(p^k)\quad\text{and}\quad PB_1(q)\supset PB_1(q^k).$$

Definition (Kohlberg and Mertens 1986; Okada 1984). Let $(A,B)\in BG(m,n)$ and let C be a non-empty closed subset of $E(A,B)$. Then C is called *strictly perfect* for (A,B) if, for each sequence $\{(e^1(k),\varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors converging to zero, there exists a sequence $\{(p^k,q^k)\}_{k\in\mathbb{N}}$ such that $(p^k,q^k)\in E(A,B;\varepsilon^1(k),\varepsilon^2(k))$ for all $k\in\mathbb{N}$, with a limit points in C. A strictly perfect set that does not properly contain another strictly perfect set, is called a stable set. A strategy pair $(p,q)\in\Delta_m\times\Delta_n$ is called a *strictly perfect equilibrium* for (A,B) if the set $\{(p,q)\}$ is stable.

Some (nice) properties of stable sets are summarized in

Theorem 4.2. (Kohlberg and Mertens 1986; van Damme 1987).

(i) Each strictly perfect set contains a stable set.
(ii) For each game, there is at least one stable set (existence).
(iii) Stable sets consist of perfect equilibria only (admissability).
(iv) A stable set corresponds to a strictly perfect set in any game that is obtained by deleting dominated pure strategies (iterated dominance).
(v) There is a stable set that lies within a maximally connected subset of the set of Nash equilibria (connectedness).

For some games, the properties of Theorem 4.2 suffice to determine all stable sets.

Example 5. Let $(A,B)\in BG(2,3)$ be given by

$$(A,B)=\begin{bmatrix}(0,0) & (2,2) & (1,1)\\ (4,4) & (2,2) & (0,0)\end{bmatrix}$$

Then, $PE(A,B)=\{(e_2,e_1)\}\cup([\tfrac{1}{2}e_1+\tfrac{1}{2}e_2,e_1\ ]\times\{e_2\})$.

Let C be stable. Suppose $C\subset[\tfrac{1}{2}e_1+\tfrac{1}{2}e_2,e_1]\times\{e_2\}$. Let (A^1,B^1) be the 2×2 bimatrix game that is obtained from (A,B) by deleting the (dominated) third column. Then, C corresponds to a strictly perfect perfect set for (A^1,B^1) (cf. (iv)), therefore C contains a stable set for (A^1,B^1) (cf. (i)) and, in particular, C contains perfect equilibria for (A^1,B^1) (cf. (iii)). However, the equilibria in C are not perfect for (A^1,B^1). Hence, $(e_2,e_1)\in C$. Then, using (ii) and (v), $C=\{(e_2,e_1)\}$.

A systematic way to determine the stable sets for any $2\times n$ bimatrix game, is described in

Theorem 4.3. *Let* $(A,B)\in BG(2,n)$.

(i) Let $I([1])=\emptyset$ *or* $I([2])=\emptyset$. *Then each stable set for* (A,B) *consists of one point. In particular, each perfect equilibrium for* (A,B) *is strictly perfect.*
(ii) Let $I([1])=\emptyset$ *and* $I([2])\neq\emptyset$. *Then each stable set for* (A,B) *consists of one or two points. In particular, a perfect equilibrium* (p,q) *is strictly perfect if and only if one of the following three assertions hold:*

(ii.1) $p\in\mathring{\Delta}_2$, $PB_2(p,[1])\neq\emptyset$ *and* $PB_2(p,[2])\neq\emptyset$.
(ii.2) $p=e_1$ *and* $I([1])\cap I_v\neq\emptyset$.
(ii.3) $p=e_2$ *and* $I([2])\cap I_1\neq\emptyset$.

Further, with (p^1, q^1), $(p^2, q^2) \in PE(A,B)$, *we have that* $\{(p^1, q^1), (p^2, q^2)\}$ *is stable if and only if the following two assertions hold:*

(ii.4) Either $p^1 \in \mathring{\Delta}_2$, $PB_2(p^1, [1]) \neq \emptyset$ *and* $PB_2(p^1, [2]) = \emptyset$ *or,* $p^1 = e_2$ *and* $I([2]) \cap I_1 = \emptyset$.

(ii.5) Either $p^2 \in \mathring{\Delta}_2$, $PB_2(p^2, [2]) \neq \emptyset$ *and* $PB_2(p^2, [1]) = \emptyset$ *or,* $p^2 = e_1$ *and* $I([1]) \cap I_v = \emptyset$.

