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On Strictly Perfect Sets

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It is shown that for a bimatrix game the set of extreme equilibria is a strictly perfect set and that every minimal strictly perfect set is finite. Moreover, it is proved that there are finitely many equivalence classes of minimal strictly perfect sets, each of which can be associated with a collection of faces of maximal Nash subsets for the game. Further, it is shown that the set of strictly perfect equilibria, if non-empty, is the finite union of faces of maximal Nash subsets. *Journal of Economic Literature* Classification Number: C72 © 1994 Academic Press, Inc.

1. INTRODUCTION

For a noncooperative game in normal form there may exist Nash equilibria which are not stable against small perturbations in the data of the game. In order to overcome this problem many refinements of the equilibrium concept have been formulated in the literature. Roughly, these can be divided into two types. Refinements of the first type are defined by requiring stability with respect to perturbations in the payoffs, whereas

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those of the second type focus on perturbations in strategies. For both types there is a need for a set-valued concept rather than a single point concept, because of existence problems.

In one of the first papers on the subject, Wu and Jiang (1962) introduced the concept of essential equilibrium which is of the first type: an equilibrium is essential for a game if every small perturbation in the payoffs of this game leads to a game that has an equilibrium close to it. However, not every game possesses an essential equilibrium. Jiang (1964) investigated the essentiality of a subset of the equilibrium set and showed that for every game there exists an essential connected subset of equilibria (see also Kohlberg and Mertens, 1986).

A similar situation holds for refinements of the second type. Selten (1975) defined perfect equilibria by requiring stability against some mistakes the players can make in choosing their strategies. Every game possesses a perfect equilbrium. This is not true for equilibria that are stable against *all* mistakes the players can make. Such equilibria are called strictly perfect and were introduced by Okada (1981). For the set-valued equivalent of a strictly perfect equilibrium a proof of existence was given by Kohlberg and Mertens (1986). However, since every closed subset of the equilibrium set which contains a strictly perfect equilibrium is a strictly perfect set, it is natural to impose a minimality condition. Kohlberg and Mertens showed that every strictly perfect set contains a minimal strictly perfect set.

Another stability requirement illustrating the need for a set-valued concept also leads to a refinement of the second type. Okada (1983) introduced robust equilibria, based on the idea that an equilibrium strategy for a player should be a best reply to all strategies close to the equilibrium strategies of the other player(s). There are games without a robust equilibrium. The concept of absorbing retracts of Kalai and Samet (1984) generalizes the robustness concept for closed and convex subsets of the strategy spaces. Kalai and Samet showed that every game possesses a minimal absorbing (persistent) retract and that the intersection of this set with the equilibrium set is always non-empty.

The two types of refinements are related. Van Damme (1987) showed that the perturbations in the strategies, as in the definition of the strictly perfectness concept, can be viewed as special perturbations in the payoffs. This implies that every essential set is strictly perfect. The main results in this paper are proved with the help of the special perturbations in the payoffs coming from the definition of a strictly perfect set. One of these results concerns a relation between persistent retracts and strictly perfect sets. In fact, persistency as well as essentiality implies strict perfectness. Kohlberg and Mertens (1986) showed that minimal strictly perfect sets possess many desirable properties. It also appeared that a minimal strictly perfect set need not be connected. In this paper we show that for bimatrix games minimal strictly perfect sets are *finite*. Since the proof relies on the typical structure of the equilibrium set for bimatrix games, a similar proof for a game with more than two players cannot be given. Note that the stable sets, as recently introduced by Mertens (1989) as well as Hillas (1990), also possess the properties formulated by Kohlberg and Mertens. Moreover, these stable sets appeared to be *connected*.

In Section 3 we give the definition of a strictly perfect set and give the related perturbed payoff matrices. We show that an absorbing retract (Kalai and Samet, 1984) for a bimatrix game is also an absorbing retract for the game with the perturbed matrices.

In Section 4 we give two examples of sets that are always strictly perfect. The first example relies on the fact that for a bimatrix game the equilibrium set is the finite union of maximal Nash subsets, which are poplytopes (Jansen, 1981). We study the behavior of extreme points of maximal Nash subsets under the special perturbations of the matrices and show that the set of all these points is strictly perfect. Clearly this set is finite so that we have an elementary proof of the fact that every bimatrix game possesses a minimal strictly perfect set. We also show that the intersection of an absorbing retract and the equilibrium set is strictly perfect.

In Section 5 we study, more generally, the behavior of faces of maximal Nash subsets under the special perturbations in the payoff matrices. From this we conclude that every strictly perfect set contains a finite strictly perfect set and hence that every minimal strictly perfect set is finite. Further we show that the set of strictly perfect equilibria, if non-empty, is the finite union of faces of maximal Nash subsets.

In Section 6 we show that there are finitely many equivalence classes of minimal strictly perfect sets and that each equivalence class can be associated with a collection of faces of maximal Nash subsets.

Notation. $\mathbb{N} := \{1, 2, 3 \ldots\}$. Let $t \in \mathbb{N}$. We denote by $\mathbb{R}^t$ the vector space of t-tuples of real numbers and $\Delta_t := \{p \in \mathbb{R}^t | \, p \geq 0, \sum_{i=1}^t p_i = 1\}$. The unit vectors in $\mathbb{R}^t$ are denoted by $e_1, e_2, \ldots, e_t$. For $x, y \in \mathbb{R}^t$, we denote $x \cdot y := \sum_{i=1}^t x_i y_i$ and $x \geq y$ ($x > y$) if $x_k \geq y_k$ ($x_k > y_k$) for all $k \in \{1, \ldots, t\}$. $\mathbf{1}_t$ denotes the vector in $\mathbb{R}^t$ with all coordinates equal to one. Let $S \subset \mathbb{R}^t$. Then conv(S) denotes the convex hull of S. If S is a convex set, then ext(S) denotes the set of extreme points of S and relint(S) is the relative interior of S (i.e., the interior of S with respect to its affine hull).

