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A CLASSIFICATION OF 2x2 BIMATRIX GAMES

by

Peter BORM

ABSTRACT

A classification of 2x2 bimatrix games is presented based only on different types of best reply multifunctions and structures of the Nash equilibrium set. Fifteen classes are distinguished. It is found that within each class the structure of the set of quasi-strong equilibria is the same, just as the structure of the set of perfect equilibria. Using a simple probability model the classification enables us to compute the probability that a random bimatrix game belongs to a certain class. Only four classes have a positive probability of appearance. In this way important features for 2x2 bimatrix games are recognized.

1. *INTRODUCTION*

This paper deals with 2x2 bimatrix games : (mixed extensions of) non-cooperative two-person games (in normal form) with only two (pure) strategies for each player. For these games several ways of classification have been suggested. Rapoport, Guyer and Gordon [7] have studied various criteria for the classification of non-degenerate games, in which the four combinations of pure strategies lead to four different payoffs for each player. Basically, on the grounds of symmetry, they proposed 78 classes. One of the criteria that led to further classification was the stability of the natural outcome : the combination of those pure strategies which maximize both players' smallest payoff. Regarding degenerate games too, Beniest [1] proposed 12 classes. Here, payoff functions were algebraically related to the Nash equilibrium set, but no real game theoretic implications were given.

The classification we present here considers degenerate games too and is primarily based on all possible combinations of best reply multifunctions without discrimination between the players. Different structures of the Nash equilibrium set lead to further classification, resulting in 15 classes. This number is not too large to be inconvenient and at the same time it is large enough to assure sufficient uniformity within each class. By construction the behaviour of the players and the structure of the Nash equilibrium set are the same within each class. Moreover, the structures of the set of quasi-strong equilibria (Harsanyi [5]) and perfect equilibria (Selten [8]) are the same within each class too. In this context it may be noted that the classification of Beniest doesn't have this last property (see section 6). At the same time the classification enables us to understand what possible features a randomly chosen bimatrix game would have. This is the case because we can compute the probability of appearance for each class with the aid of a simple probability model. In this way we obtain insight in the importance of the several features which are possible for 2x2 bimatrix games.

From this papers' classification it can be verified that for all 2x2 bimatrix games the set of quasi-strong equilibria is non empty, which is still an open question for general mxn bimatrix games ($m \in \mathbb{N}$, $n \in \mathbb{N}$). Presumably the classification can be helpful in the approach of some other problems too, e.g. in studying games with incomplete information, as can be seen in Borm [2].

The organization of this paper is as follows. Section 2 discusses the relation between Nash equilibria and best reply structure in general mxn bimatrix games ($m \in \mathbb{N}, n \in \mathbb{N}$). In this context some properties of quasi-strong and perfect equilibria are recalled. In section 3 we consider all possible best reply structures for 2x2 bimatrix games. By introducing best reply combinations and payoff combinations a well-ordered classification as given in section 4, is possible. In section 5 probabilities of appearance are assigned to the classes. The paper is concluded with some remarks.

2. NASH EQUILIBRIA, REFINEMENTS AND BEST REPLY MULTIFUNCTIONS

By (X, Y, K, L) we denote a two-person game (in normal form) with strategy spaces X and Y for player 1 and player 2, respectively and payoff-functions $K : X \times Y \to \mathbb{R}$ and $L : X \times Y \to \mathbb{R}$, respectively. Let A and B be real mxn matrices ($m \in \mathbb{N}$, $n \in \mathbb{N}$)

$$A = \left[a_{ij}\right]_{i=1}^{m}{}_{j=1}^{n}, \qquad B = \left[b_{ij}\right]_{i=1}^{m}{}_{j=1}^{n} \quad (a_{ij} \in \mathbb{R}, b_{ij} \in \mathbb{R})$$

The mxn bimatrix game (A,B) is described by $(\Delta_m, \Delta_n, K, L)$ as above, with

$$\Delta_t := \left\{(p_1, \ldots, p_t) \in \mathbb{R}^t \;\middle|\; \sum_{i=1}^{t} p_i = 1, p_i \in [0,1] \text{ for } i \in \{1, \ldots, t\}\right\} \quad (t \in \mathbb{N})$$

$$K(p,q) = pAq, \; L(p,q) = pBq \qquad (p \in \Delta_m, q \in \Delta_n).$$

The k-th pure strategy e_k for a player is defined as the vector x in Δ_m (or Δ_n) with the k-th coordinate equal to 1. A bimatrix game (A, B) can be represented as follows

$$\begin{bmatrix} (a_{11}, b_{11}) & \cdots & (a_{1n}, b_{1n}) \\ \vdots & & \vdots \\ (a_{m1}, b_{m1}) & \cdots & (a_{mn}, b_{mn}) \end{bmatrix}$$

Here, rows and columns correspond to the players' pure strategies, player 1 choosing the rows and player 2 choosing the columns. If player 1 and player 2 choose e_i and e_j, respectively, player 1 recieves a_{ij}, player 2 b_{ij}.