Proof. (i) Without loss of generality, we assume $I([1]) = \emptyset$. Let $(p,q) \in PE(A,B)$. We show that (p,q) is strictly perfect. Let $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k \in \mathbb{N}}$ be a sequence of mistake vectors that converges to zero.

If $p \in \mathring{\Delta}_2$, it follows that $I([12]) = \{1, 2, \ldots, n\}$ and, by defining $p^k := p$ and

$$q^k := \left(1 - \sum_{l \in \{1,2,\ldots,n\}} \varepsilon_l^2(k)\right) q + \sum_{l \in \{1,2,\ldots,n\}} \varepsilon_l^2(k) e_l, \quad (4.1)$$

Lemma 4.1(i) implies that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for large $k \in \mathbb{N}$. For pure p, a similar result is found by defining q^k as in (4.1), $p^k := (1 - \varepsilon_2^1(k))e_1 + \varepsilon_2^1(k)e_2)$ if $p = e_1$, and $p^k := \varepsilon_1^1(k)e_1 + (1 - \varepsilon_1^1(k))e_2$ if $p = e_2$.

In all three cases $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ converges to (p,q). Hence, (p,q) is strictly perfect.

(ii) Let $I([1]) \neq \emptyset$ and $I([2]) \neq \emptyset$. Assume $A = [a_{rs}]_{r=1}^{2}{}_{s=1}^{n}$.

(a) Let $(p,q) \in \Delta_2 \times \Delta_n$ be a strictly perfect equilibrium. We show that one of assertions (ii.1), (ii.2) or (ii.3) is satisfied.

Let $p \in \mathring{\Delta}_2$. Suppose (ii.1) does not hold. Then, without loss of generality, $PB_2(p, [1]) = \emptyset$. Choose $j \in I([2])$ and define, for $k \in \mathbb{N}$, $\varepsilon_1^1(k) = \varepsilon_2^1(k) = \dfrac{1}{k+1}$ and

$$\varepsilon_l^2(k) := \begin{cases} \dfrac{1}{k+1} & \text{if } l = j \\ \left(\dfrac{1}{k+1}\right)^2 & \text{otherwise} \end{cases} \quad (4.2)$$

Since (p,q) is strictly perfect, there is a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ converging to (p,q) such that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for all $k \in \mathbb{N}$. Using Lemma 4.1(ii), it follows that $PB_2(p^k) \subset PB_2(p)$ for large k. Consequently, $q_l^k = \left(\dfrac{1}{k+1}\right)^2$ for all $l \in I([1])$ and $e_2 A q^k - e_1 A q^k > 0$ for large k. However, this would imply $1 \notin PB_1(q^k)$ and $p^k = \dfrac{1}{k+1} e_1 + \dfrac{k}{k+1} e_2$ for large k. This contradicts the fact that $\{p^k\}_{k \in \mathbb{N}}$ converges to p.

Let $p = e_1$. For each sequence $\{p^k\}_{k \in \mathbb{N}} \subset \mathring{\Delta}_2$ converging to e_1, we have that $PB_2(p^k) = I_v \subset PB_2(e_1)$ for large k. Then, using the same line of argument as above it follows that $I([1]) \cap I_v \neq \emptyset$. Analogously, $I([2]) \cap I_1 \neq \emptyset$ if $p = e_2$.

(b) Let $(p,q) \in PE(A,B)$ be such that one of the assertions (ii.1), (ii.2) or (ii.3) is satisfied. We show that (p,q) is strictly perfect. Let $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k \in \mathbb{N}}$ be a sequence of mistake vectors that converges to zero.

If $p \in \mathring{\Delta}_2$, we can choose $i \in PB_2(p, [1])$ and $j \in PB_2(p, [2])$, and, by defining $p^k := p$ and

$$\begin{aligned} q^k := (1 - \eta(k))q &+ \sum_{l \in I([12])} \varepsilon_l^2(k) e_l \\ &+ \sum_{l \in I([2])} \varepsilon_l^2(k)(q_l(i,l))^{-1} q(i,l) \\ &+ \sum_{l \in I([1])} \varepsilon_l^2(k)(q_l(l,j))^{-1} q(l,j); \end{aligned} \quad (4.3)$$

where $\eta(k) \in [0,1)$ is such that $\sum_{l=1}^{n} q_l^k = 1$.

