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THE ROLE OF MIDDLEMEN IN EFFICIENT AND STRONGLY PAIRWISE STABLE NETWORKS

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The Role of Middlemen in Efficient and Strongly Pairwise Stable Networks*

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Abstract

We examine the strong pairwise stability concept in network formation theory under collective network benefits. Strong pairwise stability considers a pair of players to add a link through mutual consent while permitting them to unilaterally delete any subset of links under their control. We examine the properties of strongly pairwise stable networks and find that players in middleman positions, who have the power to break up the network into multiple components, play a critical role in such networks. We show that for the component-wise egalitarian rule there is no conflict between the efficient and stable networks when these middlemen have no incentive to break up the network. Finally, we examine efficiency and stability in middleman-free networks.

Keywords: Networks; pairwise stability; critical link; middleman.

JEL Classification: C71, C72.

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1 Introduction

Networks play a significant role in the economic life of individuals. Their functions include such wide ranging tasks as the dissemination of information, the creation of synergies, facilitating affordable forms of economic exchange, and the enforcement of norms.

The literature on networks in economics has witnessed an upsurge both in terms of interest and clarity of the issues related to the fundamental principles of network formation with the publication of the seminal contribution of Jackson and Wolinsky (1996) on link-based stability concepts in a game theoretic approach to network formation. The literature currently covers theories of the formation of diverse network structures such as networks between acquaintances (Brueckner 2003, Gilles and Sarangi 2004), trade networks (Goyal and Joshi 1999, Kranton and Minehart 2001, Furusawa and Konishi 2002), labor markets as contact networks (Montgomery 1991, Calvo-Armengol and Jackson 2004), information exchange networks (Bala and Goyal 2000, Haller and Sarangi 2003), and the Internet (Badasyan and Chakrabarti 2004).[1]

The most fundamental insight put forward by Jackson and Wolinsky (1996) is that there is a profound tension between efficiency and stability in game theoretic models of network formation. Indeed, networks that generate maximal collective values — indicated as *efficient networks* — are usually not stable in the sense that players have incentives to delete existing links or create new links. Since their influential paper many authors have discussed this fundamental tension between efficiency and stability of social networks.[2]

In this paper we examine the role of middlemen in attaining efficiency in stable networks. Our paper examines the conflict between efficiency and stability in networks with middlemen and identifies circumstances under which this tension is resolved. As a corollary we also examine the implications for networks that do not have middlemen.

Middlemen are individuals with positional power who can disrupt a network by disconnecting it. They can play a variety of roles in networks from acting as matchmakers who reduce costs of waiting by bringing together buyers and sellers (Rubinstein and Wolinsky 1987), to experts who who can resolve information asymmetries (Klein and Leffler 1981, Biglaiser 1993) or just disseminate information about quality (Biglaiser and Friedman 1994).

To understand the role of middlemen in networks we use a link-based stability concept

[1]There is also a relatively large literature on networks in other disciplines like sociology, operations research, and physics. Here we refer to, for example, Barabasi (2002) and Watts (2003). The difference of the treatment in economics is that nodes are agents capable of volition, and hence capable of forming and deleting links.

[2]For a discussion of this strand of the literature we refer to the excellent review by Jackson (2003).

called *strong pairwise stability* formalized by Gilles and Sarangi (2004).[3] Unlike pairwise stability (Jackson and Wolinsky 1996) where a pair of players can either add or sever a single link at a time, strong pairwise stability takes gives players the ability to delete one *or more* of the links in which they participate. The creation of a link requires mutual consent and hence as in pairwise stability players considering forming only one link at a time. On the other hand, breaking a relationship in the network is an unilateral act and, consequently, under strong pairwise stability a player can delete any subset of her links.

The ability to delete multiple links is a realistic modification of pairwise stability that provides us with a more natural stability concept. This stability concept is also a hybrid between pairwise stability and the notion of so-called Nash networks since like pairwise stability it considers the addition of a single link at a time while permitting the deletion of multiple links by a player at the same time as in Nash equilibrium. It can be shown that for certain normal form game-theoretic models of network formation, Nash equilibria are characterized by stability against the removal of sets of links by individual players. (Gilles and Sarangi 2004, Propositions 3.1 and 3.10) This is also recognized by Goyal and Joshi (2003) and Bloch and Jackson (2004) who discuss pairwise stable equilibria. This concept combines the Nash equilibrium property with stability against pairs of players forming additional links. This notion is therefore closely related to strong pairwise stability. It should also be clear that pairwise stable equilibrium networks can only be investigated in the context of (non-cooperative) network formation games.

Further, it has been argued that pairwise stability is a relatively weak concept since it admits a relatively large number of networks. On the other hand the notion of strong stability (Jackson and van den Nouweland 2004) or strong group stability (Konishi and Utku Ünver 2003) of networks are in many ways too strong; admitting too few networks. Thus there is a need for intermediate notions of stability.[4] We argue that strong pairwise stability is one such concept.