A *Nash equilibrium* in (A, B) is a pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$ such that for all strategies $p \in \Delta_m$ and $q \in \Delta_n$ we have

$$\hat{p}A\hat{q} \geqslant pA\hat{q}, \; \hat{p}B\hat{q} \geqslant \hat{p}Bq.$$

In other words, if a player unilaterally deviates from an equilibrium his payoff will not increase. The set of all Nash equilibria is denoted by $E(A, B)$. For the characterization of $E(A, B)$ in terms of best reply multifunctions, as given in proposition 1, we need the following definitions.
In the following let $\hat{p} \in \Delta_m$, $\hat{q} \in \Delta_n$.

$$C(\hat{p}) := \{i \in \{1, \ldots, m\} \mid \hat{p}_i > 0\}, \text{ the carrier of } \hat{p}$$

$$C(\hat{q}) := \{j \in \{1, \ldots, n\} \mid \hat{q}_j > 0\}$$

$$PB_1(\hat{q}) := \{i \in \{1, \ldots, m\} \mid e_i A\hat{q} = \max_{k \in \{1,\ldots,m\}} e_k A\hat{q}\}, \text{ representing the set of pure best responses of player 1 to } \hat{q}$$

$$PB_2(\hat{p}) := \{j \in \{1, \ldots, n\} \mid \hat{p}Be_j = \max_{k \in \{1,\ldots,n\}} \hat{p}Be_k\}$$

$$B_1(\hat{q}) := \{p \in \Delta_m \mid pA\hat{q} = \max_{p' \in \Delta_m} p'A\hat{q}\}, \text{ the set of best responses of player 1 to } \hat{q}$$

$$B_2(\hat{p}) := \{q \in \Delta_n \mid \hat{p}Bq = \max_{q' \in \Delta_n} \hat{p}Bq'\}.$$

$B_1(\hat{q})$ and $B_2(\hat{p})$ are well-defined, because the function $p \mapsto pA\hat{q}$, for instance, is continuous on the compact set Δ_m.
For A and B arbitrary sets, with $P(B) = \{S \mid S \subset B\}$, we call $F : A \to P(B)$ a multifunction if $F(a) \neq \emptyset$ for all $a \in A$. Hence, $B_1 : \Delta_n \to P(\Delta_m)$ and $B_2 : \Delta_m \to P(\Delta_n)$ are called best reply multifunctions. Obviously

$$B_1(\hat{q}) = \text{Conv } \{e_k \mid k \in PB_1(\hat{q})\}, \; B_2(\hat{p}) = \text{Conv } \{e_k \mid k \in PB_2(\hat{p})\} \tag{1}$$

Furthermore we define :

$$G_2 := \{(p,q) \in \Delta_m \times \Delta_n \mid q \in B_2(p)\}, \text{ the graph of } B_2$$

$$G_1^t := \{(p,q) \in \Delta_m \times \Delta_n \mid p \in B_1(q)\}, \text{ the transponed graph of } B_1.$$

Henceforth, we shall write G_1 instead of G_1^t.

Without proof we state

Proposition 1.

Let (A, B) be an mxn bimatrix game, $p \in \Delta_m$, $q \in \Delta_n$. The following four assertions are equivalent

(a) $(p,q) \in E(A,B)$

(b) $p \in B_1(q)$, $q \in B_2(p)$

(c) $C(p) \subset PB_1(q)$, $C(q) \subset PB_2(p)$

(d) $(p,q) \in G_1 \cap G_2$

Two important refinements of the Nash equilibrium concept are quasi-strong equilibria introduced by Harsanyi (1973), and perfect equilibria introduced by Selten (1975). For both refinements, characterizations closely related to the best reply structure will be given here (cf. van Damme [3] and Tijs [9]).

In the rest of this section let (A, B) be an mxn bimatrix game with $(p,q) \in \Delta_m \times \Delta_n$ ($m \in \mathbb{N}$, $n \in \mathbb{N}$).

Definition. (p,q) is called *quasi-strong* in (A,B) if

$$C(p) = PB_1(q),\ C(q) = PB_2(p)$$

By proposition 1 we have

(p,q) is quasi-strong in (A,B) $\Rightarrow$ (p,q) is an equilibrium in (A,B).

The set of all quasi-strong equilibria in (A,B) is denoted by QS(A,B). For defining perfect pairs in (A,B) we have to say what we mean by perturbed games of (A,B).

Definition. The (ε, δ)-perturbed game $(A,B)^{\varepsilon,\delta}$ of (A,B) is the game given by $(\Delta_m^\varepsilon, \Delta_n^\delta, K, L)$ with :

$$\varepsilon \in \mathbb{R}^m,\ \varepsilon > 0,\ \sum_{i=1}^{m} \varepsilon_i \leqslant 1;\ \delta \in \mathbb{R}^n,\ \delta > 0,\ \sum_{j=1}^{n} \delta_j \leqslant 1$$

$$\Delta_m^\varepsilon := \{p \in \Delta_m \mid p_i \geqslant \varepsilon_i \text{ for all } i \in \{1, \ldots, m\}\}$$

$$\Delta_n^\delta := \{q \in \Delta_n \mid q_j \geqslant \delta_j \text{ for all } j \in \{1, \ldots, n\}\}$$

$$K(p,q) = pAq;\ L(p,q) = pBq\ (p \in \Delta_m^\varepsilon,\ q \in \Delta_n^\delta)$$

Definition. (p,q) is called *perfect* in (A,B) if there is a sequence $\{(\varepsilon(k), \delta(k));\ k \in \mathbb{N}\}$ converging to $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$, and a sequence $\{(p^k, q^k);\ k \in \mathbb{N}\}$ of elements of $\Delta_m \times \Delta_n$ converging to (p,q), such that for all $k \in \mathbb{N}$:

$$(p^k, q^k) \in E((A,B)^{\varepsilon(k),\ \delta(k)}).$$

It is easy to verify that for all $(p,q) \in E((A,B)^{\varepsilon,\delta})$ we have :

$$e_i \notin B_1(q) \Rightarrow p_i = \varepsilon_i;\ e_j \notin B_2(p) \Rightarrow q_j = \delta_j.$$

Using this and characterization (c) in proposition 1 we obtain :

(p,q) is perfect in (A,B) $\Rightarrow$ (p,q) is an equilibrium in (A,B).

