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Borm, P.E.M.; Gijsberts, A.N.

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On Constructing Games with a Convex Set of Equilibrium Strategies

By P. E. M. Borm and A. N. Gijsberts[1]

Abstract: Necessary and sufficient conditions on a convex set C (of strategy pairs) are given for the existence of a $2 \times n$ bimatrix game with equilibrium set C. This is done with the use of a geometric-combinatorial solution method for $2 \times n$ bimatrix games.

Zusammenfassung: Es werden notwendige und hinreichende Bedingungen an die konvexe Menge C der Strategiepaare für die Existenz eines $2 \times n$ Bimatrix Spieles mit Gleichgewichtsmenge C aufgegeben. Dies wird durch eine geometrisch-kombinatorische Lösungsmethode für $2 \times n$ Bimatrix Spiele erreicht.

Key words: Game Theory, $2 \times n$ bimatrix games, convex equilibium set, construction problem.

1 Introduction

The construction problem for matrix games is completely solved. For these games Bohnenblust, Karlin and Shapley (1950) and Gale and Sherman (1950) established a relationship between the dimensions of the sets of optimal strategies. It was proved that this relationship is the most important ingredient for the construction of a matching game.

For bimatrix games Millham (1972) and Kreps (1974) have paid attention to the construction problem. Milham considered bimatrix games with special properties and Kreps solved the construction problem for bimatrix games with a unique equilibrium point.

Bimatrix games with a convex set of equilibrium strategies were already studied by Nash (1951) who called them *solvable.* It was argued that of all bimatrix games these games resemble matrix games most. In this paper, the geometric-combinatorial

[1] P. E. M. Borm and A. N. Gijsberts, Dept. of Mathematics, University of Nijmengen, The Netherlands.

approach of Borm, Gijsberts and Tijs (1988) enables us to solve the construction problem for the class of $2 \times n$ bimatrix games with a convex set of equilibrium strategies. It is found that even this apparently simple case requires a rather complex analysis. However, to our opinion the resulting necessary and sufficient conditions on a convex set C (of strategy pairs) to be the equilibrium set of a $2 \times n$ bimatrix game and more especially the way they come forward, certainly give insight into some of the conditions which will play a role in the general $(2 \times n)$ construction problem.

The organization of this paper is as follows. Some definitions and concepts concerning bimatrix games are recalled in section 2. After briefly describing the geometric-combinatorial approach of Borm, Gijsberts and Tijs (1988) in section 3, we survey some general results about maximal Nash subsets in section 4. The construction problem for $2 \times n$ bimatrix games with a convex equilibrium set is tackled in section 5 by distinguishing three different cases. The paper concludes with a few remarks which may be useful for a study of the general $(2 \times n)$ construction problem.

Notation: Let $S \subset \mathbb{R}^t$. Then $\mathrm{Conv}(S)$ denotes the convex hull of S. If S is finite, then $|S|$ denotes the number of elements of S. Furthermore, if S is convex, then $\mathrm{Ext}(S)$ denotes the set of extreme elements of S. Here, $x \in S$ is called extreme if for all for $x^1, x^2 \in S$ with $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$ we have that $x^1 = x^2$. Finally, for $a, b \in \mathbb{R}$ we define

$$[a, b] := \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}, (a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

2 General Bimatrix Games

In this section we recall some definitions and establish notation.

Let $A = [a_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ be real $m \times n$ matrices. Let the set of probability vectors in $\mathbb{R}^t$ be denoted by $\Delta_t := \{(p_1, \ldots, p_t) \in \mathbb{R}^t \mid \sum_{i=1}^{t} p_i = 1, p_i \geqslant 0$ for all $i \in \{1, \ldots, t\}\}$. The (mixed extension of the) $m \times n$ *bimatrix game* (A, B) is defined as the two-person game (in strategic form) with (*mixed*) strategy spaces Δ_m and Δ_n for player 1 and player 2, respectively and (expected) payoff functions $K : \Delta_m \times \Delta_n \to \mathbb{R}$ and $L : \Delta_m \times \Delta_n \to \mathbb{R}$, respectively with $K(p, q) = pAq$ and $L(p, q) = pBq$ for all $(p, q) \in \Delta_m \times \Delta_n$. The k-th *pure* strategy e_k for a player is defined as the vector x in Δ_m (or Δ_n) with the k-th coordinate equal to one.

The $m \times n$ bimatrix game (A, B) can be represented by

$$(A, B) = \begin{bmatrix} (a_{11}, b_{11}) & \cdots & (a_{1n}, b_{1n}) \\ \vdots & & \vdots \\ (a_{m1}, b_{m1}) & \cdots & (a_{mn}, b_{mn}) \end{bmatrix}$$

where rows and columns correspond to the players' pure strategies, player 1 choosing the rows and player 2 choosing the columns. If player 1 chooses row $i \in \{1, \dots, m\}$ and player 2 column $j \in \{1, \dots, n\}$, they obtain a payoff equal to a_{ij} and b_{ij}, respectively. Mixed strategies induce a probability vector on the cells of the bimatrix and expected payoffs can be determined accordingly.

A strategy pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$ is called a *Nash equilibrium* of (A, B) if

$$\hat{p}A\hat{q} \geqslant pA\hat{q} \quad \text{and} \quad \hat{p}B\hat{q} \geqslant \hat{p}Bq \tag{1}$$

for all $p \in \Delta_m$ and $q \in \Delta_n$. Nash (1951) proved that the set $E(A, B)$ of all Nash equilibria of (A, B) is non-empty for all bimatrix games (A, B).

Furthermore, for $p \in \Delta_m$, we define $C(p) := \{i \in \{1, \dots, m\} | p_i > 0\}$, the *carrier* of p, and let $PB_2(p) := \{j \in \{1, \dots, n\} | pBe_j \geqslant pBe_k \text{ for all } k \in \{1, \dots, n\}\}$ represent the set of *pure best replies* of player 2 to p. Defining, for $q \in \Delta_n$, $C(q)$ and $PB_1(q)$ in a similar way, it is easily checked that $(\hat{p}, \hat{q})$ is a Nash equilibrium of (A, B) if and only if

$$C(\hat{p}) \subset PB_1(\hat{q}) \quad \text{and} \quad C(\hat{q}) \subset PB_2(\hat{p}). \tag{2}$$

3 A Solution Method for $2 \times n$ Bimatrix Games

We now concentrate on $2 \times n$ bimatrix games (A, B) with $A = [a_{ij}]_{i=1}^{2}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{2}{}_{j=1}^{n}$. For these games we especially focus on the geometric-combinatorial approach (GC-approach) as introduced by Borm, Gijsberts and Tijs (1988). In this section we will briefly recall how the GC-approach can be used to determine the complete equilibrium set for each $2 \times n$ bimatrix game.

It may be noted here that the GC-approach differs from and has some advantages to the geometric application of the Lemke-Howson method (cf. Shapley (1974)) to $2 \times n$ bimatrix games. The Lemke-Howson method only applies to "nondegenerate" games, it uses more information about the best reply structure of the underlying game and it only guarantees the finding of at least one equilibrium point (but possibly not all).

Let (A, B) be a $2 \times n$ bimatrix game. To shorten descriptions, the strategy set Δ_2 of player 1 is identified with the closed interval $[0, 1]$ by identifying $(p, 1-p) \in \Delta_2$ with $p \in [0, 1]$. So, for example, if $p \in [0, 1]$ and $q \in \Delta_n$, then pBq is a short notation for $(p, 1-p)Bq$.