Given that strong pairwise stability allows an agent to delete multiple links and even disconnect the network, it shifts the focus from individual links (as in pairwise stability) to the player herself. Thus, it provides a natural modelling tool for studying the role of middlemen in networks. It allows us to focus on their positional power in the network. With the exception of Kalai, Postlewaite, and Roberts (1978) the issue of middlemen in networks

[3]We remark that Jackson and Wolinsky (1996) already indicated without formalizing, several generalizations of their pairwise stability concept, including what we call strong pairwise stability in this paper. Bloch and Jackson (2004) also use the notion of strong pairwise stability, but label it as pairwise stability*. Closely related to this is also the notion of pairwise stable equilibrium studied by Goyal and Joshi (2003). For a discussion of these two concepts we refer mainly to Bloch and Jackson (2004).

[4]For a survey of the recent theoretical developments in the networks literature we refer the reader to Jackson (2003), Goyal (2004) and Slikker and van den Nouweland (2001).

remains largely unexplored. They measure the power of middlemen in core allocations of a 3-person exchange economy. Interestingly they find that players occupying a middleman position need not always be better off. One of their main findings is that, if preferences are strictly monotonic and trade through the middleman is beneficial to the grand coalition, then there do exist points in the core where the middleman is better off.

Here, instead of investigating when middlemen are better off, we look at the relationship between stability and efficiency and find some similarities with the results of Kalai, Postlewaite, and Roberts (1978). We show that for strong pairwise stability the coincidence of efficiency and stability occurs for component-wise egalitarian payoffs. Jackson and Wolinsky (1996) showed that so-called critical links have to be neutralized in order to establish pairwise stable and efficient networks. Here we establish that middlemen in the network have to be secured in the sense that they have no incentives to break communication in the network.

A middleman occupies a critical position in the network in that she can disconnect communication lines by removing certain links under her control. A secure middleman will not disrupt the functioning of the social network because they tend to lose more. In a related study, anthropologist Jean Ensminger has argued that those who occupy central positions in the social network of the Orma tribe in Kenya behave more fairly in dictator and trust games since they have more to lose. Ensminger further argues that persons occupying the middlemen positions in the Orma social network can act as agents of change for social norms. Consequently, since fairness and reputation matter more in market-based economies, the middlemen try to instill these values since they benefit the most from markets (Ensminger and Knight 1997, Knight and Ensminger 1998). Thus when the incentives of the middlemen are aligned with those of the others players it is possible to generate maximal collective values in the network.

We find that for middleman-free networks, the component-wise egalitarian rule ensures that efficient networks are also strongly pairwise stable. The intuition for this is quite simple. Since no one has any positional advantage, the (component) egalitarian rule is adequate to resolve the tension between stability and efficiency. In a sense if any player attempts to exploit the network, given that all players occupy the same position in the network, the others can easily disconnect a player that tries to expropriate more than his fair share.

The rest of this paper is organized as follows. Sections 2 and 3 introduce network modelling principles and the different stability concepts mentioned above. Section 4 is devoted to strongly pairwise stable networks and its relation to earlier work in the literature. Section 5 is about networks with middlemen and in Section 6 we present networks without middlemen. Section 7 concludes.

2 Modelling principles

In this section we define the formal elements used in describing network formation, including some concepts borrowed from graph theory. This is followed by the description of generation of (collective) value and its allocation in a network.

2.1 Networks and network components

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. Two distinct players $i, j \in N$ are *linked* if i and j are mutual partners in some social or economic activity. This could range from an exchange network to a group involved in an economically productive relationship to an ethnic social network that provides information about new job openings. The two players forming a link are assumed to be "equals" within the relationship, as no player has the power to coerce the other into forming or staying in the relationship. Thus we restrict our attention only to *undirected* networks or graphs. We allow for the possibility that these relationships have spillover effects on the network relations between other players. This is captured by the formal description of such network benefits.

Formally, an (undirected) link between i and j is defined as the set $\{i, j\}$ and we use the shorthand notation ij to denote this link. Clearly ij is equivalent to ji.

The player set N permits a total of $\frac{1}{2}n(n-1)$ potential links. The collection of these potential links on N is denoted by

$$g_N = \{ij \mid i, j \in N \text{ and } i \neq j\} \tag{1}$$

A *network* g is now defined as any collection of links $g \subset g_N$. The collection of all networks on N is denoted by $\mathbb{G}^N = \{g \mid g \subset g_N\}$ and consists of $2^{\frac{1}{2}n(n-1)}$ networks. The network g_N composed of all possible links is called the *complete network* on N and the network $g_0 = \varnothing$ consisting of no links is the *empty network* on N.