2. Preliminaries

Let A and B be two real $m \times n$ matrices. The $m \times n$ *bimatrix game* (A, B) is defined as the two-person game where Player 1 and Player 2 independently choose the *strategies* $p \in \Delta_m$ and $q \in \Delta_n$, respectively, and obtain the payoffs pAq and pBq accordingly.

A pair of strategies $(p, q) \in \Delta_m \times \Delta_n$ is called an *equilibrium* of (A, B) if $xAq \leq pAq$ and $pBy \leq pBq$ for all pairs of strategies $(x, y) \in \Delta_m \times \Delta_n$. The set of all equilibria of (A, B) is denoted by $E(A, B)$. For a strategy $p \in \Delta_m$ we denote by $C(p) := \{i \in \{1, 2, \ldots, m\} | p_i > 0\}$ the *carrier* of p, and by $PB_2(B, p) := \{j \in \{1, 2, \ldots, n\} | pBe_j = \max_{l \in \{1, 2, \ldots, n\}} pBe_l\}$ the *pure best answers* of Player 2 to p. For $q \in \Delta_n$, $C(q)$ and $PB_1(A, q)$ are defined analogously.

It is a well-known fact that a strategy pair $(p, q) \in \Delta_m \times \Delta_n$ is an equilibrium of (A, B) if and only if $C(p) \subset PB_1(A, q)$ and $C(q) \subset PB_2(B, p)$.

Nash (1951) showed that $E(A, B)$ is non-empty for all games (A, B). Jansen (1981) showed that $E(A, B)$ is the finite union of maximal Nash subsets, where $T \subset E(A, B)$ is a *Nash subset* if every pair of elements in T is interchangeable, i.e., $(p, q), (x, y) \in T$ implies $(p, y), (x, q) \in T$ and a Nash subset T is called *maximal* if it is not properly contained in another Nash subset. Moreover, Jansen (1981) showed that a maximal Nash subset is a convex polytope (i.e., the convex hull of a finite number of elements). This last property and in particular the facial structure of maximal Nash subsets are used in the rest of the paper.

Maximal Nash subsets can be characterized by the carriers and pure best answers belonging to a strategy pair in its relative interior (Jansen, 1981).

LEMMA 1. *Let T be a maximal Nash subset for the bimatrix game (A, B) and let $(\mathring{p}, \mathring{q}) \in \operatorname{relint}(T)$. Then $(p, q) \in T$ if and only if*

$$\begin{aligned} & C(p) \subset C(\mathring{p}), PB_2(B, p) \supset PB_2(B, \mathring{p}) \\ \textit{and} \quad & C(q) \subset C(\mathring{q}), PB_1(A, q) \supset PB_1(A, \mathring{q}). \end{aligned} \tag{1}$$

By a *face* for the bimatrix game (A, B) we mean a face of a maximal Nash subset in $E(A, B)$. (A subset F of a polytope $P \subset \mathbb{R}^t$ is called a face of P if $F = P$ or if there is a supporting hyperplane H of P such that $F = H \cap P$.) For such faces we have a result similar to Lemma 1.

LEMMA 2. *Let F be a face for the $m \times n$ bimatrix game (A, B) and let $(\mathring{p}, \mathring{q}) \in \operatorname{relint}(F)$. Then $(p, q) \in F$ if and only if (1) is satisfied.*

Proof. The proof of the "only if"-part is left to the reader.

For the "if"-part we let the pair of strategies (p, q) satisfy the inclusions of (1). Since F is a face of some maximal Nash subset T, there are a pair $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and a real number c such that

$$\begin{aligned} (x, y) \cdot (\tilde{p}, \tilde{q}) &= c \qquad \text{for all } (\tilde{p}, \tilde{q}) \in F \\ (x, y) \cdot (\tilde{p}, \tilde{q}) &< c \qquad \text{for all } (\tilde{p}, \tilde{q}) \in T \backslash F \end{aligned} \tag{2}$$

By Lemma 1 we have $(p, q) \in T$. Suppose $(p, q) \notin F$. Then we define for an arbitrary positive real number α,

$$p(\alpha) := (1 + \alpha)\mathring{p} - \alpha p.$$

One easily shows that $p(\alpha) \in \Delta_m$, $C(p(\alpha)) = C(\mathring{p})$, and $PB_2(B, p(\alpha)) = PB_2(B, \mathring{p})$ for small α.

Similarly, $q(\alpha) := (1 + \alpha)\mathring{q} - \alpha q \in \Delta_n$, $C(q(\alpha)) = C(\mathring{q})$, and $PB_1(A, q(\alpha)) = PB_1(A, \mathring{q})$ for small α. Hence Lemma 1 implies that $(p(\alpha), q(\alpha)) \in T$ for small α. Then we obtain that for such α,

$$\begin{aligned}(x, y) \cdot (p(\alpha), q(\alpha)) &= (1 + \alpha)(x, y) \cdot (\mathring{p}, \mathring{q}) - \alpha(x, y) \cdot (p, q) \\ &> (1 + \alpha)c - \alpha c = c,\end{aligned}$$

which contradicts (2). Hence $(p, q) \in F$. $\triangleleft$

3. Strictly Perfect Sets and Perturbed Bimatrix Games

In this section we define strictly perfect sets for bimatrix games. These are subsets of the equilibrium set having a stability property with respect to perturbations in strategies. Kohlberg and Mertens (1986) defined and showed existence of minimal strictly perfect sets for n-person games. From the definition of a strictly perfect set for a bimatrix game we derive a class of bimatrix games obtained by a specific perturbation of the matrices of the original game. This class plays an important role in the proofs in the rest of this paper. We start with a definition of the game which results if all strategies for both players in a bimatrix game are perturbed.