The GC-approach is based upon a geometric representation of the marginal functions of player 2 together with a labeling procedure. Here, for $j \in \{1, \dots, n\}$, the *marginal function* $g_j : [0, 1] \to \mathbb{R}$ is defined by

$$g_j(p) = pBe_j$$

for all $p \in [0, 1]$. Furthermore, the labeling procedure is on the pure strategies of player 2. There are three labels, [1], [2] and [12], and for $j \in \{1, \dots, n\}$ we define

$$\mathrm{lab}(e_j) = [1] \Leftrightarrow PB_1(e_j) = \{1\}$$

$$\mathrm{lab}(e_j) = [2] \Leftrightarrow PB_1(e_j) = \{2\}$$

$$\mathrm{lab}(e_j) = [12] \Leftrightarrow PB_1(e_j) = \{1, 2\}$$

In other words, the label of e_j represents the pure best replies of player 1 to e_j. In geometric representations it will be convenient to put the labels on the corresponding marginal functions. Note that in this way a marginal function may have more than one label (cf. example 1). For the determination of Nash equilibria it turns out that we only have to consider the (piecewise linear) maximum function $g : [0, 1] \to \mathbb{R}$, defined by

$$g(p) = \max_{j \in \{1, \dots, n\}} g_j(p)$$

for all $p \in [0, 1]$, and the labels of the pure strategies which consitute the maximum function.

The following example, which can also be found in Borm, Gijsberts and Tijs (1988), illustrates how the above framework can be applied geometrically.

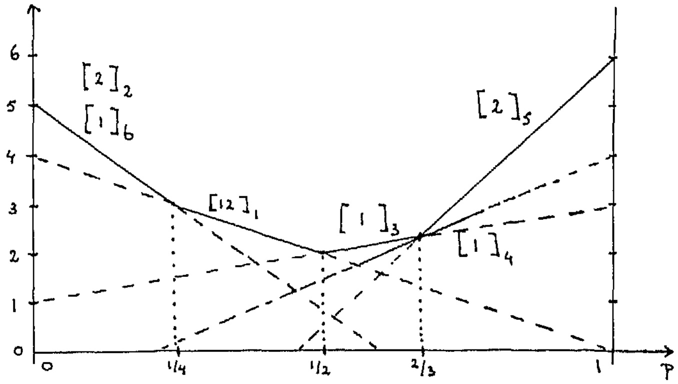

Fig. 1

Example 1: Let the 2 x 6 bimatrix game (A, B) be determined by

$$(A, B) = \begin{pmatrix} (1,0) & (0,-3) & (4,3) & (0,4) & (1,6) & (2,-3) \\ (1,4) & (2,5) & (2,1) & (-1,-1) & (3,-5) & (0,5) \end{pmatrix}$$

Figure 1 shows the labeling on the maximum function. Here, each label is provided with a subindex $j \in \{1, \ldots, n\}$ to represent the corresponding pure strategy of player 2.

In order to formalize descriptions we need the following definitions. Let $p \in [0, 1]$. Then $PB_2(p, [1]) := \{j \in \{1, \ldots, n\} | j \in PB_2(p), \text{ lab}(e_j) = [1]\}$ represents the set of pure best replies to p having label [1]. The sets $PB_2(p, [2])$ and $PB_2(p, [12])$ can be defined analogously. Note that $PB_2(p) = PB_2(p, [1]) \cup PB_2(p, [2]) \cup PB_2(p, [12])$. Furthermore, let the set $S(p)$ of *solutions* for player 2 to p (in (A, B)) be defined by

$$S(p) := \{q \in \Delta_n \,|\, (p, q) \in E(A, B)\} \tag{3}$$

Jansen (1981) showed that $S(p)$ is a polytope, i.e. the convex hull of finitely many points. A description of this solution set in terms of the GC-approach is provided in theorem 3.1. Here, the set $PS(p)$ of *pure solutions* to p is defined by

$$PS(p) := \{e_k \in \Delta_n \,|\, (p, e_k) \in E(A, B)\} \tag{4}$$

Then, by definition of the labeling procedure, it follows that

$$PS(0) = \{e_k \in \Delta_n \,|\, k \in PB_2(0, [2]) \cup PB_2(0, [12])\} \tag{5}$$

$$PS(1) = \{e_k \in \Delta_n \mid k \in PB_2(1, [1]) \cup PB_2(1, [12])\} \tag{6}$$

and, for $0 < p < 1$,

$$PS(p) = \{e_k \in \Delta_n \mid k \in PB_2(p, [12])\} \tag{7}$$

Convexity of $S(p)$ implies that $\mathrm{Conv}(PS(p)) \subset S(p)$. However, there need not be equality. For this we make the following observation. For all $i, j \in \{1, \ldots, n\}$ with $\mathrm{lab}(e_i) = [1]$ and $\mathrm{lab}(e_j) = [2]$ we can define the *transit strategy* $q(i, j)$ as the unique strategy $q \in \Delta_n$ satisfying

$$C(q) = \{1, 2\} \quad \text{and} \quad e_1 Aq = e_2 Aq. \tag{8}$$

It is not difficult to give an explicit description of $q(i, j)$ in terms of the payoff matrix A. This leads to the definition of the set $TS(p)$ of *transit solutions* to p as given by

$$TS(p) = \{q(i, j) \in \Delta_n \mid i \in PB_2(p, [1]), j \in PB_2(p, [2]\} \tag{9}$$

Using (2) it is clear that $TS(p) \subset S(p)$ and (5), (6) and (7) imply that $\mathrm{Conv}(PS(p)) \cap TS(p) = \emptyset$. Moreover, it can be shown that each extreme point of $S(p)$ must either be a transit solution or a pure solution (e.g. by using the well-known fact that $|C(p)| \leqslant 2$ implies that $|C(q)| \leqslant 2$ for all $q \in \mathrm{Ext}(S(p))$). Summarizing we find

Theorem 3.1: Let (A, B) be a $2 \times n$ bimatrix game. Let $p \in [0, 1]$ be a strategy of player 1. Then

$$\mathrm{Ext}(S(p)) = PS(p) \cup TS(p) \tag{10}$$

and consequently

$$S(p) = \mathrm{Conv}(PS(p) \cup TS(p)) \tag{11}$$

Theorem 3.1 enables us to determine the equilibrium set of a $2 \times n$ bimatrix game (A, B) in finitely many steps. This can be seen as follows. Let the line segments which constitute the maximum function be called *facets*. So, if there are ν facets G_1,

$G_2, \ldots, G_\nu$ where $\nu \leqslant n$, we can find $\nu + 1$ strategies $p(0), p(1), \ldots, p(\nu)$ such that $p(0) := 0, p(\nu) := 1$ and

$$G_k = \mathrm{Conv}(\{(p(k-1), g(p(k-1))), (p(k), g(p(k)))\})$$

for all $k \in \{1, \ldots, \nu\}$. These strategies $p(0), p(1), \ldots, p(\nu)$ are called *decisive*. Then it is clear that with $k \in \{1, \ldots, \nu\}$ and $p^1, p^2 \in (p(k-1), p(k))$ we have that $PB_2(p^1) = PB_2(p^2)$ and consequently, using (2), that

$$S(p^1) = S(p^2)$$

So the Nash equilibrium set $E(A, B)$ can be determined in (at most) $2\nu + 1$ steps since

$$E(A, B) = \bigcup_{k=0}^{\nu} \{p(k)\} \times S(p(k)) \cup \bigcup_{k=1}^{\nu} ((p(k-1), p(k)) \times S(\tfrac{1}{2} p(k-1) + \tfrac{1}{2} p(k))) \tag{12}$$