Let $\pi \colon N \to N$ be a permutation on N. For every network $g \in \mathbb{G}^N$ the corresponding permutation is denoted by $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\} \in \mathbb{G}^N$. Two networks $g, h \in \mathbb{G}^N$ have the same topology if there exists a permutation $\pi \colon N \to N$ such that $h = g^\pi$. This is denoted as $g \sim h$. For $g \in \mathbb{G}^N$ the corresponding *network topology* is denoted by $\overline{g} = \{h \in \mathbb{G}^N \mid h \sim g\}$. Clearly a network topology is a mathematical *equivalence class* with regard to the binary relationship $\sim$. It is obvious that the collection of all networks $\mathbb{G}^N$ is partitioned into network topologies.

For every network $g \in \mathbb{G}^N$, and every player $i \in N$, we denote i's *neighborhood* in g by $N_i(g) = \{j \in N \mid j \neq i \text{ and } ij \in g\}$. Player i therefore (directly) interacts with those in her *link set* $L_i(g) = \{ij \in g \mid j \in N_i(g)\} \subset g$. We also define $N(g) = \cup_{i \in N} N_i(g)$ and let $n(g) = \# N(g)$ with the convention that if $N(g) = \varnothing$, we let $n(g) = 1$.[5]

A *path* in g connecting players i and j is a set of distinct players $\{i_1, i_2, \ldots, i_p\} \subset N(g)$ with $p \geqslant 2$ such that $i_1 = i$, $i_p = j$, and $\{i_1 i_2, i_2 i_3, \ldots, i_{p-1} i_p\} \subset g$. A path between two distinct players $i, j \in N$ (assuming that a path exists between i and j) is *shortest* if it consists of a minimal number of players. Note that a shortest path between i and j contains one and only one member of the neighborhood set $N_i(g)$, as well as one and only one member of the neighborhood set $N_j(g)$. The set of all shortest paths is denoted by $P_{ij}(g)$. If there is no path between i and j, then $P_{ij}(g) = \varnothing$.

Let $t_{ij}(g)$ denote the *geodesic distance* between i and j, which is defined as follows: If $P_{ij}(g) = \varnothing$, then $t_{ij}(g) = \infty$. Otherwise, $t_{ij}(g) = \left| N(p_{ij}(g)) \right| - 1$.

The network $g' \subset g$ is a *component* of g if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$. In other words, a component is simply a maximally connected subnetwork of g. We denote the class of network components of the network g by $C(g)$. The set of players that are not connected in the network g are collected in the set of (fully) disconnected or isolated players in g denoted by

$$N_0(g) = N \setminus N(g) = \{i \in N \mid N_i(g) = \varnothing\}.$$

Furthermore, we define

$$\Gamma(g) = \{N(h) \mid h \in C(g)\} \cup \{\,\{i\} \mid i \in N_0(g)\} \tag{2}$$

as the partitioning of the player set N based on the component structure of the network g.[6]

2.2 Collective network benefits and efficiency

We describe the benefits or "utilities" generated by participation in a network through a *collective network benefit functions* given by $v \colon \mathbb{G}^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. Following Jackson and Wolinsky (1996), we refer to such functions as "network value" functions. A network value function v assigns a total benefit $v(g) \in \mathbb{R}$ to the network $g \in \mathbb{G}^N$. The space

[5] We emphasize here that if $N(g) \neq \varnothing$, we have that $n(g) \geqslant 2$. Namely, in those cases the network has to consist of at least one link.

[6] We therefore distinguish a link-based partitioning of a network g into components, denoted by $C(g)$, from a node-based partitioning denoted by $\Gamma(g)$. Both conventions are necessary to analyze the role middlemen who represent a special type of node with multiple links to others that could potentially lead to different components.

of all network value functions v such that $v(\varnothing) = 0$ is denoted by $\mathbb{V}^N$. It is clear that $\mathbb{V}^N$ is a $\left(2^{\frac{1}{2}n(n-1)} - 1\right)$-dimensional Euclidean vector space.

Let $v \in \mathbb{V}^N$ be some network value function. We now define two useful properties of such a network value function:

(i) The network value function v is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$. Component additivity requires that the total value generated in a network is the sum of the values generated in each component. An immediate consequence of component additivity is the fact that isolated players $i \in N_0(g)$ generate no value.

(ii) The network value function v is *anonymous* if $v(g^\pi) = v(g)$ for all permutations π and networks g. Anonymity implies that the benefits $v(g)$, depend only on the topology of the network g.

Finally, we define the notion of network efficiency using the collective benefits generated by the network. A network $g \in \mathbb{G}^N$ is *efficient* with respect to value function v if $v(g) \geqslant v(g')$ for all $g' \subset g_N$.[7]

2.3 Allocation rules

Next, we discuss the problem of allocating these collective network benefits or "values" amongst the members of a network. The payoff to an individual player is given by an *allocation rule* $Y \colon \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ which determines how the collective value is distributed over the individual players. Thus $Y_i(g, v)$ is the payoff to player i from the network g under the value function v. We now define some appealing properties for an allocation rule.