Let (A, B) be an $m \times n$ bimatrix game. For a *mistake vector* $(\varepsilon, \delta) \in \mathbb{R}^m \times \mathbb{R}^n$, i.e.,

$$\varepsilon > 0, \sum_{i=1}^{m} \varepsilon_i < 1 \qquad \text{and} \qquad \delta > 0, \sum_{j=1}^{n} \delta_j < 1,$$

the (ε, δ)-*perturbed game* $(A, B, \varepsilon, \delta)$ corresponding to (A, B) is defined to be the game which only differs from (A, B) in the sense that the strategy spaces are restricted to

$$\Delta_m(\varepsilon) := \{p \in \Delta_m | p \geq \varepsilon\} \qquad \text{and} \qquad \Delta_n(\delta) := \{q \in \Delta_n | q \geq \delta\}.$$

For the (ε, δ)-perturbed game $(A, B, \varepsilon, \delta)$ equilibria can be defined in an obvious way.

Dealing with the existence of such equilibria, we note that the perturbed game $(A, B, \varepsilon, \delta)$ is not a bimatrix game, but it is equivalent to one (cf. Theorem 2.4.3. in van Damme (1987)). In order to show this, we define the $m \times n$ matrix $A(\delta)$ by

$$e_iA(\delta) := (1 - \sum_{j=1}^{n} \delta_j)e_iA + (e_iA\delta)\mathbf{1}_n \qquad \text{for } i \in \{1, 2, \ldots, m\}$$

and the $m \times n$ matrix $B(\varepsilon)$ by

$$B(\varepsilon)e_j := \left(1 - \sum_{i=1}^{m} \varepsilon_i\right) Be_j + (\varepsilon Be_j)\mathbf{1}_m \qquad \text{for } j \in \{1, 2, \ldots, n\}.$$

For $(p, q) \in \Delta_m \times \Delta_n$ we define the corresponding *perturbed strategies*

$$\begin{aligned} p(\varepsilon) &:= \left(1 - \sum_{i=1}^{m} \varepsilon_i\right) p + \varepsilon \in \Delta_m(\varepsilon), \\ q(\delta) &:= \left(1 - \sum_{j=1}^{n} \delta_j\right) q + \delta \in \Delta_n(\delta). \end{aligned} \tag{3}$$

The correspondence described in (3) yields in fact a one-to-one correspondence between strategy pairs in $\Delta_m \times \Delta_n$ and strategy pairs in $\Delta_m(\varepsilon) \times \Delta_n(\delta)$. Moreover,

LEMMA 3. *Let (A, B) be an $m \times n$ bimatrix game and let $(\varepsilon, \delta) \in \mathbb{R}^m \times \mathbb{R}^n$ be a mistake vector. Then $(p(\varepsilon), q(\delta))$ is an equilibrium of the game $(A, B, \varepsilon, \delta)$ if and only if $(p, q) \in E(A(\delta), B(\varepsilon))$.*

Proof. The proof follows immediately if one notes that for $(p, q) \in \Delta_m \times \Delta_n$ we have

$$p(\varepsilon)Aq(\delta) = p(\varepsilon)A(\delta)q = \left(1 - \sum_{i=1}^{m} \varepsilon_i\right) pA(\delta)q + \varepsilon A(\delta)q$$

and

$$p(\varepsilon)Bq(\delta) = pB(\varepsilon)q(\delta) = \left(1 - \sum_{j=1}^{n} \delta_j\right) pB(\varepsilon)q + pB(\varepsilon)\delta. \qquad \triangleleft$$

A closed set $S \subset E(A, B)$ is called a *strictly perfect set* if for any open set $V \supset S$ there exists a neighbourhood U of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that for each mistake vector (ε, δ) in U there exists a Nash equilibrium of $(A, B, \varepsilon, \delta)$ that also lies in V.

A strictly perfect set that does not properly contain another strictly perfect set is called a *minimal strictly perfect set.*

The following equivalence will be used frequently in this paper.

LEMMA 4. *Let (A, B) be an $m \times n$ bimatrix game. A closed set $S \subset E(A, B)$ is strictly perfect if and only if for every sequence $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero there is a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ with a limit point in S such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$.*

Proof. We only show that S is strictly perfect if the condition mentioned in the theorem is satisfied. Suppose that S is not a strictly perfect set. Then, since S is closed, there exists an open neighborhood V of S such that for every neighborhood U of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ there is a mistake vector (ε, δ) in U such that no equilibrium of $(A, B, \varepsilon, \delta)$ lies in V.

Consider the sequence $\{U_k\}_{k\in\mathbb{N}}$ of open balls in $\mathbb{R}^m \times \mathbb{R}^n$ with radius $1/k$ and center $(0, 0)$. For all $k \in \mathbb{N}$ we can find a mistake vector $(\varepsilon^k, \delta^k) \in U_k$ such that no equilibrium of $(A, B, \varepsilon^k, \delta^k)$ lies in V.

