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PARETO EQUILIBRIA IN MULTIOBJECTIVE GAMES

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Abstract. This paper describes a model for two-person games in which the players can have several possibly conflicting objectives. To solve these games Pareto equilibria are introduced and their existence is shown. More specifically, it is found that Pareto equilibria correspond to Nash-equilibria of uniobjective games derived from a strictly positive weighting of the various objective functions, and conversely. Continuity properties are studied too and a geometric representation method for small multiobjective games is illustrated.

1. INTRODUCTION

In this paper we focus on multiobjective games. In game theory this area of research is as much as unexplored though the very rich field of multiobjective programming in decision theory seems a promising start for an extension towards competitive situations as modelled in game theory. For a study on multiobjective programming we refer to the conference book of *Cochrane* and *Zeleny* (1973) which promoted this new branch in mathematical programming and to the books of *Cohon* (1978) and *Zeleny* (1974).

Why do we aim for a multiobjective approach? Following *Cohon* (1978) and *Zeleny* (1974) we think the most important reason is that multiobjective models better apply to real-world situations. For decision-making processes *Cohon* (1978) gives the following real-life examples of decision problems in the public sector which must be concerned with society's various objectives:facility location problems, water pollution problems, water resource problems, etc. Some of these decision problems can directly be extended to game-theoretical problems. For example, consider the facility location problem. Suppose that we have a firm which has to decide upon the location of a new settlement. This decision will involve several objectives: costs of purchase of the building-ground, difficulties in getting a building-license, building-costs, costs of labour, location (reachability, distance to the consumers-market, distance to the supply industries), public interest (labour, pollution ,...), reputation etc. However, we now also suppose that there is a competing second firm with similar intentions. For each of the firms some of the above objectives will be correlated with the location of the new settlement of the other firm. Hence we deal with a game in which it will be difficult for the firms to find a unifying 'trade-off' function between the various objectives which could serve as payoff-function in this game.

Another reason to consider multiobjective models is the following. We do not have to make a 'trade-off' between the objectives in advance.
We take into account all the information we have and explore the *full* range of possibilities. Only at that moment, having an overall-picture, we have to make a 'trade-off' and find a conflict-resolution.

The aim of this paper is to present a first appoach towards a thorough mathematical study on multiobjective games. We concentrate on a model for two-person multiobjective games using vector payoffs. As a possible solution concept we consider Pareto equilibria which are related to undominated elements of special polytopes. The necessary definitions are given in section 2.
That Pareto equilibria always exist is shown in section 3. More specifically, Pareto equilibria are linked to Nash equilibria of a corresponding so called 'trade-off game' in which each player has attached strictly positive weights to all his objectives and consequently has a unifying payoff-function which takes all objectives into account. The paper concludes with a discussion of continuity properties in section 4 and some remarks in section 5. Throughout the paper a geometric solution method for 'small' multiobjective games is illustrated.

A final word about notations. $\mathbf{N} := \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. For $x \in \mathbf{R}^t$ and $y \in \mathbf{R}^t$, $t \in \mathbf{N}$, we say that $x \geq y$ $(x > y)$ if $x_k \geq y_k$ $(x_k > y_k)$ for all $k \in \{1, \ldots, t\}$. For $S \subset \mathbf{R}^t$, $t \in \mathbf{N}$, Conv(S) denotes the convex hull of S.

2. THE MODEL, PARETO EQUILIBRIA

We consider two-person multiobjective games in which player 1 and player 2 can choose between m and n possible actions, respectively, while taking into account r and s objectives, respectively. Formally we have the following
Definition. Let $m, n, r, s \in \mathbf{N}$. Let $A = (A^1, \ldots, A^r)$ and $B = (B^1, \ldots, B^s)$ be two vectors of real $m \times n$ matrices. The $r \times s$ *multiobjective* $m \times n$ *bimatrix game* (A, B) is now defined as the game $\langle \Delta_m, \Delta_n, K, L \rangle$ with vector payoffs, where

(i) Δ_m and Δ_n are the strategy spaces of player 1 and player 2, respectively, with

$$\Delta_t := \{(x_1, \ldots, x_t) \in \mathbf{R}^t \mid \sum_{k=1}^{t} x_k = 1,\ x_k \geq 0 \text{ for all } k \in \{1, \ldots, t\}\} \quad (t \in \mathbf{N}).$$