If $p = e_1$, we can choose $i \in I_v \cap I([1])$ and, by defining $p^k := (1 - \varepsilon_2^1(k))e_1 + \varepsilon_2^1(k)e_2$ and

$$\begin{aligned} q^k := (1 - \xi(k))q &+ \sum_{l \in I([1]) \cup I([12])} \varepsilon_l^2(k) e_l \\ &+ \sum_{l \in I([2])} \varepsilon_l^2(k)(q_l(i,l))^{-1} q(i,l), \end{aligned} \quad (4.4)$$

where $\xi(k) \in [0,1)$ is such that $\sum_{l=1}^{n} q_l^k = 1$. Of course, one can define a similar sequence if $p = e_2$.

In all cases $(p^k, q^k) \in E(A, B, \varepsilon^1(k), \varepsilon^2(k))$ for large k and $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ converges to (p,q). Hence, (p,q) is strictly perfect.

(c) Let $(p^1, q^1), (p^2, q^2) \in PE(A,B)$ be such that (ii.4) and (ii.5) are satisfied. We show that $\{(p^1, q^1), (p^2, q^2)\}$ is a stable set.

Using symmetry considerations, we may, without loss of generality, assume that $p^1 \in \mathring{\Delta}_2$, $PB_2(p^1, [1]) \neq \emptyset$ and $PB_2(p^1, [2]) = \emptyset$, while $p^2 = e_1$ and $I([1]) \cap I_v = \emptyset$. Choose $i \in PB_2(p^1, [1])$.

By (a) it suffices to show that $\{(p^1, q^1), (e_1, q^2)\}$ is strictly perfect.

Let $\{(\varepsilon^1(k), \varepsilon_2(k))\}_{k \in \mathbb{N}}$ be a sequence of mistake vectors that converges to zero. For all $k \in \mathbb{N}$, let

$$\begin{aligned} \gamma(\varepsilon^2(k)) := &\sum_{l \in I([1])} \varepsilon_l^2(k)(a_{1l} - a_{2l}) \\ &+ \sum_{l \in I([2])} \varepsilon_l^2(k)(a_{1l} - a_{2l}). \end{aligned} \quad (4.5)$$

Then, for all $k \in \mathbb{N}$ such that $\gamma(\varepsilon^2(k)) \leq 0$, we define $\bar{p}^k := p^1$ and

$$\bar{q}^k := (1 - \varrho(k))q^1 + \sum_{l=1}^{n} \varepsilon_l^2(k) e_l - \gamma(\varepsilon^2(k))(a_{1i} - a_{2i})^{-1} e_i, \quad (4.6)$$

where $\varrho(k) \in [0,1)$ is such that $\sum_{l=1}^{n} q_l^k = 1$. Further, for all $k \in \mathbb{N}$ such that $\gamma(\varepsilon^2(k)) > 0$, we define $\bar{p}^k := (1 - \varepsilon_2^1(k))e_1 + \varepsilon_2^1(k)e_2$ and

$$\bar{q}^k := (1 - \sum_{l=1}^{n} \varepsilon_l^2(k))q^2 + \sum_{l=1}^{n} \varepsilon_l^2(k)e_l. \quad (4.7)$$

It is easily verified that $(\bar{p}^k, \bar{q}^k) \in E(A,B;\varepsilon^1(k),\varepsilon^2(k))$ for large $k \in \mathbb{N}$. Since the sequence $\{\bar{p}^k, \bar{q}^k\}_{k\in\mathbb{N}}$ has (at most) two limit points (p^1,q^1) and (e_1,q^2), the set $\{(p^1,q^1),(e_1,q^2)\}$ is strictly perfect.

(d) Let C be a stable set with $|C| \geq 2$. For each sequence $\{(\varepsilon^1(k),\varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors converging to zero, the "indicator" $\gamma(\varepsilon^2(k))$ of (4.5) plays a decisive role: C has to contain both a perfect equilibrium (p^1,q^1) that is proof against mistakes of the type $\gamma(\varepsilon^2(k)) \leq 0$ and a perfect equilibrium (p^2,q^2) that is proof against mistakes of the type $\gamma(\varepsilon^2(k)) > 0$. This is achieved only if (p^1,q^1) satisfies (ii.4) and (p^2,q^2) satisfies (ii.5). Using (c), it follows that $C = \{(p^1,q^1),(p^2,q^2)\}$. □

In particular, Theorem 4.3 implies that each stable set for a $2\times n$ bimatrix game consists of finitely many perfect equilibria. This result has been generalized towards $m\times n$ bimatrix games by Jansen et al. (1992).