The set of perfect equilibria in (A,B) is denoted by PE(A,B). Because $E((A,B)^{\varepsilon,\delta}) \neq \emptyset$ (using an existence theorem for Nash equilibria, e.g. theorem 7.3 in Tijs [9]) and $\Delta_m \times \Delta_n$ is compact, we have $PE(A,B) \neq \emptyset$.

The following proposition will be useful in studying perfect equilibria in 2x2 bimatrix games. The proof can be found in van Damme [3].

Proposition 2. Let (A,B) be an mxn bimatrix game, $p \in \Delta_m$, $q \in \Delta_n$. The following two assertions are equivalent

(a) (p,q) is perfect in (A,B)

(b) There is a sequence $\{(p^k, q^k); k \in \mathbb{N}\}$ of elements of $\mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p,q), such that for all $k \in \mathbb{N}$:

$p \in B_1(q^k)$, $q \in B_2(p^k)$

where

$\mathring{\Delta}_t := \{p \in \Delta_t \mid p_i > 0 \text{ for all } i \in \{1, \ldots, t\}\}$ $(t \in \mathbb{N})$

Some useful properties are recalled in proposition 3. The proof is straightforward and therefore omitted.

Proposition 3. Let (A,B) be an mxn bimatrix game, $p \in \Delta_m$, $q \in \Delta_n$. *Then we have*

$$\text{(i)} \quad \left\{ \begin{array}{l} p \in \mathring{\Delta}_m, q \in \mathring{\Delta}_n \\ (p,q) \in E(A,B) \end{array} \right. \Rightarrow (p,q) \in QS(A,B) \cap PE(A,B) .$$

$$\text{(ii)} \quad \left\{ \begin{array}{l} (p,q) \text{ strong in } (A,B), \\ \text{i.e. } B_1(q) = \{p\}, B_2(p) = \{q\} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} p \text{ pure, } q \text{ pure} \\ (p,q) \in QS(A,B) \cap PE(A,B) \end{array} \right.$$

3. 2x2 BIMATRIX GAMES

We now consider 2x2 bimatrix games $(A,B) = \begin{bmatrix} (a_{11}, b_{11})(a_{12}, b_{12}) \\ (a_{21}, b_{21})(a_{22}, b_{22}) \end{bmatrix}$ with $a_{ij}, b_{ij} \in \mathbb{R}$.

Using proposition 1, for these games we can determine E(A,B) as follows. Identify $(p, 1-p) \in \Delta_2$ with $p \in [0,1]$ and consequently the product $\Delta_2 \times \Delta_2$ of strategy spaces with $[0,1] \times [0,1]$. Draw G_1 and G_2 and determine the intersection.

EXAMPLE

Consider $(A,B) = \begin{bmatrix} (3,4) & (0,0) \\ (0,0) & (2,1) \end{bmatrix}$

Best reply multifunctions and equilibria are represented in figure 1

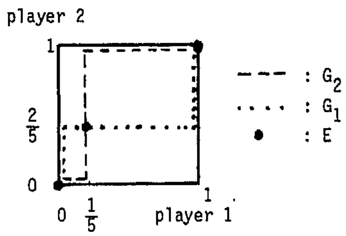


Figure 1

Let us now examine the shape of G_1 and G_2.

Lemma 1. Let (A,B) be as above. For all $x \in [0,1]$ and $i \in \{1,2\}$ exactly one of the following three assertions holds :

(a) $B_i(x) = \{0\}$

(b) $B_i(x) = \{1\}$

(c) $B_i(x) = [0,1]$.

Proof. Since we identify $(x, 1-x) \in \Delta_2$ with $x \in [0,1]$, the lemma follows directly from (1). □

Lemma 2. Let (A,B) be as above. For $i \in \{1,2\}$ the following two assertions are equivalent

(a) There are $x_1, x_2 \in [0,1]$ such that

$$x_1 \neq x_2,\ B_i(x_1) = [0,1],\ B_i(x_2) = [0,1]$$

(b) For all $x \in [0,1]$ we have

$$B_i(x) = [0,1].$$

Proof. (b) ⇒ (a) is trivial. We prove (a) ⇒ (b) for i = 2

Let $x_1, x_2 \in [0,1]$, $x_1 \neq x_2$, $B_2(x_1) = [0,1]$, $B_2(x_2) = [0,1]$. Then

$$(x_1,\ 1-x_1)\ Be_1 = (x_1,\ 1-x_1)\ Be_2;\ (x_2,\ 1-x_2)\ Be_1 = (x_2,\ 1-x_2)\ Be_2$$

or equivalently,

$$(b_{21} - b_{22}) + x_1(b_{11} - b_{21} - b_{12} + b_{22}) = 0$$

$$(b_{21} - b_{22}) + x_2(b_{11} - b_{21} - b_{12} + b_{22}) = 0.$$

Hence;

$$b_{21} = b_{22},\ b_{11} = b_{12}.$$

As a consequence we have $B_2(x) = [0,1]$ for all $x \in [0,1]$. □

Lemma 3. Let (A,B) be as above and $i \in \{1,2\}$. Let $x_1, x_2 \in [0,1]$ such that $x_1 \neq x_2$, $B_i(x_1) = \{0\}$, $B_i(x_2) = \{1\}$. Then there is an $x_0 \in [0,1]$ such that

$$B_i(x_0) = [0,1].$$

Proof. We prove the lemma for $i = 2$. Without loss of generality we suppose $x_1 < x_2$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = (b_{21} - b_{22}) + x(b_{11} - b_{21} - b_{12} + b_{22}) \quad (x \in [0,1]).$$