Finally, it can be noted that, for $k \in \{0, \ldots, r-1\}$ and $p \in (p(k), p(k+1))$,

$$PB_2(p) \subset PB_2(p(k)), PB_2(p) \subset PB_2(p(k+1)) \tag{13}$$

and

$$S(p) \subset S(p(k)), S(p) \subset S(p(k+1)) \tag{14}$$

Example 2: For the 2×6 bimatrix game (A, B) of example 1, theorem 3.1 and figure 1 imply that $E(A, B) = \bigcup_{k \in \{1, \ldots, 5\}} C_k$, with

$$C_1 = \{0\} \times \mathrm{Conv}(\{e_2, q(6,2)\}) = \{0\} \times \mathrm{Conv}(\{e_2, \tfrac{1}{2} e_2 + \tfrac{1}{2} e_6\})$$

$$C_2 = [0, \tfrac{1}{4}] \times \{q(6,2)\} = [0, \tfrac{1}{4}] \times \{\tfrac{1}{2} e_2 + \tfrac{1}{2} e_6\}$$

$$C_3 = \{\tfrac{1}{4}\} \times \mathrm{Conv}(\{e_1, q(6,2)\}) = \{\tfrac{1}{4}\} \times \mathrm{Conv}(\{e_1, \tfrac{1}{2} e_2 + \tfrac{1}{2} e_6\})$$

$$C_4 = [\tfrac{1}{4}, \tfrac{1}{2}] \times \{e_1\}$$

$$C_5 = \{\tfrac{2}{3}\} \times \mathrm{Conv}(\{q(3,5), q(4,5)\}) = \{\tfrac{2}{3}\} \times \mathrm{Conv}(\{\tfrac{1}{2} e_3 + \tfrac{1}{2} e_5, \tfrac{2}{3} e_4 + \tfrac{1}{3} e_5\})$$

4 Maximal Nash Subsets

Let (A, B) be an $m \times n$ bimatrix game and let $C \subset E(A, B)$. Two equilibrium points $(p, q), (p', q') \in C$ are called *C-interchangeable* if both (p, q') and $(p', q) \in C$. C is called a *Nash subset* for the game (A, B) if every pair of equilibrium points in C is C-interchangeable. A Nash subset C is called *maximal* if there is no Nash subset C' such that C is properly contained in C'. The term maximal Nash subset was first introduced by Heuer and Millham (1976). The importance of this concept is due to

Theorem 4.1: The set of equilibrium points of a bimatrix game is a (not necessarily disjoint) union of a finite number of maximal Nash subsets.

A proof can be found in Jansen (1981). Here it is also shown that a maximal Nash subset is in fact the Cartesian product of two polytopes and that a maximal Nash subset can be characterized as a convex (sub)set of equilibria not properly contained in any other convex subset of the set of equilibrium points. For our purposes the following result of Chin, Parthasarathy and Raghavan (1974) plays an important role.

Theorem 4.2: The set of equilibrium points of a bimatrix game is convex if and only if it is a maximal Nash subset.

So, if the Nash equilibrium set of a bimatrix game is convex it is the unique maximal Nash subset of the game. Restricting our attention to $2 \times n$ bimatrix games it is not difficult to prove

Theorem 4.3: Let (A, B) be a $2 \times n$ bimatrix game. Let C be a maximal Nash subset of (A, B). Then exactly one of the following two assertions holds

(i) There is a decisive strategy $p(k), k \in \{0, 1, \ldots, \nu\}$, such that $C = \{p\} \times S(p)$.

(ii) There are decisive strategies $p(k-1)$ and $p(k), k \in \{1, \ldots, \nu\}$, such that

$$C = [p(k-1), p(k)] \times S(p) \text{ for each } p \in (p(k-1), p(k)).$$

So for $2 \times n$ bimatrix games there are two different types of maximal Nash subsets. The first type, as in theorem 4.3(i), we will call *vertical,* the second type, as in theorem 4.3(ii), *horizontal.* If the strategy $p(k)$ in theorem 4.3(i) is pure we speak of a *pure vertical* Nash subset, otherwise of an *inner vertical.*

5 The Construction Problem

5.1 Preliminaries

The construction problem for $2 \times n$ bimatrix games with a convex equilibrium set can be formulated as follows

(P) What are necessary and sufficient conditions on a convex set $C \subset [0, 1] \times \Delta_n$ for the existence of a $2 \times n$ bimatrix game with $E(A, B) = C$?

Surely this convex set C must be a polytope, so for describing C it suffices to give its finitely many extreme points. A first necessary condition on Ext(C) is given by the notion of feasibility. Here, a set $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ is called *feasible* if the following four assertions hold.

(F.1) $|T_1| = 1$ or $|T_1| = 2$.

(F.2) $T_2 \neq \emptyset$

(F.3) $|C(q)| \leqslant 2$ for all $q \in T_2$.

(F.4) $C(q) \neq C(q')$ for all $q, q' \in T_2$ with $q \neq q'$.

Note that for a feasible set $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ we have that $|T_1 \times T_2| < \infty$ since $|T_1| < \infty$ by (F.1) and $|T_2| < \infty$ by (F.3) and (F.4). Furthermore, if C is a maximal Nash subset of a $2 \times n$ bimatrix game, then Ext(C) is feasible. Condition (F.2) is satis-fied trivially, (F.1) follows from theorem 4.3 and the same theorem together with the fact that $\text{Ext}(S(p)) = PS(p) \cup TS(p)$ for all $p \in [0, 1]$ imply (F.3) and (F.4). Now we are able to reformulate the problem (P) into

(P^*) What are necessary and sufficient conditions on a feasible set $T_1 \times T_2$ for the existence of a $2 \times n$ bimatrix game with $E(A, B)$ is convex and $\text{Ext}(E(A, B)) = T_1 \times T_2$?

In the following sections the construction problem (P^*) will be tackled by distinguishing three different cases.

Case 1: $|T_1| = 2$, this situation corresponds to construction of a $2 \times n$ bimatrix game with its unique maximal Nash subset being a horizontal one.

Case 2: $T_1 = \{p\}$ with $p \in (0, 1)$, on constructing a $2 \times n$ bimatrix game with its unique maximal Nash subset being an inner vertical one.

Case 3: $T_1 = \{p\}$ with $p \in \{0, 1\}$, on constructing a $2 \times n$ bimatrix game with its unique maximal Nash subset being a pure vertical one.

Let us now introduce some further terminology on feasible sets. Let $T_1 \times T_2$ be feasible. Then T_2 can be partitioned in the following way (cf. (F.3)). Let $P := \{q \in T_2 \mid |C(q)| = 1\}$ be the set of *pure extremals* in T_2 and let $M := \{q \in T_2 \mid |C(q)| = 2\}$ be the set of *mixed extremals* in T_2.