Recall that $\pi \colon N \to N$ is a permutation. Let v^π be defined by $v^\pi(g^\pi) = v(g)$.

(i) An allocation rule Y is *anonymous* if for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$. Anonymity of the allocation rule simply means that the payoff of a player depends solely on their position in the network rather than the label of the players.

(ii) An allocation rule Y is *balanced* if $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g. [8]

(iii) An allocation rule Y is *component balanced* if $\sum_{i \in N(h)} Y_i(g, v) = v(h)$ for every g and $h \in C(g)$ and every component additive v.

Remark 2.1 *We note that component balance implies balance for every component additive network value function. Also, component balance along with component additivity implies that fully disconnected players in $N_0(g)$ always have an allocated payoff of zero.*

[7] In the literature these are also referred to as strongly efficient networks. See for instance Jackson and Wolinsky (1996).

[8] Balance is also known as "efficiency" in the literature.

Let $v \in \mathbb{V}^N$. The *component-wise egalitarian allocation rule* is defined by

$$Y_i^{ce}(g, v) = \frac{v(h_i)}{n(h_i)} \tag{3}$$

where $h_i \in C(g)$ such that $i \in N(h_i)$ and $h_i = \emptyset$ if there is no $h \in C(g)$ such that $i \in N(h)$. Under this allocation rule, the value generated by a component is split equally among the members of that component.

Remark 2.2 *The component-wise egalitarian allocation rule Y^{ce} is the unique allocation rule Y that is component balanced and assigns an equal payoff to all players in the same component of a network, i.e., for all $(g, v) \in \mathbb{G}^N \times \mathbb{V}^N$ it holds that*

$$Y_i(g, v) = Y_j(h, v) \tag{4}$$

for every $h \in C(g)$ and all $i, j \in N(h)$.

Finally we mention that $Y^{ce}(\cdot, v)$ is balanced for every component additive $v \in \mathbb{V}^N$. The component-wise egalitarian payoff rule is not balanced for arbitrary network value functions. Equation (4) implies also that the component-wise egalitarian allocation rule is anonymous.

3 Stability properties

A network is a collection of links. It is the culmination of a process in which players establish links or sever existing links. In this section we discuss the principles underlying network formation and their stability from a link-based perspective. The central tenet of our approach is that in principle, the formation of each link must be considered separately. Each link in the network involves a pair of players and its formation requires the mutual consent of those two players. Thus the creation of a link has to be considered one at a time. However, each player can delete a link unilaterally. Therefore we consider stability with respect to the deletion of links and the addition of links separately.

We first introduce some auxiliary notation. Denote by $g + ij$ the network obtained by adding link ij to the existing network g, i.e., $g + ij = g \cup \{ij\}$. Similarly, $g - ij$ denotes the network that results from deleting link ij from the existing network g, i.e., $g - ij = g \setminus \{ij\}$.

Let Y be some allocation rule. We discuss three fundamental network stability properties that encapsulate the network formation principles described above.

(i) A network $g \in \mathbb{G}^N$ is *link deletion proof* (LDP) if for every player $i \in N$ and every neighbor $j \in N_i(g)$, it holds that $Y_i(g - ij, v) \leqslant Y_i(g, v)$. Link deletion proofness requires that each individual player has no incentive to sever an existing link with one of her neighbors.

(ii) A network $g \in \mathbb{G}^N$ is *strong link deletion proof* (SLDP) if for every player $i \in N$ and every set of neighbors $M \subset N_i(g)$, it holds that $Y_i(g \setminus h_M, v) \leqslant Y_i(g, v)$ where $h_M = \{ij \in g \mid j \in M\} \subset L_i(g)$. Strong link deletion proofness requires that each player has no incentive to sever links with one or more of his neighbors. Obviously, SLDP implies LDP.

(iii) A network $g \in \mathbb{G}^N$ is *link addition proof* if for all players $i, j \in N$, it holds that $Y_i(g + ij, v) > Y_i(g, v)$ implies $Y_j(g + ij, v) < Y_j(g, v)$. Link addition proofness states that there are no incentives to form additional links. This is founded on a process of mutual consent in link formation. Indeed, when one player would like to add a link, the other player could have strong objections.[9]

Jackson and Wolinsky (1996) introduced link deletion proofness and link addition proofness, although they did not explicitly define these concepts as such. Strong link deletion proofness was introduced recently by Gilles and Sarangi (2004).