Then we have that f is continuous and $f(x_1) < 0$, $f(x_2) > 0$. So there is an $x_0 \in (x_1, x_2)$ such that $f(x_0) = 0$ or, equivalently, $B_2(x_0) = [0,1]$. □

Theorem 1. Let $i \in \{1,2\}$. For a 2x2 bimatrix game (A,B) exactly one of the following four assertions holds

(i.1) $B_i(x) = [0,1]$ for all $x \in [0,1]$

(i.2) There is a $k \in \{0,1\}$ such that
$B_i(x) = \{k\}$ for all $x \in [0,1]$

(i.3) There are $k,t \in \{0,1\}$ such that
$B_i(k) = [0,1]$
$B_i(x) = \{t\}$ for all $x \in [0,1]$, $x \neq k$

(i.4) There are $k, t \in \{0,1\}$, $k \neq t$ and $x_0 \in (0,1)$ such that
$B_i(x_0) = [0,1]$
$B_i(x) = \{k\}$ for all $x \in [0,x_0)$
$B_i(x) = \{t\}$ for all $x \in (x_0,1]$

Proof. Consider

(i.3)* There are $k \in \{0,1\}$ and $x_0 \in (0,1)$ such that
$B_i(x_0) = [0,1]$
$B_i(x) = \{k\}$ for all $x \in [0,1]$, $x \neq x_0$.

Using lemma 1, 2 and 3 it is sufficient to prove that (1.3)* is not possible. Without loss of generality we take $i = 2$.
Suppose (2.3)*. By substituting $x = 1$ and $x = 0$ we have

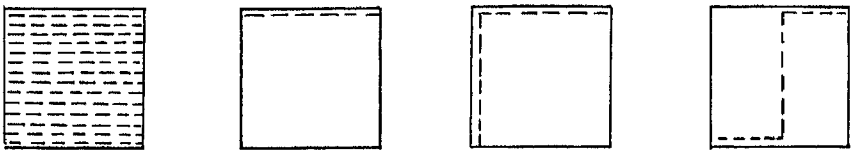
$$b_{11} > b_{12},\ b_{21} > b_{22} \ \text{ if } \ k = 1, \text{ or}$$
$$b_{11} < b_{12},\ b_{21} < b_{22} \ \text{ if } \ k = 2.$$

Because $x_0 \in (0,1)$ we then have

$$b_{21} + x_0(b_{11} - b_{21}) > b_{22} + x_0(b_{12} - b_{22}) \text{ or}$$
$$b_{21} + x_0(b_{11} - b_{21}) < b_{22} + x_0(b_{12} - b_{22}).$$

But this is a contradiction with $B_2(x_0) = [0,1]$. □

Representations of (2.t), $t \in \{1, 2, 3, 4\}$, are given in figure 2.



(t = 1) (t = 2) (t = 3) (t = 4)

Figure 2

Theorem 1 leads to the following

Definition. A 2x2 bimatrix game (A,B) has *best reply combination* [s;t] if the assertions (1.s) and (2.t) hold ($s,t \in \{1, 2, 3, 4\}$).

The four basic shapes of G_1 and G_2 as given in theorem 1 can be translated in conditions on the payoff matrices A and B. For this purpose we introduce the following notation. For $i, j, u, v \in \{0, 1, 2\}$ let

$$A[ij] = \begin{cases} a_{11} = a_{21} & \text{if } i = 2 \\ a_{11} > a_{21} & \text{if } i = 1 \\ a_{11} < a_{21} & \text{if } i = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_{12} = a_{22} & \text{if } j = 2 \\ a_{12} > a_{22} & \text{if } j = 1 \\ a_{12} < a_{22} & \text{if } j = 0 \end{cases}$$

and

$$B[{}^u_v] = \begin{cases} b_{11} = b_{12} & \text{if } u = 2 \\ b_{11} > b_{12} & \text{if } u = 1 \\ b_{11} < b_{12} & \text{if } u = 0 \end{cases} \quad \text{and} \quad \begin{cases} b_{21} = b_{22} & \text{if } v = 2 \\ b_{21} > b_{22} & \text{if } v = 1 \\ b_{21} < b_{22} & \text{if } v = 0. \end{cases}$$

This has the following interpretation. In A[ij], $i \in \{0, 1, 2\}$ represents the strategy that player 1 should use if his opponent chooses e_1 (the first column). He should play $x = 1$ if $i = 1$, $x = 0$ if $i = 0$ and arbitrarily if $x = 2$. Analogously $j \in \{0, 1, 2\}$ represents the strategy that player 1 should use if his opponent chooses e_2. In $B[{}^u_v]$, u and v in a similar way represent the strategies that player 2 should use if his opponent chooses e_1 and e_2, respectively.