Now let $i \in C(M)$, where for $S \subset \Delta_t$ the carrier $C(S)$ of S is defined by $C(S) := \bigcup_{x \in S} C(x)$. Then the set $D(i) \subset C(M)$ is defined by

$$D(i) := \{j \in C(M) \mid \text{there is a } q \in M \text{ such that } C(q) = \{i, j\}\} \tag{15}$$

Note that (F.4) implies that, with $j \in D(i)$, there is a unique mixed extremal $q \in M$ such that $C(q) = \{i, j\}$. This unique mixed extremal will be denoted by $m(\{i, j\})$. Having these definitions it follows that

$$M = \bigcup_{i \in C(M)} \{m(\{i, j\}) \mid j \in D(i)\}. \tag{16}$$

5.2 The Horizontal Case

For the horizontal case we first derive the two necessary conditions (H.1) and (H.2) as given in

Lemma 5.1: Let (A, B) be a $2 \times n$ bimatrix game such that $E(A, B)$ is convex and $\mathrm{Ext}(E(A, B)) = T_1 \times T_2$ where $T_1 = \{p^1, p^2\} \subset [0, 1]$ with $p^1 < p^2$ and $T_2 \subset \Delta_n$. Then the following two assertions hold

(H.1) $T_2 = P$ *(pure extremality)*

(H.2) $|C(T_2)| - |T_1 \cap \{0, 1\}| \leqslant n - 2$ *(attachment condition)*

Proof: Since $E(A, B)$ is convex it is the unique maximal Nash subset (cf. theorem 4.2) and from theorem 4.3 we infer that

$$E(A, B) = [p^1, p^2] \times S(p)$$

for all $p \in (p^1, p^2)$. Hence, using theorem 4.1,

$$T_2 = \mathrm{Ext}(S(p)) = PS(p) \cup TS(p)$$

for all $p \in (p^1, p^2)$. Let $S := S(p), PS := PS(p)$ and $TS := TS(p)$ for all $p \in (p^1, p^2)$.

Then $M = \{q \in T_2 \mid |C(q)| = 2\} = \{q \in PS \cup TS \mid |C(q)| = 2\} = TS$ and, consequently, $P = PS$.

First we show pure extremality or, equivalently, that the set M of mixed extremals is empty. Suppose $M \neq \emptyset$. Then, since $M = TS$, we can find $i, j \in \{1, \ldots, n\}$ such that $\mathrm{lab}(e_i) = [1]$, $\mathrm{lab}[e_j] = [2]$ and $q(i, j) \in TS$. Now suppose $p^1 = 0$. Then (5) and (7) imply that $e_j \in PS(p^1)$ but $e_j \notin S$. This contradicts the fact that $E(A, B) = [p^1, p^2] \times S$. So $p^1 > 0$. Since p^1 has to be decisive (cf. theorem 4.3) there exists a $k \in \{1, \ldots, n\}$ such that $k \in PB_2(p^1)$ and $k \notin PB_2(p)$ for all $p \in (p^1, p^2)$. If $\mathrm{lab}(e_k) = [12]$, then $e_k \in PS(p^1)$ but $e_k \notin S$. If $\mathrm{lab}(e_k) = [1]$, then $q(k, j) \in TS(p^1)$ but $q(k, j) \notin S$. And, if $\mathrm{lab}(e_k) = [2]$, then $q(i, k) \in TS(p^1)$ but $q(i, k) \notin S$. All three cases contradict the fact that $E(A, B) = [p^1, p^2] \times S$. Therefore we may conclude that M is empty.

The attachement condition (H.2) has to do with connecting the horizontal maximal Nash subset to the axes $p = 0$ and $p = 1$. In view of (H.1) we have that $T_2 \subset \{e_j \mid j \in \{1, \ldots, n\}\}$. So it is clear that $|C(T_2)| \leqslant n$. However, as we have already noted above, if $p^1 > 0$ there exists a $k \in \{1, \ldots, n\}$ with $k \in PB_2(p^1)$ and $k \notin PB_2(p)$ for all $p \in (p^1, p^2)$, so especially $k \notin C(T_2)$. Similarly, if $p^2 < 1$, then there exists an $l \in \{1, \ldots, n\}$ with $l \in PB_2(p^2)$ and $l \notin C(T_2)$. Combining these arguments together with the fact that $(PB_2(p^1) \cap PB_2(p^2)) \subset PB_2(p)$ for all $p \in (p^1, p^2)$ we find that

$$|C(T_2)| - |T_1 \cap \{0, 1\}| \leqslant n - 2 \qquad \square$$

The conditions (H.1) and (H.2) are also sufficient. This is seen in

Theorem 5.1: Let $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ be a feasible set where $T_1 = \{p^1, p^2\}$ with $p^1 < p^2$. If both (H.1) and (H.2) are satisfied, then there exists a $2 \times n$ bimatrix game (A, B) such that $E(A, B)$ is convex and $\mathrm{Ext}(E(A, B)) = T_1 \times T_2$.

Proof: We explicitely construct a corresponding $2 \times n$ bimatrix game (A, B) for the case $0 < p^1 < p^2 < 1$ (the construction for the case with $p^1 = 0$ or $p^2 = 1$ can be done in a similar way). According to (H.1) and (H.2) we have that $T_2 \subset \{e_1, \ldots, e_n\}$ and $|T_2| \leqslant n-2$. Without loss of generality (else renumber) we may assume that $T_2 = \{e_1, \ldots, e_t\}$ with $t \leqslant n-2$. Now let $A = [a_{ij}]_{i=1}^{2}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{2}{}_{j=1}^{n}$ be defined by

$$\begin{cases} a_{1n-1} = a_{2n} = 1 \\ a_{ij} = 0 \quad \text{else} \end{cases} \quad \text{and} \quad \begin{cases} b_{1n-1} = 2 - 1/p^1 & b_{2n-1} = b_{1n} = 2, b_{2n} = 2 - 1/p^2 \\ b_{ij} = 1 & \text{if } j \in \{1, \ldots, t\} \\ b_{ij} = 0 & \text{else} \end{cases}$$

i.e.

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & & t & t+1 & & n-2 & n-1 & n \end{array} \\ A = & \begin{bmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & & t & t+1 & & n-2 & n-1 & n \end{array} \\ B = & \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 & 2 - 1/p^1 & 2 \\ 1 & \ldots & 1 & 0 & \ldots & 0 & 2 & 2 - 1/p^2 \end{bmatrix} \end{array}$$

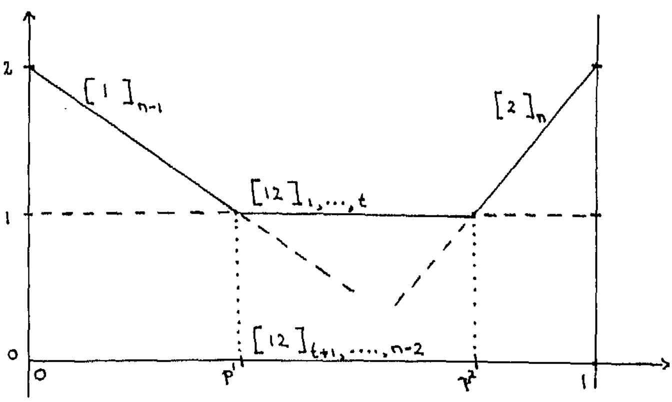

Fig. 2

Applying the GC-approach (cf. figure 2), theorem 3.1 implies that $E(A, B) = [p^1, p^2] \times \mathrm{Conv}(\{e_1, \ldots, e_t\})$. Hence, $E(A, B)$ is convex and $\mathrm{Ext}(E(A, B)) = T_1 \times T_2$. □

5.3 The Inner Vertical Case

In lemma 5.2 we derive two necessary conditions (IV.I) and (IV.2).