Example 6. For the 2×6 bimatrix game (A,B) of example 1 each stable set has only one element. The stable sets are

$$\{(\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, q)\} \quad \text{with}$$

$$q \in \operatorname{Conv}\{(e_2, \tfrac{1}{2},0,0,\tfrac{1}{2},0,0),(\tfrac{1}{3},0,0,0,0,\tfrac{2}{3})\}.$$

The unique stable set for the 2×3 bimatrix game (A,B) of Example 4 is given by

$$\{(\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, e_2), (\tfrac{3}{4}e_1 + \tfrac{1}{4}e_2, e_2)\}.$$

As is seen in the 2×3 bimatrix game of Example 4 and 6, there need not be a stable set that contains a proper equilibrium. However, defining a *stable component* as a maximally connected subset of the set of Nash equilibria that contains a stable set (cf. (v) of Theorem 4.2), we have

Theorem 4.4. *For $2\times n$ bimatrix games the following two assertions hold.*

(i) Each strictly perfect equilibrium is proper.
(ii) Each stable component contains a proper equilibrium.

Proof. (i) This is a direct consequence of Theorem 4.3 and Theorem 3.1.
(ii) Let $D \subset E(A,B)$ be a stable component for $(A,B) \in BG(2,n)$. Because of (i) we may assume that there is no strictly perfect equilibrium within D. Then, Theorem 4.3 implies that $I([1]) \neq \emptyset$, $I([2]) \neq \emptyset$ and that there are $(p^1,q^1),(p^2,q^2) \in PE(A,B)$ satisfying (ii.4) and (ii.5) of Theorem 4.3 with $\{(p^1,q^1),(p^2,q^2)\} \subset D$. Without loss of generality we may assume that $p^1 = p(r)$ and $p^2 = p(s)$ for some $r,s \in \{0,1,\ldots,v\}$ with $r<s$. Since D is connected and there are no strictly perfect equilibria within D, we have that $I_k \cap I([12]) \neq \emptyset$ for all $k \in \{r+1,\ldots,s\}$. By distinguishing cases corresponding to the conditions in (ii.4) and (ii.5), one can verify that there exists a perfect equilibrium $(\bar{p},\bar{q})$ for the game $(\bar{A},\bar{B})$ of Theorem 3.1 such that $\bar{p} \in [p(r),p(s)]$. Hence, $(\bar{p},e_j) \in PR(A,B) \cap D$ for all $j \in I_k \cap I([12])$ if $k \in \{r+1,\ldots,s\}$ is such that $\bar{p} \in [p(k-1),p(k)]$. □

5. Persistent retracts

This section provides a characterization of persistent retracts (Kalai and Samet 1984) for $2\times n$ bimatrix games. In particular it is found that all persistent equilibria are perfect. This is not the case for general $m\times n$ bimatrix games (cf. van Damme 1987).

Definition (Kalai and Samet 1984). Let $(A,B) \in BG(m,n)$. Let $R = R_1 \times R_2$ be a non-empty, closed and convex subset of $\Delta_m \times \Delta_n$. R is called an *absorbing retract* if there exist open neighbourhoods U_1 of R_1 and U_2 of R_2 such that for all $p' \in U_1$ there is a $q \in R_2$ with $q \in B_2(p')$, and, for all $q' \in U_2$ there is a $p \in R_1$ with $p \in B_1(q')$.

An absorbing retract that does not properly contain another absorbing retract is called a *persistent retract.* Nash equilibria that are contained in a persistent retract are called *persistent.* By $PERS(A,B)$ we denote the set of all persistent equilibria for (A,B).

To determine the persistent retracts for a particular bimatrix game, one can use the fact that persistent retracts have the special form of selection retracts.

Definition. Let $(A,B) \in BG(m,n)$. A non-empty finite set $F \subset \Delta_n$ is called a *selection* for player 2 in (A,B) if

$$q \in F \Rightarrow Be_j = Be_s \quad \text{for all } j,s \in C(q) \quad (5.1)$$

and

$$q^1,q^2 \in F, q^1 \neq q^2 \Rightarrow Bq^1 \neq Bq^2. \quad (5.2)$$

Selections for player 1 are defined analogously.