(ii) K and L are the vector payoff-functions of player 1 and player 2, respectively, i.e.

$$K(p,q) := (pA^1q, \ldots, pA^rq), \;\; L(p,q) = (pB^1q, \ldots,, pB^sq) \quad (p \in \Delta_m, \; q \in \Delta_n)$$

To shorten descriptions we often set

$$pAq = (pA^1q, \ldots, pA^rq), \quad pBq = (pB^1q, \ldots, pB^sq) \qquad (p \in \Delta_m, \;\; q \in \Delta_n)$$

The set of all $r \times s$ multiobjective $m \times n$ bimatrix games is denoted by $MG(r \times s, m \times n)$.

Note that we consider *mixed* strategies which correspond to probability vectors over the possible actions. *Pure* strategies $e_k \in \Delta_t$ correspond to the choice of action $k \in \{1, \ldots, t\}$ with probability one. The set of *completely mixed* strategies is denoted by $\mathring{\Delta}_t$, $\mathring{\Delta}_t := \{(x_1, \ldots, x_t) \in \mathbf{R}^t \mid \sum_{k=1}^{t} x_k = 1, \, x_k > 0 \text{ for all } k \in \{1, \ldots, t\}\}$. Further we have in the definition identified the set of r (s) objectives of player 1 (2) with the set $\{1, \ldots, r\}$ ($\{1, \ldots, s\}$) and the payoff-vector pAq (pBq) measures the payoff of the outcome $(p,q) \in \Delta_m \times \Delta_n$ with respect to all objectives of player 1 (2) separately.

It is clear that for $r = 1$ and $s = 1$ we are back in the 'old' setting of $m \times n$ bimatrix games. However, some other cases can be reduced to this setting too. For example, if $A^1 = \ldots = A^r$ and $B^1 = \ldots = B^s$ (for both players all objectives are equivalent) or if $e_1A^k = \ldots = e_mA^k$ for all $k \in \{2, \ldots, r\}$ and $B^ke_1 = \ldots B^ke_n$ for all $k \in \{2, \ldots, n\}$ (The strategy of a player only effects his payoff with respect to one of his objectives.)

For introducing the concept of Pareto equilibria we need the following notion of undominatedness.

Definition. Let $P \subset \mathbf{R}^t$, $t \in \mathbf{N}$. Then $z \in P$ is called *undominated* in P if

$$P \cap \{x \in \mathbf{R}^t \mid x \geq z\} = \{z\}$$

The idea is now the following. Fixing a certain strategy of player 2 we can look at the polytope (i.e. the convex hull of finitely many points) of possible vector payoffs to player 1 and determine its undominated elements together with the corresponding strategies for player 1. These strategies will be called *Pareto-best replies* for player 1. For player 2 we can do the same. Pairs of strategies which are Pareto-best replies to one another are called *Pareto equilibria.*

Definition. Let $(A, B) \in MG(r \times s, m \times n)$. Let $p \in \Delta_m$, $q \in \Delta_n$.
The polytopes $P_A(q) \subset \mathbf{R}^r$ and $P_B(p) \subset \mathbf{R}^s$ are defined by

$$P_A(q) := \text{Conv}\{e_1Aq, \ldots, e_mAq\}, \; P_B(p) := \text{Conv}\{pBe_1, \ldots, pBe_n\}$$

The sets $PAR_1(q)$ and $PAR_2(p)$ of *Pareto-best replies* of player 1 and player 2 to q and p, respectively, are defined by

$$PAR_1(q) := \{p \in \Delta_m \mid pAq \text{ is undominated in } P_A(q)\}, \text{ and}$$
$$PAR_2(p) := \{q \in \Delta_n \mid pBq \text{ is undominated in } P_B(p)\}$$

The pair (p,q) is called a *Pareto equilibrium* if

$$p \in PAR_1(q),\ q \in PAR_2(p)$$

The set of all Pareto equilibria in (A,B) is denoted by $PAR(A,B)$.