Lemma 5.2: Let (A, B) be a $2 \times n$ bimatrix game such that $E(A, B)$ is convex and $\mathrm{Ext}(E(A, B)) = T_1 \times T_2$ where $T_1 = \{p\}$ with $0 < p < 1$ and $T_2 \subset \Delta_n$. Then the following two assertions hold.

(IV.1) $M \neq \emptyset$ *(mixed extremal existence)*

(IV.2) $C(M) \cap C(P) = \emptyset$ *(no overlap)*

Proof: $E(A, B)$ is the unique maximal Nash subset, so theorem 4.3 implies that

$$E(A, B) = \{p\} \times S(p)$$

and consequently

$$T_2 = \mathrm{Ext}(S(p)) = PS(p) \cup TS(p)$$

Moreover, $P = \{q \in T_2 \mid |C(q)| = 1\} = PS(p)$ and $M = \{q \in T_2 \mid |C(q)| = 2\} = TS(p)$. Hence, since $C(PS(p)) \subset \{k \in \{1, \ldots, n\} \mid \mathrm{lab}(e_k) = [12]\}$ and $C(TS(p)) \subset \{k \in \{1, \ldots, n\} \mid \mathrm{lab}(e_k) = [1]$ or $\mathrm{lab}(e_k) = [2]\}$, we have that $C(M) \cap C(P) = \emptyset$. So (IV.2) holds. Further, if we can show that $TS(p) \neq \emptyset$, then also (IV.1) holds.

Suppose $TS(p) = \emptyset$. As we know from theorem 4.3, p is a decisive strategy. Therefore, we can choose a $k \in \{1, \ldots, \nu - 1\}$ such that $p = p(k)$. Since $TS(p(k)) = \emptyset$ and $S(p') = \emptyset$ for all $p' \in [0, 1]$ with $p' \neq p$, expression (13) and theorem 3.1 imply that either $PB_2(p') = PB_2(p', [1])$ or $PB_2(p') = PB_2(p', [2])$ for all strategies $p' \in (p(k-1), p(k)) \cup (p(k), p(k+1))$. Without loss of generality we assume the first. Then, if $p(k+1) < 1$, the fact that $S(p(k+1)) = \emptyset$, (13) and theorem 3.1 imply that $PB_2(p(k+1)) = PB_2(p(k+1), [1])$ and so $PB_2(p') = PB_2(p', [1])$ for all $p' \in (p(k+1), p(k+2))$. Repeating this procedure we find that $PB_2(1) \supset PB_2(p') = PB_2(p', [1])$ for all $p' \in (p(\nu - 1), 1)$. However, this would imply that $PS(1) \neq \emptyset$ (cf. (6)). □

The following lemma shows how mixed extremals are related. These relationships play an important role for the inner vertical case since, by (IV.1), $M \neq \emptyset$.

Here, for $i \in C(M)$ and $j \in D(i)$ the *pure strategy ratio* λ_{ij} of $m(\{i, j\})$ is defined by

$$\lambda_{ij} := m_j(\{i, j\}) \cdot (m_i(\{i, j\}))^{-1} \tag{17}$$

Lemma 5.3: Let (A, B) be a $2 \times n$ bimatrix game such that $C \subset E(A, B)$ is a maximal Nash subset with $\mathrm{Ext}(C) = T_1 \times T_2 \subset [0, 1] \times \Delta_n$ where $T_2 = P \cup M$. Then

(M.1) there exists a $D \subset C(M)$ such that

(i) $D(k) \in \{D, C(M)\backslash D\}$ for all $k \in C(M)$ *(partition property)*

(ii) there is an $i_0 \in D$ such that for all $i \in D$ there exists a constant $c(i, i_0) \in \mathbb{R}$ with

$$\lambda_{ij}(\lambda_{i_0 j})^{-1} = c(i, i_0) \quad \text{for all } j \in C(M)\backslash D \quad \textit{(proportionality)} \tag{18}$$

Proof: Theorem 4.3 implies that either there exists a $p^1 \in [0, 1]$ such that $C = \{p^1\} \times S(p^1)$ or the exist $p^1, p^2 \in [0, 1]$, $p^1 < p^2$, such that $[p^1, p^2] \times S(p')$ for all $p' \in (p^1, p^2)$. So in both cases we can find a strategy p such that

$$T_2 = \mathrm{Ext}(S(p)) = PS(p) \cup TS(p).$$

Let $p \in [0, 1]$ be as above. Then feasibility implies that

$$M = TS(p) = \{q(i, j) \in \Delta_n \,|\, i \in PB_2(p, [1]), j \in PB_2(p, [2])\}$$

Let $k \in C(M)$. Then either $k \in PB_2(p, [1])$ or $k \in PB_2(p, [2])$. If $k \in PB_2(p, [1])$, then $D(k) = \{l \in C(M) |$ there is a $q \in M$ with $C(q) = \{k, l\}\} = PB_2(p, [2])$.

Similarly, if $k \in PB_2(p, [2])$, then $D(k) = PB_2(p, [1])$. So, choosing $D = PB_2(p, [1])$, (i) results.

Furthermore, for all $i \in D$ and $j \in C(M)\backslash D$, it is clear that the mixed extremal $m(\{i, j\})$ is equal to the transit strategy $q(i, j)$. So

$$e_1 Am(\{i, j\}) = e_2 Am(\{i, j\}) \quad \text{for all } i \in D,\ j \in C(M)\backslash D \tag{19}$$

If we substitute $a_{1i} - a_{2i} (> 0)$ by α_i and $a_{2j} - a_{1j} (> 0)$ by α_j and $m_j(\{i, j\}) \cdot (m_i(\{i, j\}))^{-1}$ by the pure strategy ratio λ_{ij} for all $i \in D$ and $j \in C(M) \backslash D$, then (19) can be reformulated as

$$\alpha_i = \lambda_{ij} \alpha_j \quad \text{for all } i \in D \text{ and } j \in C(M) \backslash D \tag{20}$$

Having $i_0 \in D$ fixed, another transformation of (20) leads to

$$\begin{cases} \alpha_j = (\lambda_{i_0 j})^{-1} \alpha_{i_0} & \text{for all } j \in C(M) \backslash D \\ \alpha_i = \lambda_{ij} (\lambda_{i_0 j})^{-1} \alpha_{i_0} & \text{for all } i \in D \text{ and } j \in C(M) \backslash D \end{cases} \tag{21}$$

This implies that

$$\lambda_{ij} (\lambda_{i_0 j})^{-1} = \alpha_i (\alpha_{i_0})^{-1} \quad \text{for all } i \in D \text{ and } j \in C(M) \backslash D \tag{22}$$

Defining

$$c(i, i_0) := \alpha_i (\alpha_{i_0})^{-1} \tag{23}$$

for all $i \in D$, (ii) results. □

Remark on the proof: Note that we have even proved that for *all* $i_0 \in D$ and all $i \in D$ there exists a constant $c(i, i_0)$ etc. This seems a stronger assertion than (ii). However, under the conditions of the lemma both assertions are equivalent.