By construction $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ converges to zero. For every k we can take an equilibrium $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ such that S contains a limit point of the sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$. Therefore, in view of the compactness of $\Delta_m \times \Delta_n$, we may suppose without loss of generality that $\{(p^k, q^k)\}_{k\in\mathbb{N}}$, and hence also $\{(p(\varepsilon^k), q(\delta^k))\}_{k\in\mathbb{N}}$, converges to some $(p, q) \in S$. Since, by Lemma 3, $(p(\varepsilon^k), q(\delta^k)) \in E(A, B, \varepsilon^k, \delta^k)$ for all k, this contradicts the fact that $(p(\varepsilon^k), q(\delta^k)) \notin V$ for all k. ◁

In order to prove (in Theorem 2) a relation between strictly perfect sets and persistent equilibria (Kalai and Samet, 1984) we need the following lemma. First we give some definitions.

For an $m \times n$ bimatrix game (A, B), a closed and convex set $R \subset \Delta_m \times \Delta_n$ is called an *absorbing retract* if these exists an open neighborhood V of R such that for all $(p, q) \in V$ there exists a pair $(\hat{p}, \hat{q}) \in R$ where $\hat{p}$ is a best reply to q and $\hat{q}$ is a best reply to p. An absorbing retract which does not properly contain another absorbing retract is called a *persistent retract*. Kalai and Samet showed that every bimatrix game possesses a persistent retract and that every persistent retract contains an equilibrium. Equilibria contained in a persistent retract are called *persistent equilibria.*

LEMMA 5. *Let (A, B) be an $m \times n$ bimatrix game and let R be an absorbing retract for (A, B). Then R is also an absorbing retract for $(A(\delta), B(\varepsilon))$ for all mistake vectors $(\varepsilon, \delta) \in \mathbb{R}^m \times \mathbb{R}^n$ close to $(0, 0)$.*

Proof. Since R is an absorbing retract for (A, B), there is an open neighborhood V of R such that R *absorbs* V in the game (A, B); i.e., for all $(p, q) \in V$ there is a pair $(\hat{p}, \hat{q}) \in R$ such that $C(\hat{p}) \subset PB_1(A, q)$ and $C(\hat{q}) \subset PB_2(B, p)$. Let d be the metric on $\Delta_m \times \Delta_n$ defined by the supremum norm and let, for a positive real ρ,

$$B_\rho(R) := \{(x, y) \in \Delta_m \times \Delta_n | \inf_{(p,q)\in R} \{d((x, y), (p, q))\} \le \rho\}.$$

Since R is compact, we can choose ρ such that $B_\rho(R) \subset V$.

Let (ε, δ) be a mistake vector in $\mathbb{R}^m \times \mathbb{R}^n$ such that $\max\{\sum_{i=1}^m \varepsilon_i, \sum_{j=1}^n \delta_j\} \le \frac{1}{2}\rho$. Define $V(\varepsilon, \delta) := \{(p, q) \in V| (p(\varepsilon), q(\delta)) \in V\}$. Then $V(\varepsilon, \delta)$ is an open set and, for $(p, q) \in R$, $(p(\varepsilon), q(\delta)) \in B_\rho(R) \subset V$. This implies $(p, q) \in V(\varepsilon, \delta)$ and consequently $R \subset V(\varepsilon, \delta)$. So if we show that R absorbs $V(\varepsilon, \delta)$ in the game $(A(\delta), B(\varepsilon))$, the proof is complete.

Take $(p, q) \in V(\varepsilon, \delta)$. Then $(p(\varepsilon), q(\delta)) \in V$. Since R absorbs V in the game (A, B), there is a pair $(\hat{p}, \hat{q}) \in R$ with $C(\hat{p}) \subset PB_1(A, q(\delta))$ and $C(\hat{q}) \subset PB_2(B, p(\varepsilon))$. Since $e_iAq(\delta) = e_iA(\delta)q$ for all $i \in \{1, 2, \ldots, m\}$ and $p(\varepsilon)Be_j = pB(\varepsilon)e_j$ for all $j \in \{1, 2, \ldots, n\}$, this implies that $C(\hat{p}) \subset PB_1(A(\delta), q)$ and $C(\hat{q}) \subset PB_2(B(\varepsilon), p)$. So R absorbs $V(\varepsilon, \delta)$ in the game $(A(\delta), B(\varepsilon))$. ◁

4. Examples of Strictly Perfect Sets

In this section we introduce the set of extreme equilibria of a bimatrix game. In Theorem 1 we show that this set is strictly perfect. A direct consequence of Theorem 1 is that we have a new proof of the existence of a minimal strictly perfect set for bimatrix games. Further, in Theorem 2, we show that the equilibria contained in an absorbing retract form a strictly perfect set.

Let (A, B) be an $m \times n$ bimatrix game. We denote by $\mathscr{E}(A, B)$ the set of *extreme equilibria* of (A, B), i.e., the set of all extreme points of the maximal Nash subsets in $E(A, B)$. Clearly every equilibrium of (A, B) is a convex combination of some elements of $\mathscr{E}(A, B)$.

In order to decide whether a strategy pair (p, q) is an equilibrium of a bimatrix game we have to check the inclusions $C(p) \subset PB_1(A, q)$ and $C(q) \subset PB_2(B, p)$. For that purpose we need the 4-tuple

$$(C(p), PB_2(B, p), C(q), PB_1(A, q)).$$

We call this 4-tuple the *characteristic* of (p, q) and denote it by $\mathrm{Ch}(p, q)$. Note that in view of Lemma 2, a face for a bimatrix game can be characterized by the characteristic of an arbitrary element of its relative interior.