Proposition 4. For $i \in \{1,2\}$ and $t \in \{1, 2, 3, 4\}$ the assertions (i.t) of theorem 1 can be rewritten as follows :

(1.1) $\Leftrightarrow$ A[22]
(1.2) $\Leftrightarrow$ A[11] or A[00]
(1.3) $\Leftrightarrow$ A[12] or A[21] or A[02] or A[20]
(1.4) $\Leftrightarrow$ A[10] or A[01]
(2.1) $\Leftrightarrow B[{}^2_2]$
(2.2) $\Leftrightarrow B[{}^1_1]$ or $B[{}^0_0]$

$(2.3) \leftrightarrow B[{}^{1}_{2}]$ or $B[{}^{2}_{1}]$ or $B[{}^{0}_{2}]$ or $B[{}^{2}_{0}]$

$(2.4) \leftrightarrow B[{}^{1}_{0}]$ or $B[{}^{0}_{1}]$.

Proof. We use the notation of theorem 1.

A[11] corresponds with (1.2) for k = 1, A[00] with (1.2) for k = 0,

A[12] with (1.3) for k = 0 and t = 1, A[02] with (1.3) for k = 0 and t = 0,

A[21] with (1.3) for k = 1 and t = 1, A[20] with (1.3) for k = 1 and t = 0,

A[10] with (1.4) for k = 0 and t = 1, A[01] with (1.3) for k = 1 and t = 0.

The proof is similar for $B[{}^{u}_{v}]$, with $u, v \in \{0, 1, 2\}$. □

This leads to the following

Definition. A 2x2 bimatrix game (A,B) has *payoff combination* ([ij], $[{}^{u}_{v}]$) if the conditions A[ij] and $B[{}^{u}_{v}]$ hold. (i,j,u,v ∈ {1, 2, 3, 4}).

Best reply combinations and payoff combinations are the corner-stone for the classification given in section 4.

4. THE CLASSIFICATION

In this section we distinguish 15 classes of 2x2 bimatrix games. The distinction is made without discriminating between the players. In other words, the classification does not depend upon the players' numbering. This is being achieved by considering the payoff combination ([ij], $[{}^{u}_{v}]$) equivalent to ([uv], $[{}^{i}_{j}]$) for all i,j,u,v ∈ {0, 1, 2}. Consequently, the best reply combination [s;t] is equivalent to [t;s] for all s,t ∈ {1, 2, 3, 4}. By doing this we obtain 10 different best reply combinations and 45 different payoff combinations. The classification is now made as follows. Consider the 10 classes of 2x2 bimatrix games corresponding with the different best reply combinations. Due to a different payoff combination the structure of the Nash equilibrium set within four of these classes can greatly vary. According to these various structures a further distinction is made. The result is presented in table 1. In each class, for every possible payoff combination the Nash equilibria, quasi-strong equilibria and perfect equilibria are given.

Remark. In table 1 we have

$$p_B = \frac{b_{22} - b_{21}}{b_{11} + b_{22} - b_{21} - b_{12}} \in (0,1) \text{ and } \quad q_A = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \in (0,1),$$

since p_B and q_A rise as (unique) solutions of

$(x, 1-x)\, Be_1 = (x, 1-x)\, Be_2$ with $B[{}^{1}_{0}]$ or $B[{}^{0}_{1}]$ $(x \in [0,1])$

and

$e_1A(x, 1-x) = e_2A(x, 1-x)$ with A[10] or A[01] $(x \in [0,1])$,

respectively.

CLASS	BEST REPLY COMBINATION	PAYOFF COMBINATION	E	QS	PE
c1	[1;1]	$([22], [^{2}_{2}])$	[0,1] x [0,1]	(0,1) x (0,1)	E
c2	[1;2]	$([22], [^{1}_{1}])$	[0,1] x {1}	(0,1) x {1}	E
		$([22], [^{0}_{0}])$	[0,1] x {0}	(0,1) x {0}	
c3	[1;3]	$([22], [^{1}_{2}])$	[0,1] x {1} U {0} x [0,1]	(0,1) x {1}	[0,1] x {1}
		$([22], [^{0}_{2}])$	[0,1] x {0} U {0} x [0,1]	(0,1) x {0}	[0,1] x {0}
		$([22], [^{2}_{1}])$	[0,1] x {1} U {1} x [0,1]	(0,1) x {1}	[0,1] x {1}
		$([22], [^{2}_{0}])$	[0,1] x {0} U {1} x [0,1]	(0,1) x {0}	[0,1] x {0}
c4	[1;4]	$([22], [^{1}_{0}])$	$[0,p_B]$ x {0} U $\{p_B\}$ x [0,1] U $[p_B,1]$ x {1}	$(0,p_B)$ x {0} U $\{p_B\}$ x (0,1) U $(p_B,1)$ x {1}	E
		$([22], [^{0}_{1}])$	$[0,p_B]$ x {1} U $\{p_B\}$ x [0,1] U $[p_B,1]$ x {0}	$(0,p_B)$ x {1} U $\{p_B\}$ x (0,1) U $(p_B,1)$ x {0}	
c5	[2;2]	$([11], [^{1}_{1}])$	{1} x {1}	E	E
		$([11], [^{0}_{0}])$	{1} x {0}		
		$([00], [^{0}_{0}])$	{0} x {0}		
c6	[2;3]	$([11], [^{1}_{2}])$	{1} x {1}	E	E
		$([11], [^{0}_{2}])$	{1} x {0}		
		$([00], [^{2}_{1}])$	{0} x {1}		
		$([00], [^{2}_{0}])$	{1} x {1}		
c7	[2;3]	$([11], [^{2}_{1}])$	{1} x [0,1]	{1} x (0,1)	{1} x {1}
		$([11], [^{2}_{0}])$	{1} x [0,1]	{1} x (0,1)	{1} x {0}
		$([00], [^{1}_{2}])$	{0} x [0,1]	{0} x (0,1)	{0} x {1}
		$([00], [^{0}_{2}])$	{0} x [0,1]	{0} x (0,1)	{0} x {0}
c8	[2;4]	$([11], [^{1}_{0}])$	{1} x {1}	E	E
		$([11], [^{0}_{1}])$	{1} x {0}		
		$([00], [^{1}_{0}])$	{0} x {0}		
		$([00], [^{0}_{1}])$	{0} x {1}		