It is clear that for a $2 \times n$ bimatrix game (A, B) with $E(A, B)$ is convex (and so the unique maximal Nash subset), $\text{Ext}(E(A, B)) = T_1 \times T_2 \subset [0, 1] \times \Delta_n$ satisfies (M.1). For the inner vertical case the conditions (IV.1), (IV.2) and (M.1) are also sufficient as is seen in

Theorem 5.2: Let $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ be a feasible set where $T_1 = \{p\}$ with $p \in (0, 1)$. If (IV.1), (IV.2) and (M.1) are satisfied, then there exists a $2 \times n$ bimatrix game (A, B) such that $E(A, B)$ is convex and $\text{Ext}(E(A, B)) = T_1 \times T_2$.

Proof: We explicitly construct a $2 \times n$ bimatrix game (A, B) such that $E(A, B)$ is convex and $\text{Ext}(E(A, B)) = T_1 \times T_2$. Because of (IV.1) we know that $M \neq \emptyset$. Furthermore, (IV.2) implies that $C(M) \cap C(P) = \emptyset$. Let us only consider the case that $P \neq \emptyset$ (The case $P = \emptyset$ can be treated in a similar way.) Then, withouth loss of generality, the strategies can be renumbered in such a way that $C(M) = \{1, \ldots, s\}$ and $C(P) = \{t, \ldots, n\}$ for some $s, t \in \mathbb{N}$ with $1 < s < t \leqslant n$. Moreover, having $D \subset C(M)$ and $i_0 \in D$ as in (M.1) we may also assume that it is done in such a way that $D = \{1, \ldots, r\}$ for some $r \in \mathbb{N}$ with $1 \leqslant r < s$, and that $i_0 = 1$. Let, for $i \in D = \{1, \ldots, r\}$ and $j \in C(M) \backslash D = \{r+1, \ldots, s\}$, $c(i, 1)$ be defined as in (18) and λ_{ij} be the pure strategy ratio of $m(\{i, j\})$. Now consider the $2 \times n$ bimatrix game (A, B) with $A = [a_{ij}]_{i=1}^{2}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{2}{}_{j=1}^{n}$ defined by

$$\begin{cases} a_{1i} = c(i, 1) & \text{if } i \in \{1, \ldots, r\} \\ a_{2j} = \lambda_{1j}^{-1} & \text{if } j \in \{r+1, \ldots, s\} \\ a_{kl} = 0 & \text{else} \end{cases}$$

$$\begin{cases} b_{21} = (1-p)^{-1}, b_{1s} = p^{-1} & \\ b_{kl} = 1 & \text{if } l \in \{2, \ldots, s-1\} \cup \{t, \ldots, n\} \\ b_{kl} = 0 & \text{else} \end{cases}$$

i.e.

$$\begin{array}{cc} & \begin{array}{cccccccccccccc} 1 & 2 & \ldots & r & r+1 & \ldots & s-1 & s & s+1 & \ldots & t-1 & t & \ldots & n \end{array} \\ A = & \begin{bmatrix} c(1,1) & c(2,1) & \ldots & c(r,1) & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \lambda_{1r+1}^{-1} & \cdot & \lambda_{1s-1}^{-1} & \lambda_{1s}^{-1} & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \end{array}$$

and

$$\begin{array}{cc} & \begin{array}{cccccccccccccc} 1 & 2 & \ldots & r & r+1 & \ldots & s-1 & s & s+1 & \ldots & t-1 & t & \ldots & n \end{array} \\ B = & \begin{bmatrix} 0 & 1 & \ldots & 1 & 1 & \ldots & 1 & p^{-1} & 0 & \ldots & 0 & 1 & \ldots & 1 \\ (1-p)^{-1} & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 1 & \ldots & 1 \end{bmatrix} \end{array}$$

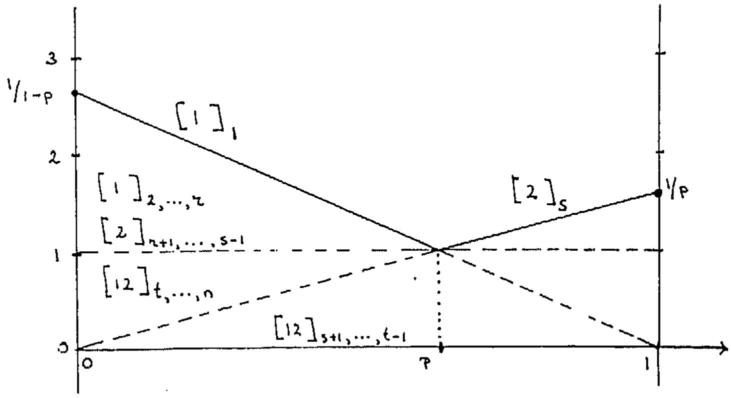

Fig. 3

Then, using figure 3, it is clear that there is no equilibrium point $(p', q) \in [0, 1] \times \Delta_n$ with $p' \neq p$ since, if $p' < p$, then $PB_2(p') = \{1\}$ while $PB_1(e_1) = \{1\}$ and, if $p' > p$, then $PB_2(p') = \{s\}$ while $PB_1(e_s) = \{2\}$. So it suffices to show that $T_2 = \text{Ext}(S(p)) = PS(p) \cup TS(p)$. Since

$$PS(p) = \{e_j \mid j \in PB_2(p, [12])\} = \{e_j \mid j \in \{t, \dots, n\}\} = P$$

we only have to show that $TS(p) = M$.

Note that

$$TS(p) = \{q(i, j) \in \Delta_n \mid i \in PB_2(p, [1]), j \in PB_2(p, [2]\}$$

$$= \{q(i, j) \in \Delta_n \mid i \in \{1, \dots, r\}, j \in \{r + 1, \dots, s\}\}$$

$$= \{q(i, j) \in \Delta_n \mid i \in D, j \in C(M) \backslash D\}$$

and, using (16),

$$M = \{m(\{i, j\}) \in \Delta_n \mid i \in D, j \in C(M) \backslash D\}$$

So we are finished if we can show that $q(i, j) = m(\{i, j\})$ for all $i \in D$ and $j \in C(M) \backslash D$, or since $C(q(i, j)) = C(m(\{i, j\})) = \{i, j\}$, that $q_i(i, j) = m_i(\{i, j\})$. This is the case because for $i \in D$ and $j \in C(M) \backslash D$,

$$
\begin{aligned}
q_i(i,j) &\overset{1}{=} (a_{2j}-a_{1j})((a_{2j}-a_{1j})+(a_{1i}-a_{2i}))^{-1} \\
&\overset{2}{=} \lambda_{1j}^{-1}(\lambda_{1j}^{-1}+c(i,1))^{-1} \\
&\overset{3}{=} \lambda_{1j}^{-1}(\lambda_{1j}^{-1}+\lambda_{ij}\cdot\lambda_{1j}^{-1})^{-1} \\
&\overset{4}{=} ((m_i(\{i,j\}))^{-1})^{-1} = m_i(\{i,j\})
\end{aligned}
$$

where $\overset{1}{=}$ follows from the fact that $e_1Aq(i,j) = e_2Aq(i,j)$ and $C(q(i,j)) = \{i,j\}$, $\overset{2}{=}$ from the definition of A, $\overset{3}{=}$ from (M.1) (ii) and $\overset{4}{=}$ from the definition of λ_{ij}. □

The following example illustrates the theorem above.