These three fundamental stability concepts can be used to define additional stability concepts. A network $g \in \mathbb{G}^N$ is *pairwise stable* if it is link deletion proof and link addition proof. Furthermore, a network $g \in \mathbb{G}^N$ is *strongly pairwise stable* if it is strong link deletion proof and link addition proof.

The main difference between the regular pairwise stability and strong pairwise stability is that individual players are allowed to remove multiple links rather than a single link under their control. This is the same as the difference between LDP and SLDP.

We first remark that strong pairwise stability is a natural link-based stability concept. Since links require mutual consent, it considers the addition of one link at a time. However, link deletion is unilateral and, hence, it allows a single player to delete multiple links at the same time. Thus, while pairwise stability can only focus on links, by permitting deletion of multiple links strong pairwise stability allows us to focus on links as well as the players who form these links. Second, Goyal and Joshi (2003) discuss positive and negative spillovers in networks in relation to strong pairwise stability and show that a large class of network topologies satisfy this property. They show that in games with positive spillovers where the players are playing against the field, a strongly pairwise stable network is either empty, or complete, or has a dominant group topology. With negative spillovers it is possible to obtain the empty networks and stars as strongly pairwise stable networks. Moreover, regular (or symmetric) and irregular networks with unequal connections are possible with negative spillovers in the playing field games.

[9] Considering one link at a time with regard to link formation in this fashion seems natural. A generalization to the simultaneous formation of multiple links wold not yield much unless it is incorporated within a coalitional framework. Such coalitional considerations are the foundation of the notion of strong stability introduced and analyzed by Jackson and van den Nouweland (2004).

We now provide a simple three player network formation game that demonstrates the natural advantages of strong pairwise stability over (regular) pairwise stability. It illustrates that pairwise stability has some serious limitations in the sense that individual players do not have the ability to delete multiple links even in situations where this is extremely desirable.

Example 3.1 Being stuck in bad company
Consider a three player situation with $N = \{1, 2, 3\}$. For simplification of notation we denote the potential links in this situation as follows: $a = 12$, $b = 13$, and $c = 23$. Hence, $\mathbb{G}^N = \{\emptyset, a, b, c, ab, ac, bc, abc\}$.
Let $\alpha > 0$. We consider an allocation rule $\overline{Y} \colon \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}$ which for every $v \in \mathbb{V}^N$ is defined by

$$\begin{aligned}
&\overline{Y}(\emptyset, v) = (0, 0, 0)\\
&\overline{Y}(a, v) = \left(\tfrac{v(a)}{2}, \tfrac{v(a)}{2}, 0\right)\\
&\overline{Y}(b, v) = \left(\tfrac{v(b)}{2}, 0, \tfrac{v(b)}{2}\right)\\
&\overline{Y}(c, v) = \left(0, \tfrac{v(c)}{2}, \tfrac{v(c)}{2}\right)\\
&\overline{Y}(ab, v) = (v(ab), 0, 0)\\
&\overline{Y}(ac, v) = \left(-\alpha v(abc), v(ac) - \tfrac{1}{2}(1 - \alpha)v(abc), \tfrac{1}{2}(1 + \alpha)v(abc)\right)\\
&\overline{Y}(bc, v) = \left(-\alpha v(abc), \tfrac{1}{2}(1 + \alpha)v(abc), v(bc) - \tfrac{1}{2}(1 - \alpha)v(abc)\right)\\
&\overline{Y}(abc, v) = \left(-\alpha v(abc), \tfrac{1}{2}(1 + \alpha)v(abc), \tfrac{1}{2}(1 + \alpha)v(abc)\right)
\end{aligned}$$

Note that $\overline{Y}$ is component balanced. Our main claim is that in general, under the allocation rule $\overline{Y}$, the complete network abc is LDP, but not SLDP:

Claim: *If $v \in \mathbb{V}^N$ such that $v(g) > 0$ for every $g \neq \emptyset$, then the network $g^\star = abc$ is link deletion proof, but not strong link deletion proof, with respect to the allocation rule $\overline{Y}$.*

The claim states that if a player does not have the possibility of removing multiple links simultaneously, he might get stuck with "bad company". Indeed, here player 1 would like to remove his links with player 2 as well as player 3, but using LDP he can only remove at most one of these two links. Under SLDP player 1 is able to remove both links and improve his situation.