CLASS	BEST REPLY COMBINATION	PAYOFF COMBINATION	E	QS	PE
c9	[3;3]	$([12], [\substack{1\\2}])$	$\{1\} \times \{1\} \cup \{0\} \times \{0\}$	$\{1\} \times \{1\}$	$\{1\} \times \{1\}$
		$([02], [\substack{2\\1}])$	$\{1\} \times \{0\} \cup \{0\} \times \{1\}$	$\{0\} \times \{1\}$	$\{0\} \times \{1\}$
		$([20], [\substack{2\\0}])$	$\{1\} \times \{1\} \cup \{0\} \times \{0\}$	$\{0\} \times \{0\}$	$\{0\} \times \{0\}$
c10	[3;3]	$([12], [\substack{2\\1}])$	$\{1\} \times [0,1]$	$\{1\} \times (0,1)$	$\{1\} \times \{1\}$
		$([02], [\substack{1\\2}])$	$\{0\} \times [0,1]$	$\{0\} \times (0,1)$	$\{0\} \times \{1\}$
		$([20], [\substack{0\\2}])$	$\{0\} \times [0,1]$	$\{0\} \times (0,1)$	$\{0\} \times \{0\}$
		$([21], [\substack{2\\0}])$	$\{1\} \times [0,1]$	$\{1\} \times (0,1)$	$\{1\} \times \{0\}$
c11	[3;3]	$([12], [\substack{2\\0}])$	$[0,1] \times \{0\} \cup \{1\} \times [0,1]$	$(0,1) \times \{0\} \cup \{1\} \times (0,1)$	$\{1\} \times \{0\}$
		$([02], [\substack{0\\2}])$	$[0,1] \times \{0\} \cup \{0\} \times [0,1]$	$(0,1) \times \{0\} \cup \{0\} \times (0,1)$	$\{0\} \times \{0\}$
		$([21], [\substack{2\\1}])$	$[0,1] \times \{1\} \cup \{1\} \times [0,1]$	$(0,1) \times \{1\} \cup \{1\} \times (0,1)$	$\{1\} \times \{1\}$
c12	[3;4]	$([12], [\substack{1\\0}])$	$\{1\} \times \{1\} \cup [0,p_B] \times \{0\}$	$\{1\} \times \{1\} \cup (0,p_B) \times \{0\}$	$\{1\} \times \{1\}$
		$([02], [\substack{0\\1}])$	$\{0\} \times \{1\} \cup [p_B,1] \times \{0\}$	$\{0\} \times \{1\} \cup (p_B,1) \times \{0\}$	$\{0\} \times \{1\}$
		$([21], [\substack{0\\1}])$	$\{1\} \times \{0\} \cup [0,p_B] \times \{1\}$	$\{1\} \times \{0\} \cup (0,p_B) \times \{1\}$	$\{1\} \times \{0\}$
		$([20], [\substack{1\\0}])$	$\{0\} \times \{0\} \cup [p_B,1] \times \{1\}$	$\{0\} \times \{0\} \cup (p_B,1) \times \{1\}$	$\{0\} \times \{0\}$
c13	[3;4]	$([12], [\substack{0\\1}])$	$[p_B,1] \times \{0\}$	$(p_B,1) \times \{0\}$	$\{1\} \times \{0\}$
		$([02], [\substack{1\\0}])$	$[0,p_B] \times \{0\}$	$(0,p_B) \times \{0\}$	$\{0\} \times \{0\}$
		$([21], [\substack{1\\0}])$	$[p_B,1] \times \{1\}$	$(p_B,1) \times \{1\}$	$\{1\} \times \{1\}$
		$([20], [\substack{0\\1}])$	$[0,p_B] \times \{1\}$	$(0,p_B) \times \{1\}$	$\{0\} \times \{1\}$
c14	[4;4]	$([10], [\substack{1\\0}])$	$\{1\} \times \{1\} \cup \{0\} \times \{0\} \cup \{p_B\} \times \{q_A\}$	E	E
		$([01], [\substack{0\\1}])$	$\{1\} \times \{0\} \cup \{0\} \times \{1\} \cup \{p_B\} \times \{q_A\}$		
c15	[4;4]	$([10], [\substack{0\\1}])$	$\{p_B\} \times \{q_A\}$	E	E

TABLE 1

To show how one can prove the results given in table 1, we examine a 2x2 bimatrix game (A,B) in c10 with payoff combination ([02], $\begin{bmatrix}1\\2\end{bmatrix}$). As can be seen from the proof of proposition 4 we have

$$B_1(0) = [0,1],\ B_1(y) = \{0\} \text{ for all } y \in (0,1] \qquad (2)$$

$$B_2(0) = [0,1],\ B_2(x) = \{1\} \text{ for all } x \in (0,1] \qquad (3)$$

Hence

$$G_1 = \{(x,y) \in [0,1] \times [0,1] \mid x = 0 \text{ or } y = 0\}$$

$$G_2 = \{(x,y) \in [0,1] \times [0,1] \mid x = 0 \text{ or } y = 1\}$$

So $E(A,B) = G_1 \cap G_2 = \{(x,y) \in [0,1] \times [0,1] \mid x = 0\} = \{0\} \times [0,1]$. Using (2) and (3) we have

$$PB_2(0) = \{0,1\},\ PB_1(y) = \{0\} \text{ for all } y \in (0,1).$$

Here we identify the index of a pure strategy with the strategy itself.