An absorbing retract $R = R_1 \times R_2$ is called a *selection retract* if there are selections F^1 for player 1 and F^2 for player 2, such that $R_1 = \operatorname{Conv}(F^1)$ and $R_2 = \operatorname{Conv}(F^2)$.

Summarizing the results of Kalai and Samet we have

Theorem 5.1 (Kalai and Samet 1984).

(i) Each persistent retract is a selection retract.
(ii) Each absoring retract contains a persistent retract.
(iii) For each game, there is at least one persistent retract.
(iv) Each persistent retract contains a proper equilibrium.

If a persistent retract consists of only one equilibrium, this equilibrium is called *robust* (Okada 1983). Persistent retracts for $2\times n$ bimatrix games are characterized in

Theorem 5.2. *Let* $(A,B)\in BG(2,n)$.

(i) Let $(p,q)\in \Delta_2\times\Delta_n$. *Then* (p,q) *is a robust equilibrium if and only if* (p,q) *is a perfect equilibrium for* (A,B) *and one of the following three assertions holds:*

(i.1) $p=e_1$ *and, either* $PB_1(q)=\{1\}$ *or* $I([2])=\emptyset$

(i.2) $p=e_2$ *and, either* $PB_1(q)=\{2\}$ *or* $I([1])=\emptyset$

(i.3) $p\in\mathring{\Delta}_2\backslash\{p(k)\}_{k\in\{1,\dots,\nu-1\}}$ *and* $I([12])=\{1,2,\dots,n\}$.

(ii) Let $R=R_1\times R_2\subset\Delta_2\times\Delta_n$ *be such that* $|R|>1$. *Then* R *is a persistent retract if and only if one of the following assertions holds:*

(ii.1) $I([12])=\{1,2,\dots,n\}$ *and there is a* $k\in\{1,\dots,\nu-1\}$ *such that*

$$R_1=\{p(k)\}\quad and\quad R_2=\mathrm{Conv}\{q^k,q^{k+1}\}$$

with $q^k\in\mathrm{Conv}\{e_l|l\in I_k\}$ *and* $q^{k+1}\in\mathrm{Conv}\{e_l|\,l\in I_{k+1}\}$.

(ii.2) $I([1])\neq\emptyset$, $I([2])\neq\emptyset$ *and*

$$R_1=\Delta_2\quad and\quad R_2=\mathrm{Conv}\{q^1,q^2,\dots,q^\nu\},$$

with $q^k\in\mathrm{Conv}\{e_l|l\in I_k\}$ *for all* $k\in\{1,\dots,\nu\}$ such that $1\in PB_1(q^1)$ *and* $2\in PB_1(q^\nu)$.

Proof. (i) Let (p,q) be a robust equilibrium for (A,B). By Theorem 5.1 (iv) we have that $(p,q)\in PE(A,B)$.

Assume $p=e_1$. Suppose (i.1) is not satisfied. Then $PB_1(q)=\{1,2\}$ and $I([2])\neq\emptyset$. Let $j\in I([2])$ and define $q(\lambda):=(1-\lambda)q+\lambda e_j$ for all $\lambda\in[0,1]$. Since $\{(e_1,q)\}$ is absorbing, we should have that $e_1\in PB_1(q(\lambda))$ for small $\lambda>0$. However, $PB_1(q(\lambda))=\{2\}$ for all $\lambda\in(0,1)$. Similarly, if $p=e_2$, then (i.2) is satisfied.

Assume $p\in\mathring{\Delta}_2$. By Theorem 5.1 (i) and (5.1), it follows that $e_1A=e_2A$ and $I([12])=\{1,2,\dots,n\}$. Suppose $p=p(k)$ for some $k\in\{1,\dots,\nu-1\}$. Define $p'(\lambda):=\lambda e_2+(1-\lambda)p$ and $p''(\lambda)=\lambda e_1+(1-\lambda)p$ for all $\lambda\in[0,1]$. Since $\{(p,q)\}$ is absorbing we should have that $q\in B_2(p'(\lambda))\cap B_2(p''(\lambda))$ for small $\lambda>0$. However, $B_2(p'(\lambda))\cap B_2(p''(\lambda))=\emptyset$ for all $\lambda\in(0,1)$.