For (ordinary) uniobjective $m \times n$ bimatrix games (A,B) (i.e. $r=1,\ s=1$) it is easily verified that the set $PAR(A,B)$ of Pareto equilibria is equal to the set $E(A,B)$ of Nash equilibria, where $(\hat{p},\hat{q}) \in \Delta_m \times \Delta_n$ is called a *Nash equilibrium* for (A,B) if $\hat{p}A\hat{q} \geq pA\hat{q}$ for all $p \in \Delta_m$ and $\hat{p}B\hat{q} \geq \hat{p}Bq$ for all $q \in \Delta_n$.

The above concepts and definitions are illustrated in

Example 1. Consider the 2×2 multiobjective 2×2 bimatrix game (A,B) as given by

$$A = (A^1, A^2) = \begin{bmatrix} (0,3) & (8,1) \\ (6,0) & (2,5) \end{bmatrix} \text{ and } B = (B^1, B^2) = \begin{bmatrix} (1,-1) & (2,1) \\ (6,3) & (5,2) \end{bmatrix}$$

In the following we will identify $(x, 1-x) \in \Delta_2$ with $x \in [0,1]$ and use $(x,1-x) \in \Delta_2$ and $x \in [0,1]$ interchangeable.

For $p \in \Delta_2$ and $q \in \Delta_2$ we find

$$\begin{cases} e_1 Aq = (8-8q, 1+2q) \\ e_2 Aq = (2+4q, 5-5q) \end{cases} \quad \begin{cases} pBe_1 = (6-5p, 3-4p) \\ pBe_2 = (5-3p, 2-p) \end{cases}$$

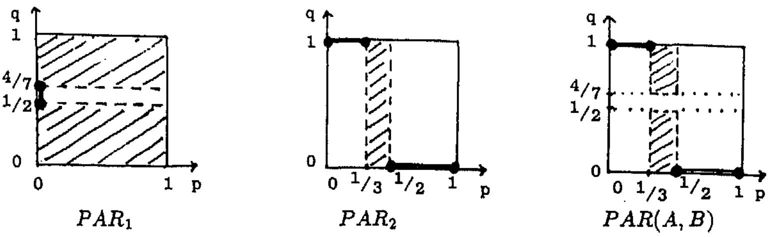
Hence,

$$PAR_1(q) = \begin{cases} \Delta_2 & \text{if } q > \frac{4}{7} \text{ or } q < \frac{1}{2} \\ \{e_2\} & \text{if } \frac{1}{2} \leq q \leq \frac{4}{7} \end{cases} \text{ and } PAR_2(p) = \begin{cases} \{e_1\} & \text{if } p \leq \frac{1}{3} \\ \Delta_2 & \text{if } \frac{1}{3} < p < \frac{1}{2} \\ \{e_2\} & \text{if } p \geq \frac{1}{2} \end{cases}$$

As can be see geometrically in figure 1 we then have that

$$PAR(A,B) = [0,\frac{1}{3}] \times \{e_1\} \cup (\frac{1}{3},\frac{1}{2}) \times [0,\frac{1}{2}) \cup (\frac{1}{3},\frac{1}{2}) \times (\frac{4}{7},1] \cup [\frac{1}{2},1] \times \{e_2\}$$

FIGURE 1.



3. AN EXISTENCE THEOREM FOR PARETO EQUILIBRIA

In section 2 we introduced Pareto equilibria. We will now show that for each multiobjective game (A, B) the set of Pareto equilibria is non-empty. This will be done by considering special bimatrix games which arise from (A, B) by a strictly positive weighting of all objectives for each player. Since we will show that Nash equilibria of those games correspond to Pareto-equilibria of the original game the existence of Pareto equilibria follows from the existence of Nash equilibria for bimatrix games (*Nash* (1951)).

Definition. Let $(A, B) \in MG(r \times s,\ m \times n)$.
For $\lambda \in \mathring{\Delta}_r$ and $\mu \in \mathring{\Delta}_s$ the (λ, μ) trade-off bimatrix game $(A(\lambda), B(\mu))$ corresponding to (A, B) is defined by

$$A(\lambda) = \sum_{k=1}^{r} \lambda_k A^k \text{ and } B(\mu) = \sum_{k=1}^{s} \mu_k B^k$$

Theorem 1. (EXISTENCE THEOREM FOR PARETO EQUILIBRIA)
Let $(A, B) \in MG(r \times s,\ m \times n)$. Let $\lambda \in \mathring{\Delta}_r$, $\mu \in \mathring{\Delta}_s$.
Then $\emptyset \neq E(A(\lambda), B(\mu)) \subset PAR(A, B)$.