Hence, $\{0\} \times (0,1) \subset QS(A,B)$, $(0,0) \notin QS(A,B)$ and $(0,1) \notin QS(A,B)$.
Now suppose $(\hat{x},\hat{y}) \in QS(A,B)$ with $\hat{x} \neq 0$. Because $B_2(\hat{x}) = 1$ we have $C(\hat{y}) = 1$ or, equivalently, $\hat{y} = 1$. But $B_1(1) = \{0\}$, so $C(\hat{x}) = \{0\}$ or, equivalently, $\hat{x} = 0$. Contradiction.
We may conclude : $QS(A,B) = \{0\} \times (0,1)$.
For computing the perfect equilibria we use proposition 2. By (2) and (3) we have

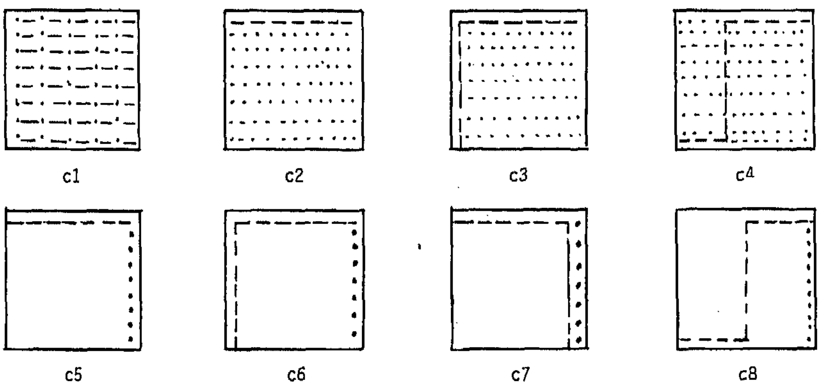
$$B_2(p^k) = \{1\} \text{ for all sequences } \{p^k;\ k \in \mathbb{N}\} \text{ in } (0,1)$$

Similarly,

$$B_1(q^k) = \{0\} \text{ for all sequences } \{q^k;\ k \in \mathbb{N}\} \text{ in } (0,1)$$

Hence, $PE(A,B) \subset \{0\} \times \{1\}$.
Because $PE(A,B) \neq \emptyset$ we may concluce that $PE(A,B) = \{0\} \times \{1\}$.

The results in the table can be visualized using representations as given in Figure 3. For each class this figure represents the first payoff combination as given in Table 1.



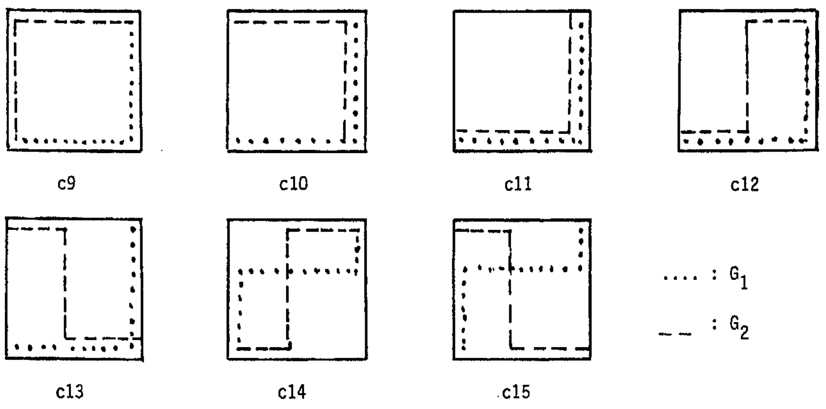


Figure 3

Theorem 2 enables us to use table 1 for any given 2x2 bimatrix game. The proof is straightforward and therefore omitted.

Theorem 2

Let $p,q \in [0,1]$. Let (A,B) be a 2x2 bimatrix game with payoff combination $([ij], [{u \atop v}])$. $(i,j,u,v \in \{0, 1, 2\})$. Then the following four assertions hold

(i) (B^t, A^t) has payoff combination $([uv], [{i \atop j}])$

(ii) $(p,q) \in E(A,B) \leftrightarrow (q,p) \in E(B^t, A^t)$

(iii) $(p,q) \in QS(A,B) \leftrightarrow (q,p) \in QS(B^t, A^t)$

(iv) $(p,q) \in PE(A,B) \leftrightarrow (q,p) \in PE(B^t, A^t)$

Corollary. For every 2x2 bimatrix game (A,B) we have $QS(A,B) \neq \emptyset$.

Although the classification is based neither upon the structure of the set of quasi-strong equilibria nor on that of the set of perfect equilibria, these structures are the same within each class. This probably can be seen as a major asset of the classification. Moreover, all possible structures can be described by a partition of the classes as follows. For the Nash equilibrium set :

{{c1}, {c2, c7, c10}, {c3, c11}, {c4}, {c5, c6, c8}, {c9}, {c12}, {c13}, {c14}, {c15}}.

For the quasi-strong equilibrium set :

{{c1}, {c2, c3, c7, c10}, {c4}, {c5, c6, c8, c9}, {c11}, {c12}, {c13}, {c14}, {c15}}.

For the perfect equilibrium set :

{{c1}, {c2, c3}, {c4}, {c5, c6, c7, c8, c9, c10, c11, c12, c13}, {c14}, {c15}}.