Proof of the claim: Let $v \in \mathbb{V}^N$ be such that $v(g) > 0$ for every $g \neq \emptyset$. That $g^\star = abc$ is not SLDP is clear since player 1 would like to remove both links a and b to arrive at network c, which yields him $\overline{Y}_1(c) = 0 > -\alpha v(abc) = \overline{Y}(abc)$.
We show that $g^\star = abc$ is LDP. Removing link a or link b would not yield a strict improve-

ment for either of the involved players, since $\overline{Y}_1(abc) = \overline{Y}_1(bc) = \overline{Y}_1(ac)$, $\overline{Y}_2(abc) = \overline{Y}_2(bc)$, and $\overline{Y}_3(abc) = \overline{Y}_3(ac)$. Finally, it is not profitable to remove link c for players 2 and 3 since

$$\overline{Y}_2(ab) = 0 < \tfrac{1}{2}(1+\alpha)v(abc) = \overline{Y}_2(abc) \quad \text{and}$$
$$\overline{Y}_3(ab) = 0 < \tfrac{1}{2}(1+\alpha)v(abc) = \overline{Y}_3(abc).$$

This implies that the complete network $g^\star = abc$ is LDP, as asserted. □

Example 3.1 clearly shows the limited applicability of regular link deletion proofness to economic situations. Unless one considers a situation under strict control or supervision, free individuals usually have the ability to sever unwanted connections and to escape situations as described in the example. From that perspective, strong pairwise stability is the more applicable stability concept.

4 Properties of strongly pairwise stable networks

We have already seen that as a modeling principle strong pairwise stability has some of advantages over pairwise stability. As a stability concept, it is still a modification of the more primitive notion of pairwise stability and therefore a comparison is appropriate. This section provides a further examination of strong pairwise stability by illustrating a few properties and applying it to some well known models in the literature. We also study the relationship between efficiency, pairwise stability and strong pairwise stability.

4.1 Boundedness of payoffs in strongly pairwise stable networks

Using the insight from Example 3.1 we can draw a further conclusion – in general SLDP networks only have bounded payoffs. This is the subject of the next proposition.

Proposition 4.1 *Let v be a component additive network value function and Y be a component balanced allocation rule. Then there exists some $V \geqslant 0$ such that $0 \leqslant Y_i(g,v) \leqslant V$ for every strong link deletion proof network $g \in \mathbb{G}^N$ and every player $i \in N$.*

Proof. Let v be a component additive network value function and Y be a component balanced allocation rule. These properties immediately imply that disconnected players are always allocated zero. Hence, if for some player i we have that $Y_i(g) < 0$ in some network $g \in \mathbb{G}^N$, then

$$Y_i\,(g \setminus L_i(g)) = Y_i(\varnothing) = 0 > Y_i(g) \tag{5}$$

In other words, by severing all links, player i can earn zero payoffs. Hence, if any player i earns less than zero payoffs in a certain network, the network is obviously not strong link deletion proof.
Next, let

$$V = \max_{g \in \mathbb{G}^N} v(g) \geqslant 0. \tag{6}$$

Consider an arbitrary strongly pairwise stable network $g \in \mathbb{G}^N$. Given that Y is component balanced, and therefore balanced, $\sum_{i \in N} Y_i(g) = v(g) \leqslant V$. From the above, $Y_i(g) \geqslant 0$ for all $i \in N$. Hence,

$$Y_i(g) = v(g) - \sum_{j \neq i} Y_j(g) \leqslant v(g) \leqslant V \tag{7}$$

for all $i \in N$. □

Note that the result does not require anonymity of v or Y.

We emphasize that the boundedness of payoffs is a property of strongly pairwise stable networks does not extend to regular pairwise stable networks. Example 3.1 shows that individual players do not have the ability to guarantee themselves autarkic existence from the other players in the network under regular pairwise stability. Hence, under pairwise stability, the (individually) lower bound of the payoff to any player is not zero, but rather whatever this player is confronted with by his fellow players. This is not the case under strong stability, where this lower bound is zero. This confirms what Goyal and Joshi (2003) find in networks with negative spillovers — strongly pairwise stable networks are either empty or stars.

4.2 The connections model

We discuss strong pairwise stability within the context of two popular and well-developed explicit models of network value functions. First, we discuss the connections model and, subsequently, we investigate strong pairwise stability in the context of unequal connections.[10]

In the connections model of Jackson and Wolinsky (1996) the links represent social relationships like friendship between individuals. Since it is unrealistic to suppose that payments could be exchanged for friendship we assume away the possibility of side payments. Consequently, $Y_i(g, v) = Y_i^{\gamma}(g)$ for all $v \in \mathbb{V}^N$. The payoff that player i receives from

[10] For elaborate discussions of other applications and models we refer to Jackson (2003) and Goyal (2004).

network g is

$$Y_i^\gamma(g) = \sum_{j \neq i} \delta^{t_{ij}(g)} - \sum_{j:\ ij \in g} c_{ij} \tag{8}$$

where $\delta \in (0,1)$ is the benefits parameter, $c_{ij} \geqslant 0$ is the cost of establishing link ij for player i and $t_{ij}(g)$ is the number of links on the shortest path between i and j. If for the connections model given in (8), it holds that all link formation costs are equal, i.e., $c_{ij} = c \geqslant 0$, then we refer to this setup as the *symmetric* connections model.