Example 2: Let

$$
[0,1]\times\Delta_7 \supset T_1\times T_2 = \{\tfrac{1}{2}\}\times\{\tfrac{2}{5}e_1+\tfrac{3}{5}e_3, \tfrac{1}{2}e_1+\tfrac{1}{2}e_4, \tfrac{1}{4}e_1+\tfrac{3}{4}e_5, \tfrac{9}{10}e_1+\tfrac{1}{10}e_6,
$$
$$
\tfrac{4}{7}e_2+\tfrac{3}{7}e_3, \tfrac{2}{3}e_2+\tfrac{1}{3}e_4, \tfrac{2}{5}e_2+\tfrac{3}{5}e_5, \tfrac{18}{19}e_2+\tfrac{1}{19}e_6, e_7\}
$$

We want to construct a 2×7 bimatrix game (A, B) with $E(A, B) = \mathrm{Conv}(T_1\times T_2)$. It is easy to check that $T_1\times T_2$ is feasible. Furthermore $T_1 = \{\tfrac{1}{2}\}$, $P = \{e_7\}$ and $M = T_2\backslash\{e_7\}$. Clearly $M\neq\emptyset$ and $C(M)\cap C(P) = \emptyset$. So (IV.1) and (IV.2) are satisfied. Checking (M.1) we see that $D(1) = D(2) = \{3,4,5,6\}$ and $D(3) = D(4) = D(5) = D(6) = \{1,2\}$. So, without loss of generality, we can choose $D = \{1,2\}$ satisfying (M.1)(i). A simple calculation shows that

$$
\lambda_{13} = \tfrac{3}{2},\quad \lambda_{14} = 1,\quad \lambda_{15} = 3,\quad \lambda_{16} = \tfrac{1}{9},
$$
$$
\lambda_{23} = \tfrac{3}{4},\quad \lambda_{24} = \tfrac{1}{2},\quad \lambda_{25} = \tfrac{3}{2},\quad \lambda_{26} = \tfrac{1}{18}
$$

So $\lambda_{2j}(\lambda_{1j})^{-1} = \tfrac{1}{2}$ for all $j\in\{3,4,5,6\}$. So with $i_0 = 1$, $c(1,1) = 1$ and $c(2,1) = \tfrac{1}{2}$, also (M.1)(ii) is satisfied. According to the proof of theorem 5.2 (we do not have to renumber) with $r = 2$, $s = 6$ and $t = 7$ we can take

$$
A = \begin{pmatrix} 1 & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tfrac{2}{3} & 1 & \tfrac{1}{3} & 9 & 0 \end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}
$$

Note that if e.g. the mixed extremal $\frac{2}{5}e_1 + \frac{3}{5}e_3$ is replaced by $\frac{1}{5}e_1 + \frac{4}{5}e_3$ we can not construct a corresponding 2×7 bimatrix game anymore. The conditions (IV.1) and (IV.2) are still satisfied but (M.1) is not, since D still has to be chosen equal to $\{1, 2\}$ or $\{3, 4, 5, 6\}$ but λ_{13} changes into 4 (the other pure strategy ratios stay the same) and consequently $\lambda_{23}(\lambda_{13})^{-1} = \frac{3}{16} \neq \frac{1}{2} = \lambda_{24}(\lambda_{14})^{-1}$.

5.4 The Pure Vertical Case

The following examples indicate that there indeed is a difference between pure verticals and inner verticals.

Example 3: Let the 2×2 bimatrix game (A, B) be determined by

$$(A, B) = \begin{bmatrix} (1,1) & (1,0) \\ (0,1) & (0,0) \end{bmatrix}$$

Then $E(A, B) = \mathrm{Ext}(E(A, B)) = \{e_1\} \times \{e_1\}$. So $T_1 \times T_2 := \{1\} \times \{e_1\}$ is a feasible set corresponding to a pure vertical maximal Nash subset. Note that $M = \emptyset$. So in the pure vertical case *mixed extremal existence* need not hold.

Example 4: Let the 2×2 bimatrix game (A, B) be determined by

$$(A, B) = \begin{bmatrix} (1,1) & (0,1) \\ (0,0) & (1,-1) \end{bmatrix}$$

Then $E(A, B) = \{e_1\} \times \mathrm{Conv}(\{e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2\})$. So $T_1 \times T_2 := \{1\} \times \{e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2\}$ is a feasible set corresponding to a pure vertical maximal Nash subset. Note that $M = \{\frac{1}{2}e_1 + \frac{1}{2}e_2\}$ and $P = \{e_1\}$, so $C(M) \cap C(P) = \{1\}$. Hence *no overlap* need not hold in the pure vertical case.

Both examples justify a seperate treatment of the pure vertical case. Of course, for a feasible set $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ which corresponds to a pure vertical maximal Nash subset, the mixed extremals in T_2 must satisfy the condition (M.1) (cf. lemma 5.3).

However, for the pure vertical case this condition can be strenghtened in the sense that the partition of $C(M)$ is completely determined by the overlap between M and P as is seen in

Lemma 5.4: Let (A, B) be a $2 \times n$ bimatrix game such that $C \subset E(A, B)$ is a pure vertical maximal Nash subset with $\text{Ext}(C) = T_1 \times T_2 \subset [0, 1] \times \Delta_n$ where $T_2 = P \cup M$. Then

(M.2) (i) $D(k) \in \{C(M) \cap C(P), C(M) \backslash C(P)\}$ for all $k \in C(M)$
(*specified partition property*)

(ii) there is an $i_0 \in C(M) \cap C(P)$ such that for all $i \in C(M) \cap C(P)$ there exists a constant $c(i, i_0) \in \mathbb{R}$ with

$\lambda_{ij}(\lambda_{i_0 j})^{-1} = c(i, i_0)$ for all $j \in C(M) \backslash C(P)$ (*specified proportionality*)

Proof: Let $T_1 = \{0\}$. The case $T_1 = \{1\}$ can be treated in a similar way. Because of lemma 5.3 we only have to show that

$$D(k) \in \{C(M) \cap C(P), C(M) \backslash C(P)\}$$

for all $k \in C(M)$. So let $k \in C(M)$. As we have seen before,

$$M = TS(0) = \{q(i, j) \in \Delta_n | i \in PB_2(0, [1]), j \in PB_2(0, [2])\}$$

$$P = PS(0) = \{e_t \in \Delta_n | t \in PB_2(0, [12]) \cup PB_2(0, [2])\}$$

Consequently, either $k \in PB_2(0, [1])$ or $k \in PB_2(0, [2])$. If $k \in PB_2(0, [1])$, then $D(k) = PB_2(0, [2]) = C(M) \cap C(P)$. Else, if $k \in PB_2(0, [2])$, then $D(k) = PB_2(0, [1]) = C(M) \backslash C(P)$. □

Sufficiency of the condition (M.2) is stated in

Theorem 5.3: Let $T_1 \times T_2 \subset [0, 1] \times \Delta_n$ be a feasible set with $T_1 = \{p\}$ and $p \in \{0, 1\}$. If (M.2) is satisfied, then there exists a $2 \times n$ bimatrix game (A, B) such that $E(A, B)$ is convex and $\text{Ext}(E(A, B)) = T_1 \times T_2$.

Proof: The construction of a corresponding $2 \times n$ bimatrix game (A, B) proceeds in a similar way as in the proof of theorem 5.2. Let $p = 1$ (the case $p = 0$ can be treated analogously.