The relative interior of a face consisting of an extreme equilibrium is this equilibrium itself. Therefore Lemma 2 implies

LEMMA 6. *Let* (A, B) *be an* $m \times n$ *bimatrix game. Let* $(\mathring{p}, \mathring{q}) \in E(A, B)$. *Then* $(\mathring{p}, \mathring{q}) \in \mathscr{E}(A, B)$ *if and only if* $\mathrm{Ch}(\mathring{p}, \mathring{q}) \neq \mathrm{Ch}(p, q)$ *for all* $(p, q) \in \Delta_m \times \Delta_n$ *with* $(p, q) \neq (\mathring{p}, \mathring{q})$.

We use this result in the proof of

THEOREM 1. *The set of extreme equilibria of a bimatrix game is a strictly perfect set.*

Proof. Let (A, B) be an $m \times n$ bimatrix game. Throughout this proof we have multiplied, for convenience, the entries of $A(\delta)$ by $(1 - \sum_{j=1}^{n} \delta_j)^{-1}$ and the entries of $B(\varepsilon)$ by $(1 - \sum_{i=1}^{m} \varepsilon_i)^{-1}$.

Let $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ be a sequence of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero and let $(p^k, q^k) \in \mathscr{E}(A(\delta^k), B(\varepsilon^k))$ for all $k \in \mathbb{N}$. Suppose $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ converges to $(p, q) \in \Delta_m \times \Delta_n$. Then continuity arguments imply $(p, q) \in E(A, B)$. By Lemma 4 the proof is complete if we can show that $(p, q) \in \mathscr{E}(A, B)$.

So suppose $(p, q) \notin \mathscr{E}(A, B)$. Then, in view of Lemma 6, there exists a strategy pair $(\tilde{p}, \tilde{q}) \neq (p, q)$ such that $\mathrm{Ch}(\tilde{p}, \tilde{q}) = \mathrm{Ch}(p, q)$. Then we can take $\hat{p} \in \mathrm{conv}\{p, \tilde{p}\}$ close enough to p such that

$$\max_{i\in C(p)} |\hat{p}_i - p_i| < \tfrac{1}{2} \min_{i\in C(p)} p_i.$$

Note that $C(\hat{p}) = C(p)$ and $PB_2(B, \hat{p}) = PB_2(B, p)$. Define

$$\hat{p}^k := p^k + (\hat{p} - p).$$

Then $\{\hat{p}^k\}_{k\in\mathbb{N}}$ converges to $\hat{p}$.

For $i \notin C(p)$ we find $\hat{p}_i^k = p_i^k$. For $i \in C(p)$ and for large k we have $p_i^k > \frac{1}{2} \min_{l\in C(p)} p_l > 0$, and so $\hat{p}_i^k > 0$. Hence $C(\hat{p}^k) = C(p^k)$ and $\hat{p}^k \in \Delta_m$ for large k. Consider for $j \in \{1, 2, \ldots, n\}$

$$\begin{aligned}\hat{p}^k B(\varepsilon^k)e_j &= p^k B(\varepsilon^k)e_j + (\hat{p} - p)B(\varepsilon^k)e_j \\ &= p^k B(\varepsilon^k)e_j + (\hat{p} - p)Be_j.\end{aligned}$$

Since $PB_2(B, \hat{p}) = PB_2(B, p) \supset PB_2(B(\varepsilon^k), p^k)$ for large k, we have that $\hat{p}^k B(\varepsilon^k)e_r = \hat{p}^k B(\varepsilon^k)e_s$ for all $r, s \in PB_2(B(\varepsilon^k), p^k)$. We show that, for large k, $j \notin PB_2(B(\varepsilon^k), p^k)$ implies $j \notin PB_2(B(\varepsilon^k), \hat{p}^k)$. Let $j_0 \in PB_2(B(\varepsilon^k), p^k)$ and $j \notin PB_2(B(\varepsilon^k), p^k)$. We consider two cases.

(i) $j \in PB_2(B, p) = PB_2(B, \hat{p})$. Then

$$\begin{aligned}\hat{p}^kB(\varepsilon^k)e_j - \hat{p}^kB(\varepsilon^k)e_{j_0} \\ &= p^kB(\varepsilon^k)e_j + (\hat{p} - p)Be_j - p^kB(\varepsilon^k)e_{j_0} - (\hat{p} - p)Be_{j_0} \\ &= p^kB(\varepsilon^k)e_j - p^kB(\varepsilon^k)e_{j_0} < 0.\end{aligned}$$

(ii) $j \notin PB_2(B, p) = PB_2(B, \hat{p})$. Then, for large k,

$$\begin{aligned}\hat{p}^kB(\varepsilon^k)e_j - \hat{p}^kB(\varepsilon^k)e_{j_0} &= p^kB(\varepsilon^k)e_j + (\hat{p} - p)Be_j - \hat{p}^kB(\varepsilon^k)e_{j_0} \\ &= (\hat{p}Be_j - \hat{p}Be_{j_0}) + (p^kB(\varepsilon^k)e_j - pBe_j) \\ &\quad + (\hat{p}Be_{j_0} - \hat{p}^kB(\varepsilon^k)e_{j_0}) < 0,\end{aligned}$$

since the last two terms converge to zero as k goes to infinity. Hence (i) and (ii) imply $PB_2(B(\varepsilon^k), \hat{p}^k) = PB_2(B(\varepsilon^k), p^k)$.