Jackson and Wolinsky (1996) characterize the collection of pairwise stable networks in the symmetric connections model. The next proposition shows that all pairwise stable networks in the symmetric connections model are also strong pairwise stable. This is a consequence of the additive nature of the (connections) allocation rule.

Proposition 4.2 *Let $n \geqslant 3$. Every pairwise stable equilibrium in the symmetric connections model is strong pairwise stable.*

For a proof of Proposition 4.2 we refer to the appendix of this paper.

An interesting extension of the symmetric connections model to a spatial setting has been developed by Johnson and Gilles (2000). In their model, player $i \in N$, is located at x_i and the set $X = \{x_1, \ldots, x_n\} \subset [0,1]$ with $x_1 = 0$ and $x_n = 1$ represents the spatial distribution of players. Without loss of generality assume that $x_i < x_j$ if $i < j$. Thus, the distance between the players $i, j \in N$ is given by $d_{ij} = \left| x_i - x_j \right| \leqslant 1$. This allows for the link establishment costs being determined by the spatial distance between players instead of having a fixed cost per link.

It is easy to verify that both Proposition 1 and 2 (which characterize the pairwise stable networks) of Johnson and Gilles (2000) are satisfied by strong pairwise stability. The class of acyclic pairwise stable (empty network and the chain) networks identified in their paper are also strong pairwise stable. Arguments similar to the one given above can be used to demonstrate this.

4.3 Unequal connections

Goyal and Joshi (2003) develop a framework to discuss unequal connections in which they allow explicitly for the possibility of positive and negative spillovers arising due to links between the players. They consider *playing the field games* where spillovers depend on the number of links all the other players have, and *local spillovers* where the externalities depend on the number of links of a potential partner. Here we restrict attention to the local spillover games. The (gross) payoffs of player i satisfy local spillovers if for any network g and any additional link ij it holds that

$$Y_i(g + ij) - Y_i(g) = \Psi(\eta_i(g), \eta_j(g)). \tag{9}$$

where $\eta_i(g)$ and $\eta_j(g)$ denote the number of neighbors of player i and j respectively. Thus, marginal returns depend on the number of links a player has, as well as the cardinality of the neighborhood set of a potential partner. The identity of the potential partner is crucial in local spillovers games since they may all have a different number of links. Each link has a cost $c > 0$ which must be subtracted to obtain the net benefits of a link. We say that marginal returns satisfy positive spillovers with respect to own links ($PSOL$) as well as links of the potential partners ($PSPL$), if $\Psi(\eta_i, \eta_j)$ is increasing in both η_i and η_j. We now show that in this setting a pairwise stable network may not be strongly pairwise stable.

Example 4.3 Let $c > 0$ and let $N = \{1, 2, 3, 4\}$. Consider the complete network g_N on N, where the marginal returns are given by $\Psi(2, 2) = 1.2c$, $\Psi(1, 1) = c$ and $\Psi(0, 0) = 0.5c$. This network satisfies both $PSOL$ and $PSPL$.
It is easy to check that no player wishes to break a single link and hence the network is pairwise stable. But by deleting 3 links simultaneously, a player is better off since $3c - 2.7c > 0$. Hence, g_N is not strongly pairwise stable. □

Note that it is also possible to construct other such examples as long as the marginal benefits satisfy the $PSOL$ property. This is because the marginal link may outweigh the links costs, while the earlier links fail to do so. Hence, by deleting a subset of links a player might be able to obtain a higher payoff.

4.4 Component-wise egalitarian payoffs

A major focus of the networks literature has been on the conflict between stability and efficiency in social and economic networks. Jackson and Wolinsky (1996) identify conditions under which this conflict is resolved for the component-wise egalitarian rule. This is an appealing allocation rule since it splits the value of a network equally among all members of the component. In this section we revisit the earlier work on the tension between stability and efficiency using strong pairwise stability. We begin by introducing the notion of a critical link.

Definition 4.4 *A link $ij \in g \in \mathbb{G}^N$ is **critical** in the network g if $\#\Gamma(g) < \#\Gamma(g - ij)$.*

In other words, a link is critical if after its removal either the number of components of the network increases, or the number of disconnected players increases. It means that there is no alternative path to replace such a critical link. A critical link is also known as a "bridge" in the sociological literature on networks.

Let $h \in C(g)$ denote a component that contains a critical link in the network $g \in \mathbb{G}^N$ and let $h_1 \subset h$ and $h_2 \subset h$ denote components obtained from h by severing that critical link. (Note that it may be the case that $h_1 = \varnothing$ or $h_2 = \varnothing$.) We now define the notion of critical link monotonicity introduced by Jackson and Wolinsky (1996) in their discussion of the properties of component egalitarian allocation rule Y^{ce}.