Conversely, if $(p,q)\in PE(A,B)$ is such that one of the conditions (i.1), (i.2) or (i.3) is satisfied, then it is straightforward to verify that $\{(p,q)\}$ is an absorbing retract. Hence, (p,q) is robust.

(ii) Trivially, if R satisfies (ii.1), then R is a persistent retract.

Let R satisfy (ii.2). Obviously, R is an absorbing retract. By Theorem 5.1 (ii) we can find a persistent retract $\bar{R}=\bar{R}_1\times\bar{R}_2$ with $\bar{R}\subset R$. Since $\bar{R}$ is a selection retract (cf. Theorem 5.1 (i)) and $e_1A\neq e_2A$, we may, without loss of generality, assume that $e_1\in\bar{R}_1$. Absorbingness implies that $q^\nu\in\bar{R}_2$. Since $e_2Aq^\nu\geq e_1Aq^\nu$ and $I([2])\neq\emptyset$, it follows that $e_2\in\bar{R}_1$ and, by convexity, $\bar{R}_1=\Delta_2$. Therefore, $\{q^1,q^2,\dots,q^\nu\}\subset\bar{R}_2$ and $\bar{R}_2=R$. Hence, $\bar{R}=R$ and R is a persistent retract.

Conversely, let R be a persistent retract. Suppose (ii.1) does not hold. We show that (ii.2) is satisfied.

Since $|R|>1$, part (i) implies that $I([12])\neq\{1,2,\dots,n\}$. If $I([1])\neq\emptyset$ and $I([2])=\emptyset$, then absorbingness implies that $e_1\in R_1$ and so, there exists a $q^\nu\in\mathrm{Conv}(\{e_l|l\in I_\nu\})$ with $q^\nu\in R_2$. However, using part (i), this would imply that $(e_1,q^\nu)\in R$ is a robust equilibrium. Similarly, it can not be the case that $I([2])\neq\emptyset$ and $I([1])=\emptyset$. Hence, $I([1])\neq\emptyset$ and $I([2])\neq\emptyset$. Without loss of generality, we may assume that $e_1\in R_1$. Consequently, there is a $q^\nu\in\mathrm{Conv}(\{e_l|l\in I_\nu\})$ with $q^\nu\in R_2$. Because of part (i), $2\in PB_1(q^\nu)$. Since $I([2])\neq\emptyset$, absorbingness implies that $e_2\in R_1$ and so $R_1=\Delta_2$. Hence, there are $q^1,\dots,q^{\nu-1}\in R_2$ with $q^k\in\mathrm{Conv}(\{e_l|l\in I_k\})$ for all $k\in\{1,\dots,\nu-1\}$. Moreover, part (i) implies that $1\in PB_1(q^1)$. So (ii.2) is satisfied. □

An immediate consequence of Theorem 5.2 is

Corollary 5.3. *For* $2\times n$ *bimatrix games the following two assertions hold:*

(i) Each persistent equilibrium is perfect.

(ii) Each robust equilibrium is strictly perfect.

Example 7. The persistent retracts for the 2×6 bimatrix game (A,B) of example 1 are the sets $\Delta_2\times\mathrm{Conv}\{e_1,q,e_3\}$ with $q\in\mathrm{Conv}\{e_2,e_4\}$. Consequently,

$$PERS(A,B)=$$

$$\{\tfrac{1}{4}e_1+\tfrac{3}{4}e_2\}\times\mathrm{Conv}\{\tfrac{1}{2},0,0,\tfrac{1}{2},0,0),e_2\}$$

$$\cup\,[\tfrac{1}{4}e_1+\tfrac{3}{4}e_2,\tfrac{3}{4}e_1+\tfrac{1}{4}e_2]\times\{e_2\}$$

Finally, Fig. 3 provides a full description of the relations between various refinements of the Nash equilibrium

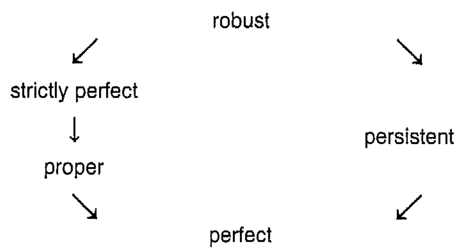

Fig. 3 (see text)

concept for $2\times n$ bimatrix games, in the sense that each implication that is not present in the diagram does not hold.

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