5. *PROBABILITIES*

To get an impression of the importance of class ci ($i \in \{1, \ldots, 15\}$) we look at the probability that a random 2x2 bimatrix game belongs to ci, thereby using the probability model below. The results are stated in theorem 3.
Let $N \in \mathbb{N}$ (sufficiently large).
Let $X_1, X_2, \ldots, X_8$ be independent random variables with

$$X_i \overset{d}{=} Un(-N, N) \text{ for all } i \in \{1, \ldots, 8\}$$

i.e. all X_i are uniformly distributed over the interval (-N, N). We assume $X_1, \ldots, X_8$ determine a 2x2 bimatrix game (A, B), where

$$(A,B) = \begin{bmatrix} (X_1, X_2) & (X_3, X_4) \\ (X_5, X_6) & (X_7, X_8) \end{bmatrix}$$

Theorem 3

Let (A,B) be a 2x2 bimatrix game.
Using the above probability model we have

$$P[(A,B) \in ci] = \begin{cases} \frac{1}{2} & \text{if } i = 8 \\ \frac{1}{4} & \text{if } i = 5 \\ \frac{1}{8} & \text{if } i \in \{14,15\} \\ 0 & \text{else} \end{cases}$$

Proof

We prove the assertion for i = 5. The proof for $i \in \{8,14,15\}$ can be done analogously. Note that if $(A,B) \in ci$ for an $i \notin \{5,8,14,15\}$, $X_k = X_t$ for some $k,t \in \{1, \ldots, 8\}$, $k \neq 1$. But for such k and t $P[X_k = X_t] = 0$.
Using table 1 we have the equivalence of the following two assertions

(a) $(A,B) \in c5$

(b) (A,B) has one of the following four payoff combinations :

$([11], [^1_1])$, $([11], [^0_0])$, $([00], [^1_1])$ or $([00], [^0_0])$

(Note the equivalent payoff combinations).

By the definition of these payoff combinations it follows that

$$P[(A,B) \in c5] = P[X_1 > X_5,\ X_3 > X_7,\ X_2 > X_4,\ X_6 > X_8] + P[X_1 > X_5,\ X_3 > X_7,\ X_2 < X_4,\ X_6 < X_8]$$

$$+ P[X_1 < X_5,\ X_3 < X_7,\ X_2 > X_4,\ X_6 > X_8] + P[X_1 < X_5,\ X_3 < X_7,\ X_2 < X_4,\ X_6 < X_8]$$

$$= 4 \ .\ (\tfrac{1}{2})^4 = \tfrac{1}{4}$$

where we use the independency of $X_1, \ldots, X_8$ and the fact that $P[X_i > X_j] = \frac{1}{2}$ for all $i,j \in \{1, \ldots, 8\}$, $i \neq j$. □

Using a general probability model, of which the above model is a special case, Golberg, Goldman and Newman [4] calculate the probability of appearance of a pure Nash equilibrium in a random mxn bimatrix game for all $m,n \in \mathbb{N}$. For $m = 2$ and $n = 2$ this probability turns out to be $\frac{7}{8}$. This result can be found also by using the classification of section 4 and theorem 3. The only class in which a 2x2 bimatrix game (A,B) does not have a pure Nash equilibrium is c15, and by theorem 3

$$1 - P[(A,B) \in 15] = \frac{7}{8}\ .$$

6. REMARKS

(i) Note that for 2x2 bimatrix games the proper equilibria (Myerson [6]), which are another important refinement of the Nash equilibrium concept, coincide with the perfect equilibria. This can be found in van Damme [3].

(ii) For all classes which have positive probability of appearance, we have that the number of Nash equilibira is finite and that all Nash equilibria are quasi-strong and perfect.

(iii) In the classification of Beniest [1] classes c2 and c7 are joined. However, there is a fundamental difference in the set of perfect equilibria between an element of c2 and c7.

(iv) Consider the space of 2x2 bimatrix games $(\mathbb{R}^{2x2})^2$ with the metric d defined by

$$d((A,B),\ (A',B')) = \max_{i,j} \{|a_{ij} - a'_{ij}|,\ |b_{ij} - b'_{ij}|\}$$

if

$$A = [a_{ij}],\ B = [b_{ij}],\ A' = [a'_{ij}],\ B' = [b'_{ij}] \in \mathbb{R}^{2x2}.$$

Consequently we may interpret class ci as a subset of this metric space $(\mathbb{R}^{2x2})^2$ $(i \in \{1,\ldots,15\})$. Using payoff combinations we can prove the following :

- ci is open w.r.t. d for all $i \in \{5,8,14,15\}$
- ci is closed w.r.t. d for all $i \in \{1, \ldots, 15\} \setminus \{5,8,14,15\}$
- $\overline{c5} = \bigcup_{i \in I} ci$, with $I = \{1,2,3,5,6,7,9,10,11\}$
- $\overline{c8} = \bigcup_{i \in J} ci$, with $J = \{1,2,3,4,6,7,8,9,10,11,12,13\}$

- $\overline{c14} = \bigcup_{i \in K} ci$, with K = {1,3,4,9,10,11,12,13,14}

- $\overline{c15} = \bigcup_{i \in L} ci$, with L = {1,3,4,9,10,11,12,13,15}

where for $X \subset (R^{2x2})^2$, $\bar{X}$ denotes the closure of X w.r.t. d. Hence, classes c5, c8, c14 and c15 are stable in the sense that small perturbations in the payoff matrices will never cause a switch in classifiation from one of these classes to another.

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