If $M = \emptyset$ we may assume (else renumber) that $C(P) = \{1, \ldots, r\}$ with $1 \leqslant r \leqslant n$. Then, defining

$$\begin{array}{c} \begin{array}{cccccc} 1 & \ldots & r & r+1 & \ldots & n \end{array} \\ A = \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \end{array}, \quad \begin{array}{c} \begin{array}{cccccc} 1 & \ldots & r & r+1 & \ldots & n \end{array} \\ B = \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix} \end{array}$$

it is clear that $E(A, B) = \{1\} \times \mathrm{Conv}(\{e_1, \ldots, e_r\}) = \{1\} \times \mathrm{Conv}(P)$.

If $M \neq \emptyset$ we have that $C(M) \cap C(P) \neq \emptyset$ by (M.2)(i). So we may assume (else renumber) that $C(M) = \{1, 2, \ldots, s\}$ and $C(P) = \{r, r+1, \ldots t\}$ for some $r, s, t \in \mathbb{N}$ with $1 \leqslant r \leqslant s \leqslant t \leqslant n$. Moreover, having $i_0 \in C(M) \cap C(P)$ as in (M.2)(ii), we may assume that $i_0 = r$ and that the constants $c(k, r) > 0$, for $k \in C(M) \cap C(P) = \{r, r+1, \ldots, s\}$ and the pure strategy ratios λ_{rl}, for $l \in C(M) \backslash C(P) = \{1, 2, \ldots, r-1\}$, are defined accordingly. Consider the $2 \times n$ bimatrix game (A, B) defined by

$$\begin{array}{c} \begin{array}{cccccccccccccc} 1 & 2 & \ldots & r-1 & r & r+1 & \ldots & s & s+1 & \ldots & t & t+1 & \ldots & n \end{array} \\ A = \begin{bmatrix} 0 & 0 & \ldots & 0 & c(r,r) & c(r+1,r) & \ldots & c(s,r) & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \lambda_{r1}^{-1} & \lambda_{r2}^{-1} & \ldots & \lambda_{rr-1}^{-1} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \end{array}$$

and

$$\begin{array}{c} \begin{array}{cccccccccccccc} 1 & 2 & \ldots & r-1 & r & r+1 & \ldots & s & s+1 & \ldots & t & t+1 & \ldots & n \end{array} \\ B = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \end{array}$$

Then

$$E(A, B) \stackrel{1}{=} \{1\} \times \mathrm{Conv}(PS(1) \cup TS(1))$$

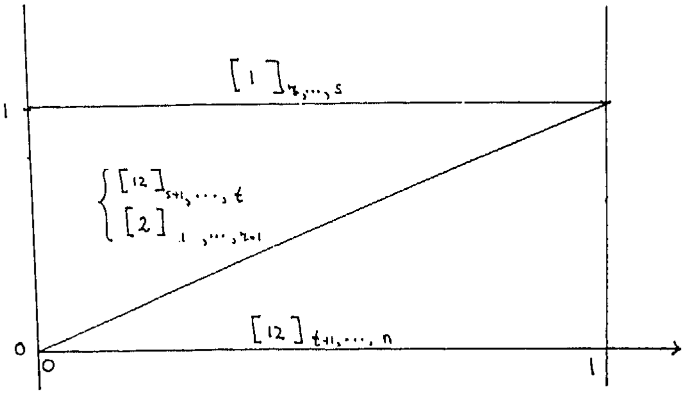

Fig. 4

and

$$\mathrm{Ext}(E(A, B)) \overset{2}{=} \{1\} \times (PS(1) \cup TS(1))$$

where

$$PS(1) \overset{1}{=} \{e_k \in \Delta_n \,|\, k \in \{r, r+1, \ldots, t\} = P$$

$$TS(1) \overset{1}{=} \{q(i, j) \in \Delta_n \,|\, i \in \{r, r+1, \ldots, s\}, j \in \{1, 2, \ldots, r-1\}\}$$

$$\overset{3}{=} \{m(\{i, j\}) \in \Delta_n \,|\, i \in \{r, r+1, \ldots, s\}, j \in \{1, 2, \ldots, r-1\}\}$$

$$\overset{4}{=} M$$

where $\overset{1}{=}$ can be verified using the GC-approach (cf. figure 4), $\overset{2}{=}$ is a consequence of theorem 3.1, $\overset{3}{=}$ follows from the fact that $m(\{i, j\}) = q(i, j)$ for all $i \in C(M) \cap C(P)$ and $j \in C(M) \backslash C(P)$ (cf. the proof of theorem 5.2) and, finally, $\overset{4}{=}$ follows from (16). □

Theorem 5.3 is illustrated in

Example 5: Let $T_1 \times T_2 \subset [0, 1] \times \Delta_7$ be defined by

$$T_1 \times T_2 = \{1\} \times \{\tfrac{2}{5}e_1 + \tfrac{3}{5}e_3, \tfrac{1}{2}e_1 + \tfrac{1}{2}e_4, \tfrac{1}{4}e_1 + \tfrac{3}{4}e_5, \tfrac{9}{10}e_1 + \tfrac{1}{10}e_6,$$
$$\tfrac{4}{7}e_2 + \tfrac{3}{7}e_3, \tfrac{2}{3}e_2 + \tfrac{1}{3}e_4, \tfrac{2}{5}e_2 + \tfrac{3}{5}e_5, \tfrac{18}{19}e_2 + \tfrac{1}{19}e_6, e_1, e_2, e_7\}$$

We want to construct a 2×7 bimatrix game such that $E(A, B) = \text{Conv}(T_1 \times T_2)$. Note that this example is more or less an extension of example 2. It is easy to check that $T_1 \times T_2$ is feasible. Furthermore, $P = \{e_1, e_2, e_7\}$, $M = T_2 \backslash \{e_1, e_2, e_7\}$ and $C(M) \cap C(P) = \{1, 2\}$. It is easy to check (M.2)(i) and, defining $i_0 \in C(M) \cap C(P)$ by $i_0 = 1$, we find that

$$\lambda_{23}(\lambda_{13})^{-1} = \lambda_{24}(\lambda_{14})^{-1} = \lambda_{25}(\lambda_{15})^{-1} = \lambda_{26}(\lambda_{16})^{-1} = \tfrac{1}{2}$$

So $c(2, 1) = \frac{1}{2}$ and, trivially, $c(1, 1) = 1$. This implies (M.2)(ii).

Then, according to the proof of theorem 5.2, having

$$\lambda_{13} = 3/2, \quad \lambda_{14} = 1, \quad \lambda_{15} = 3 \quad \text{and} \quad \lambda_{16} = 1/9$$

we can take (renumber and translate it back)

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 1 & 1/3 & 9 & 0 \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6 Remarks

For the construction of $2 \times n$ bimatrix games with a general type of equilibrium set one is especially interested in conditions on maximal Nash subsets which do not depend on the convexity of the equilibrium set. So let $C \subset E(A, B)$ be a maximal Nash subset with $\text{Ext}(C) = T_1 \times T_2 \subset [0, 1] \times \Delta_n$. What can be said about $T_1 \times T_2$? Of course $T_1 \times T_2$ is feasible and further we have

(R.1) Condition (M.1) is satisfied. Moreover, if C is pure vertical, then (M.2) is satisfied. Note that these conditions are trivially satisfied if $M = \emptyset$.

(R.2) If C is horizontal and *isolated,* then *pure extremality* holds.

(R.3) If C is horizontal or inner vertical the *no overlap* holds.

Finally, we note that in the general $2 \times n$ case also other conditions concerning the "overlap" between maximal Nash subsets will arise.

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