Proof. Let $(p, q) \in E(A(\lambda), B(\mu))$. We will prove that $p \in PAR_1(q)$, i.e.

$$\{x \in \mathbf{R}^r \mid x \geq pAq\} \cap P_A(q) = \{pAq\}$$

Suppose $z \in \mathbf{R}^r$ with $z \geq pAq$ and $z \in P_A(q)$. Then, since $z \in P_A(q)$, we can find a strategy $\overline{p} \in \Delta_m$ such that $z = \overline{p}Aq$. Consequently, $\overline{p}Aq \geq pAq$.

Further, using the equilibrium condition, we find that

$$\lambda \cdot \overline{p}Aq = \overline{p}(\sum_{k=1}^{r} \lambda_k A^k)q = \overline{p}(A(\lambda))q \leq pA(\lambda)q = p(\sum_{k=1}^{r} \lambda_k A^k)q = \lambda \cdot pAq$$

Combining these two facts with the fact that $\lambda > 0$ we find that $z = \overline{p}Aq = pAq$. Similarly one can show that $q \in PAR_2(p)$. ◇

The converse of theorem 1 also holds: each Pareto equilibrium can be found as a Nash equilibrium of some corresponding trade-off bimatrix game.
This result is stated in theorem 2 below. For the proof we need the following
Lemma 1. Let $P \subset \mathbf{R}^t$ be a polytope ($t \in \mathbf{N}$). Let $z \in P$ be undominated in P. Then there exists a vector $u \in \mathbf{R}^t$ such that

$$u > 0 \text{ and } u \cdot x \leq u \cdot z \text{ for all } x \in P$$

A remark to the proof of this lemma: under the above conditions it is easily verified that there exists a vector $u \in \mathbf{R}^t$ such that $u \geq 0$ and $u \cdot x \leq u \cdot z$ for all $x \in P$. For the rather technical proof of the fact that $u \in \mathbf{R}^t$ can also be chosen such that $u_k > 0$ for all $k \in \{1, \dots, t\}$ we refer to *Tijs* (1985) or to the forthcoming paper of *Borm, Jansen, Potters* and *Tijs* (1988). Using this lemma we can prove
Theorem 2. Let $(A, B) \in MG(r \times s, m \times n)$ with $(\hat{p}, \hat{q}) \in PAR(A, B)$. Then there exist $\lambda \in \mathring{\Delta}_r$ and $\mu \in \mathring{\Delta}_s$ such that $(\hat{p}, \hat{q}) \in E(A(\lambda), B(\mu))$.

Proof. By definition $\hat{p}A\hat{q}$ is undominated in $P_A(\hat{q})$ and $\hat{p}B\hat{q}$ is undominated in $P_B(\hat{p})$. Consequently, using lemma 1, there exist vectors $u \in \mathbf{R}^r$ and $v \in \mathbf{R}^s$ such that $u > 0$, $x > 0$ and $x \cdot u \leq (\hat{p}A\hat{q}) \cdot u$ for all $x \in P_A(\hat{q})$, $y \cdot v \leq (\hat{p}B\hat{q}) \cdot v$ for all $y \in P_B(\hat{p})$. Defining $\lambda = (\sum_{k=1}^{r} u_k)^{-1} u \in \mathring{\Delta}_r$ and $\mu = (\sum_{k=1}^{s} v_k)^{-1} \cdot v \in \mathring{\Delta}_s$, we find that

$$\hat{p}(A(\lambda))\hat{q} = \hat{p}(\sum_{k=1}^{r} \lambda_k A^k)\hat{q} = \lambda \cdot \hat{p}A\hat{q} \geq \lambda \cdot pA\hat{q} = p(A(\lambda))\hat{q} \text{ for all } p \in \Delta_m$$

and, similarly, that $\hat{p}(B(\mu))\hat{q} \geq \hat{p}(B(\mu))q$ for all $q \in \Delta_n$. But this exactly means that $(\hat{p}, \hat{q}) \in E(A(\lambda), B(\mu))$. ◇