Similarly we construct a sequence $\{\hat{q}^k\}_{k\in\mathbb{N}}$ converging to $\hat{q} \in \text{conv}\{q, \tilde{q}\}$, such that $\hat{q}^k \in \Delta_n$, $C(\hat{q}^k) = C(q^k)$, $PB_1(A(\delta^k), \hat{q}^k) = PB_1(A(\delta^k), q^k)$ for large k. Consequently we find that for k large enough $(\hat{p}^k, \hat{q}^k)$ is an equilibrium of $(A(\delta^k), B(\varepsilon^k))$ and $\text{Ch}(\hat{p}^k, \hat{q}^k) = \text{Ch}(p^k, q^k)$. However, by construction $(\hat{p}^k, \hat{q}^k) \neq (p^k, q^k)$. Therefore, using Lemma 6, this contradicts the fact that $(p^k, q^k) \in \mathscr{E}(A(\delta^k), B(\varepsilon^k))$. ◁

Since for a bimatrix game the set of extreme equilibria is finite, it contains a minimal strictly perfect set. Consequently we have a new proof of

COROLLARY 1. *Every bimatrix game possesses a minimal strictly perfect set.*

Since there exist characterizations for the set of extreme equilibria of a bimatrix game (cf. Vorob'ev, 1958; Kuhn, 1961; and Jurg and Jansen, 1989), we also have a characterization for a (finite) strictly perfect set. Since every minimal strictly perfect set consists only of perfect equilibria (cf. Kohlberg and Mertens, 1986), it is enough, in order to find a minimal strictly perfect set which is a subset of the set of extreme equilibria, to consider the extreme equilibria that are perfect.

In the following example we deal with a game for which every minimal strictly perfect set consists of extreme equilibria only.

EXAMPLE 1. Consider the 2×4 bimatrix game (A, B) given by

$$(A, B) = \begin{bmatrix} (0, -5) & (1, -2) & (0, 2) & (0, 3) \\ (0, 3) & (0, 2) & (1, -2) & (0, -5) \end{bmatrix}.$$

The maximal Nash subsets for this game are $\text{conv}\{e_2, \frac{1}{4}e_1 + \frac{3}{4}e_2\} \times \{e_1\}$, $\{(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_2 + \frac{1}{2}e_3)\}$ and $\text{conv}\{e_1, \frac{3}{4}e_1 + \frac{1}{4}e_2\} \times \{e_4\}$.

There are two minimal strictly perfect sets: $\{(e_2, e_1), (e_1, e_4)\}$ and $\{(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_2 + \frac{1}{2}e_3)\}$. (For a $2 \times n$ bimatrix game all minimal strictly perfect sets consist of one or two elements, as Borm (1990) showed.) Note that both sets are contained in $\mathscr{E}(A, B)$ and that the first set does not lie within any connected subset of $E(A, B)$.

The following theorem, which can be proved with the help of Lemmas 4 and 5 and along similar lines to the proof of Theorem 1, shows that a persistent retract always contains a minimal strictly perfect set.

THEOREM 2. *For a bimatrix game the set of equilibria contained in an absorbing retract is strictly perfect.*

5. The Finiteness of Minimal Strictly Perfect Sets

In the previous section we found that each bimatrix game possesses a strictly perfect set that is finite. In this section we show that each strictly perfect set contains a strictly perfect set which is finite. In particular this implies that for bimatrix games every minimal strictly perfect set is finite.

Let (A, B) be an $m \times n$ bimatrix game. Let C be a strictly perfect set for (A, B). For every face F for (A, B) such that $\text{relint}(F) \cap C \neq \emptyset$ we select a single equilibrium in $\text{relint}(F)$. A set of equilibria selected in this way is called a *selection for C*. Since there are only finitely many faces for (A, B), each selection for C is finite.

Using the techniques of the previous section we show that each selection for C is a strictly perfect set for (A, B). First we need a result on the convergence of faces for perturbed games corresponding to (A, B).

We start with the set $\mathscr{F}$ of non-empty closed subsets of $\Delta_m \times \Delta_n$. Let d denote the metric on $\Delta_m \times \Delta_n$ corresponding to the supremum norm. Then we can define the Hausdorff-metric d_H on $\mathscr{F}$ and since $(\Delta_m \times \Delta_n, d)$ is a compact metric space, $(\mathscr{F}, d_H)$ is a compact metric space also (Hildenbrand, 1974). Furthermore,

LEMMA 7. [cf. Hildenbrand (1974)]. *If a sequence $\{F^k\}_{k\in\mathbb{N}}$ converges in $(\mathscr{F}, d_H)$ to F, then $(x, y) \in F$ if and only if there is a sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ converging to (x, y) such that $(x^k, y^k) \in F^k$ for every $k \in \mathbb{N}$.*

Now we prove a lemma for faces that is similar to Lemma 6.

LEMMA 8. *Let (A, B) be an $m \times n$ bimatrix game. Let $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ be a sequence of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero, and for* *all $k \in \mathbb{N}$, let F^k be a face for $(A(\delta^k), B(\varepsilon^k))$. Suppose $\{F^k\}_{k\in\mathbb{N}}$ converges in $(\mathcal{F}, d_H)$ to F. Then F is a face for (A, B).*

Proof. Take $(\bar{p}, \bar{q}) \in F$. By Lemma 7 there is a sequence $\{(\bar{p}^k, \bar{q}^k)\}_{k\in\mathbb{N}}$, such that $(\bar{p}^k, \bar{q}^k) \in F^k \subset E(A(\delta^k), B(\varepsilon^k))$ for all k, that converges to $(\bar{p}, \bar{q})$. Continuity arguments imply $(\bar{p}, \bar{q}) \in E(A, B)$. Hence $F \subset E(A, B)$.

Since F^k is convex for all k, it is, using Lemma 7, straightforward to show that F is convex. Consequently we can find a face $\tilde{F}$ for (A, B) such that $F \subset \tilde{F}$ and $\text{relint}(\tilde{F}) \cap F \neq \emptyset$. If we can show that $\text{relint}(\tilde{F})\backslash F = \emptyset$, then $\tilde{F} = F$ and we are finished.

So suppose $\text{relint}(\tilde{F})\backslash F \neq \emptyset$ and take $(\tilde{p}, \tilde{q}) \in \text{relint}(\tilde{F})\backslash F$ and $(x, y) \in \text{relint}(\tilde{F}) \cap F$. Then $\text{conv}\{(x, y), (\tilde{p}, \tilde{q})\} \subset \text{relint}(\tilde{F})$. Since F is a closed and convex set, we can find a $(p, q) \in \text{conv}\{(x, y), (\tilde{p}, \tilde{q})\}$ such that conv $\{(p, q), (\tilde{p}, \tilde{q}))\} \cap F = \{(p, q)\}$. Since $(p, q), (\tilde{p}, \tilde{q}) \in \text{relint}(\tilde{F})$, Lemma 2 implies $\text{Ch}(p, q) = \text{Ch}(\tilde{p}, \tilde{q})$. In view of Lemma 7 we can find a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ with $(p^k, q^k) \in F^k$ for all k, converging to (p, q). Then, exactly as in the proof of Theorem 1, we can construct $(\hat{p}, \hat{q}) \in \text{conv}\{(p, q), (\tilde{p}, \tilde{q})\}$ and a sequence $\{(\hat{p}^k, \hat{q}^k)\}_{k\in\mathbb{N}}$ in $\Delta_m \times \Delta_n$ converging to $(\hat{p}, \hat{q})$. Similarly, we find that $\text{Ch}(\hat{p}^k, \hat{q}^k) = \text{Ch}(p^k, q^k)$ for large k. Using Lemma 2 again, we then find $(\hat{p}^k, \hat{q}^k) \in F^k$ for large k and hence Lemma 7 yields $(\hat{p}, \hat{q}) \in F$. However, by construction $(\hat{p}, \hat{q}) \notin F$. So we have a contradiction. ◁

We need Lemma 8 for the proof of

THEOREM 3. *Let C be a strictly perfect set for a bimatrix game. If Σ is a selection for C, then Σ is a strictly perfect set.*

Proof. Let C be a strictly perfect set for an $m \times n$ bimatrix game (A, B). Let Σ be a selection for C and let $\{(\varepsilon_k, \delta_k)\}_{k\in\mathbb{N}}$ be a sequence of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero. Since C is a strictly perfect set, there exists, in view of Lemma 4, a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ with a limit point $(p, q) \in C$ such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for all $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$, let $\bar{F}^k$ be the face for $(A(\delta^k), B(\varepsilon^k))$ such that $(p^k, q^k) \in \text{relint}(\bar{F}^k)$. Without loss of generality we may suppose that $\bar{F}^k$ converges in $(\mathcal{F}, d_H)$ to a closed subset $\bar{F}$ of $\Delta_m \times \Delta_n$. By Lemma 8, $\bar{F}$ is a face for (A, B) and $(p, q) \in \bar{F}$. Let F be the face for (A, B) with $(p, q) \in \text{relint}(F)$. Then $F \subset \bar{F}$. This implies that

$$\{(\bar{p}, \bar{q})\} := \text{relint}(F) \cap \Sigma \subset \bar{F}. \tag{3}$$

Hence, by Lemma 7, there exists a sequence $\{(\bar{p}^k, \bar{q}^k)\}_{k\in\mathbb{N}}$ converging to $(\bar{p}, \bar{q})$ such that $(\bar{p}^k, \bar{q}^k) \in \bar{F}^k$ for all k. Then Lemma 4 implies that Σ is a strictly perfect set. ◁

According to the definition we can always take a selection for a strictly perfect set which is a subset of this strictly perfect set. So Theorem 3 implies that for a bimatrix game every strictly perfect set contains a finite strictly perfect set. Hence

COROLLARY 2. *For a bimatrix game every minimal strictly perfect set is finite.*

If (p, q) is a *strictly perfect equilibrium* for a bimatrix game (A, B) (i.e., the set $\{(p, q)\}$ is strictly perfect) and $(p, q) \in \text{relint}(F)$, where F is a face for $(\tilde{P}, B)$, then by selecting $(A, \tilde{q}) \in F$ in (3) of the proof of Theorem 3, one can show that all elements of F are strictly perfect equilibria. Hence

THEOREM 4. *For a bimatrix game the set of strictly perfect equilibria, if non-empty, is the finite union of faces for this game.*

A similar result with respect to perfect equilibria can be found in Borm *et al.* (1988).

6. CARRIERS OF MINIMAL STRICTLY PERFECT SETS

In this section we show that, although a bimatrix game may possess infinitely many minimal strictly perfect sets, one can distinguish only a finite number of essentially different minimal strictly perfect sets.

Let S be a minimal strictly perfect set for a bimatrix game (A, B). If F is a face for (A, B) such that $(x^1, y^1), (x^2, y^2) \in \text{relint}(F) \cap S$ and $(x^1, y^1) \neq (x^2, y^2)$, then $S \setminus \{(x^1, y^1)\}$ is a selection for S, and hence, by Theorem 3, a strictly perfect set. Since this contradicts the assumption that S is a minimal strictly perfect set, for all faces F for (A, B) we have $|\text{relint}(F) \cap S| \in \{0, 1\}$.