Let us illustrate the correspondence of Pareto equilibria to Nash equilibria in trade-off bimatrix games by means of the game in example 1.
Example 1. (CONTINUED).
Identifying $(\lambda, 1-\lambda) \in \mathring{\Delta}_2$ and $(\mu, 1-\mu) \in \mathring{\Delta}_2$ with $\lambda \in (0,1)$ and $\mu \in (0,1)$, respectively, the general (λ, μ) trade-off 2×2 bimatrix game $(A(\lambda), B(\mu))$ has the following form

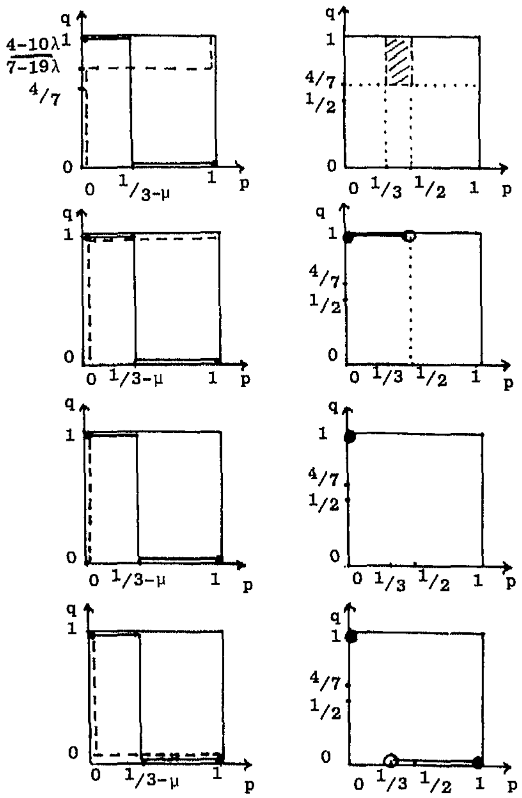
$$(A(\lambda), B(\mu)) = \begin{bmatrix} (3-3\lambda, -1+2\mu) & (1+7\lambda, 1+\mu) \\ (6\lambda, 3+3\mu) & (5-3\lambda, 2+3\mu) \end{bmatrix}$$

Calculating the set of Nash equilibria for such a game by means of the best-reply structure (which is also the Pareto-best reply structure for this uniobjective game) we have to distinguish five cases depending on $\lambda \in (0,1)$.

In figure 2 we have drawn for each of these five classes of 2×2 bimatrix games two typical best reply curves $B_1(= PAR_1)$ and $B_2(= PAR_2)$, the intersection of which corresponds to the set of Nash equilibria. Further we have represented the set of Pareto equilibria of the original game (A, B) which are generated by each class of parameters λ (and μ).

FIGURE 2

Best-reply representations for $(A(\lambda), B(\mu))$ in the figures to the left; B_1 : —, B_2 : – – –. Total contribution of the class of parameters to the Pareto equilibrium set of (A, B) in the figures to the right.

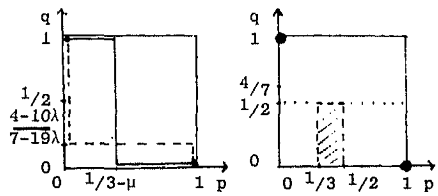


$\lambda \in (0, \frac{1}{3}), \mu \in (0,1)$

$\lambda = \frac{1}{3}, \mu \in (0,1)$

$\lambda \in (\frac{1}{3}, \frac{2}{5}), \mu \in (0,1)$

$\lambda = \frac{2}{5}, \mu \in (0,1)$



$\lambda \in (\frac{2}{5}, 1), \mu \in (0,1)$

4. CONTINUITY PROPERTIES

In this section we show that each of the multifunctions PAR_1, PAR_2 and PAR in general (i.e. $r \geq 2$, $s \geq 2$, $m \geq 2$, $n \geq 2$) are neither upper semicontinuous (USC) nor lower semicontinuous (LSC). For a detailed elaboration on both continuity concepts we refer to the book of *Hildenbrand-Kirman* (1976).