We call the set $\Phi(S)$ of those faces for which the relative interior has exactly one point in common with S the *carrier* of S.

LEMMA 9. *Let S be a minimal strictly perfect set for a bimatrix game. If Σ is a selection for S, then Σ is a minimal strictly perfect set and $\Phi(\Sigma) = \Phi(S)$.*

Proof. By Theorem 3, Σ is a strictly perfect set. Hence Σ contains a minimal strictly perfect set Σ'. Let S' consist of those elements of S that lie in the relative interior of a face $F \in \Phi(\Sigma')$. Then $|S'| = |\Sigma'| \leq |\Sigma| = |S|$. By definition S' is a selection for Σ' and, in view of Theorem 3, a strictly perfect set. Since S is a minimal strictly perfect set, this implies $S' = S$. Hence $\Sigma' = \Sigma$ and Σ is a minimal strictly perfect set. Evidently $\Phi(\Sigma) = \Phi(S)$. ◁

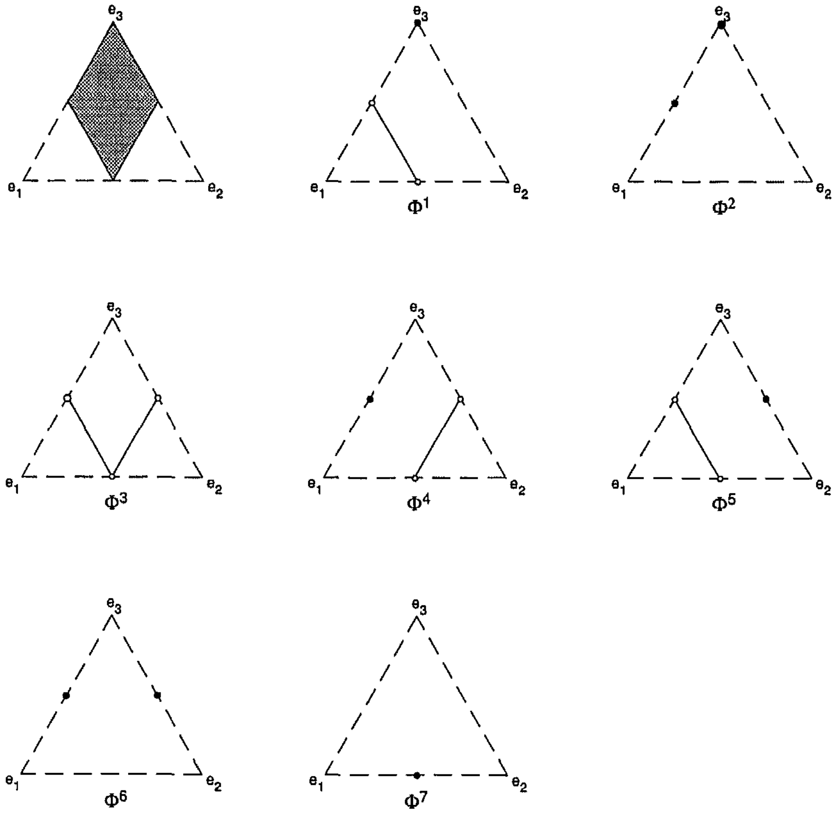

FIG. 1. The carriers for the bimatrix game (A, B).

Since two minimal strictly perfect sets S and T are essentially the same if their carriers coincide, we say that S is *equivalent* with T and denote $S \sim T$, if $\Phi(S) = \Phi(T)$. One easily checks that the relation $\sim$ is an equivalence relation for minimal strictly perfect sets. Since there are finitely many faces for a bimatrix game, there are also finitely many carriers for minimal strictly perfect sets. Hence

THEOREM 5. *For a bimatrix game there are finitely many equivalence classes of minimal strictly perfect sets.*

In the following example we give all the carriers for a 3×3 bimatrix game.

EXAMPLE 2. Consider the 3×3 bimatrix game (A, B) given by

$$(A,B) = \begin{bmatrix} (1,0) & (1,0) & (1,0) \\ (2,0) & (0,1 & (0,2) \\ (0,2) & (2,1) & (0,0) \end{bmatrix}.$$

The maximal Nash subsets for this game are $\{e_1\} \times \text{conv}\{e_3, \frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_1 + \frac{1}{2}e_3, \frac{1}{2}e_2 + \frac{1}{2}e_3\}$ and $\text{conv}\{e_1, \frac{1}{2}e_3 + \frac{1}{2}e_3\} \times \{\frac{1}{2}e_1 + \frac{1}{2}e_2\}$.

All minimal strictly perfect sets are $\{(e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2)\}$, $\{e_1, q^1), (e_1, e_3)\}$ with $q^1 \in \text{conv}\{(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_1 + \frac{1}{2}e_3\} \backslash \{\frac{1}{2}e_1 + \frac{1}{2}e_2\}$ and $\{(e_1, q^1), (e_1, q^2)\}$ with q^1 as above and $q^2 \in \text{conv}\{\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_2 + \frac{1}{2}e_3\} \backslash \{\frac{1}{2}e_1 + \frac{1}{2}e_2\}$.

The minimal strictly perfect sets are all contained in the first maximal Nash subset which is of the type $\{e_1\} \times F$ (F is the dashed area in the first picture of Fig. 1). Hence with respect to the seven carriers as described in Fig. 1, we only consider the part of the faces contained in the strategy space of the second player.

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