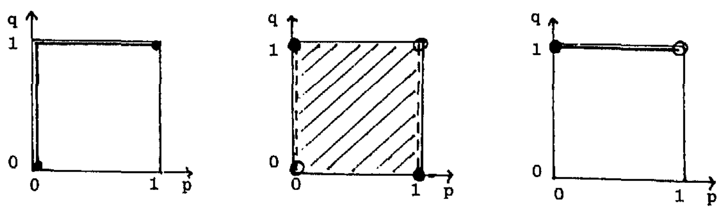
For our purposes it suffices to concentrate on $MG(2\times 2, 2\times 2)$ and therefore to consider PAR_1 (PAR_2) as multifunctions from Δ_2 into Δ_2, assigning to each strategy of player 2 (player 1) its non-empty set of Pareto-best replies for player 1 (player 2) and PAR as a multifunction from $MG(2\times 2, 2\times 2)$ into $\Delta_2 \times \Delta_2$, assigning to each 2×2 multiobjective 2×2 bimatrix game its non-empty set of Pareto equilibria. That these multifunctions need not be USC nor LSC is seen in

Example 2. Let $\varepsilon > 0$ and consider the multiobjective games $(A, -A)$ and $(A_\varepsilon, -A_\varepsilon)$ in $MG(2\times 2, 2\times 2)$ given by

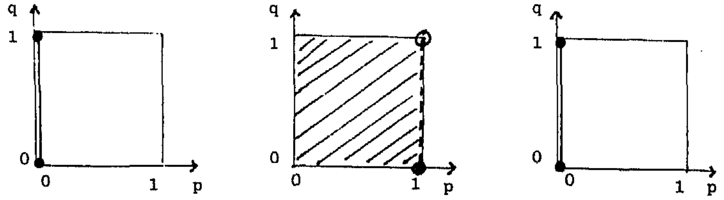
$$A = \begin{bmatrix} (2,3) & (2,2) \\ (2,3) & (3,3) \end{bmatrix} \text{ and } A_\varepsilon = \begin{bmatrix} (2,3) & (2,2) \\ (2,3+\varepsilon) & (3,3) \end{bmatrix}$$

In figure 3 we have, for both games, represented the Pareto-best reply structure and the set of Pareto equilibria.

FIGURE 3



PAR_1 PAR_2 $PAR(A, B)$



PAR_1 PAR_2 $PAR(A_\varepsilon, B_\varepsilon)$

Considering the open set $U := \{p \in [0,1] \mid p > \frac{1}{2}\}$ for which $U \cap PAR_1(1) \neq \emptyset$ we find (in $(A, -A)$) that PAR_1 is not LSC in $q = 1$. That PAR_2 (in $(A, -A)$) is not USC in $p = 0$ can be seen by means of the open set $V := \{q \in [0,1] \mid q > \frac{1}{2}\}$ for which $PAR_2(0) \subset V$. Further, by putting any resonable metric on the set $MG(2\times 2, 2\times 2)$ and considering the open sets $O \subset [0,1]\times[0,1]$, $W \subset [0,1]\times[0,1]$, respectively, where $O := [0,1] \times \{q \in [0,1] | q > \frac{1}{2}\}$ such that $PAR(A,B) \subset O$ and $W := \{p \in [0,1] \mid p > \frac{1}{2}\} \times [0,1]$ such that $PAR(A,B) \cap W \neq \emptyset$, we find that PAR is neither USC nor LSC in $(A, -A)$, respectively.

5. REMARKS

(i) The set of Pareto-best replies of player 1 to a fixed strategy q of player 2 in an $r \times s$ multiobjective $m \times n$ bimatrix game (A, B) can be fully characterized by means of the notion of *maximally undominated subsimplices* as introduced in the forthcoming paper of *Borm, Jansen, Potters* and *Tijs* (1988). According to their results the set $\mathcal{I} \subset 2^{\{1,\dots,m\}}$ can be introduced in such a way that $I \in \mathcal{I}$ if and only if the set $\text{Conv}\{e_i Aq; i \in I\} \subset P_A(q)$ consists of undominated elements only and, most importantly, such that

$$PAR_1(q) = \bigcup_{I \in \mathcal{I}} \text{Conv}\{e_i \mid i \in I\}$$

An analogous statement can be made with respect to player 2.

(ii) The set $MG(2 \times 2, 2 \times 2)$ of 2×2 multiobjective 2×2 bimatrix games can be classified in exactly the same way as is done in *Borm* (1987) for the set of 2×2 bimatrix games by using the geometric solution method which is illustrated in all examples above. Forty-seven classes result.

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