Definition 4.5 *The pair (g, v) satisfies **critical link monotonicity** if for any critical link $ij \in h$ with $h \in C(g)$ and the two associated components h_1 and h_2 of $h - ij$, we have that*

$$v(h) \geqslant v(h_1) + v(h_2) \quad \textit{implies that} \quad \frac{v(h)}{n(h)} \geqslant \max\left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right] \tag{10}$$

This constitutes a necessary and sufficient condition for the existence of efficient networks that are pairwise stable with regard to the component wise egalitarian allocation rule:

Claim 4.6 (Jackson and Wolinsky 1996, Claim, page 61)
If g is efficient relative to a component additive v, then g is pairwise stable for Y^{ce} relative to v if and only if (g, v) satisfies critical link monotonicity.

We next show that critical link monotonicity, however, is not adequate for strong pairwise stability.

Example 4.7 Let $n = 4$. Let the collective network benefits function v be given by

$$\begin{aligned} &v(\{ij\}) = 10, \\ &v(\{ij, ik\}) = 5, \\ &v(\{ij, ik, il\}) = 13, \\ &v(g_N) = 2, \text{ and} \\ &v(g) = 0 \text{ for all other } g \in \mathbb{G}^N \end{aligned}$$

Observe that v is component additive and anonymous.
Now consider the component-wise egalitarian allocation rule Y^{ce} for this particular setup. Clearly, every efficient network is a star given by $\{ij, ik, il\}$ for $i = 1, 2, 3$. However, a star is not strongly pairwise stable because it is not SLDP. In fact, $Y_i^{ce}(\{ij, ik, il\}) = 3\frac{1}{4}$, $Y_i^{ce}(\{ij, ik\}) = 1\frac{2}{3}$, and $Y_i^{ce}(\{ij\}) = 5$. Therefore, player i would sever two of his three links: $Y_i^{ce}(\{ij, ik, il\} \setminus \{il, ik\}) = Y_i^{ce}(\{ij\}) = 5 > 3\frac{1}{4} = Y_i^{ce}(\{ij, ik, il\})$.
For the star $h = \{ij, ik, il\}$ all three links are critical. Consider deletion of any link and let h_1

and $h_2 = \varnothing$ be the two associated components. Then, $v(h) = 13$, $v(h_1) = 5$, and $v(h_2) = 0$, implying that

$$\frac{v(h)}{n(h)} = 3\tfrac{1}{4}, \quad \frac{v(h_1)}{n(h_1)} = 1\tfrac{2}{3}, \quad \text{and } \frac{v(h_2)}{n(h_2)} = 0.$$

Hence the efficient network h obviously satisfies critical link monotonicity but is not strongly pairwise stable.[11] □

This naturally leads to the question: What conditions are required to make efficient networks strong pairwise stable under the component egalitarian allocation rule? Interestingly, this leads us to a condition relating to the presence of middlemen in the network. The analysis is presented below.

5 Networks with middlemen

A critical link refers to a single link between two players, whose removal results in a disconnected network. On the other hand when a single player removes multiple links leading to the disintegration of the network, we call such a player a *middleman* in the network.[12]

Definition 5.1 *A player $i \in N$ has a **middleman position** in the network $g \in \mathbb{G}^N$ if there exists some set of links $h^* \subset L_i(g)$ under the control of player i in g such that, there are at least two distinct players $j_1, j_2 \in N \setminus \{i\}$ who are connected in g and who are not connected in $g \setminus h^*$. A player with a middleman position in a network g is denoted as a **middleman** in g. The set of middlemen in the network g is denoted by $M(g) \subset N$.*

It is clear from this definition that a middleman has a critical position in a network since she can break up communication among other players in the network by deleting a well-chosen subset of her own links. A subset $h^* \subset L_i(g)$ of links that a middleman $i \in M(g)$ can delete to break up communication within a network g is called a *critical link set* for middleman i.

The following re-statement of the definition of a middleman is given without a proof. It follows immediately from the definition of a middleman position in a network.

Remark 5.2 *Let $n \geqslant 3$ and let $g \in \mathbb{G}^N$ be some network with $\#\Gamma(g) = 1$. Now, $i \in M(g)$ if and only if player $i \in N$ controls a critical link set $h^* \subset L_i(g)$ such that exactly one of the following properties holds:*

[11] In fact, in this example all of the 64 possible networks satisfy critical link monotonicity.

[12] In graph theory, the position of a middleman in the network is also referred to as a “cut node”.

(i) $\#C(g \setminus h^*) > \#C(g) = 1$.

(ii) $\#C(g \setminus h^*) = 1$ *and there is some player* $j \in N \setminus N_0(g)$ *such that* $j \in N_0(g \setminus h^*)$.

(iii) $\#C(g \setminus h^*) = 0$ *and* $N_0(g \setminus h^*) = N$.

Remark 5.2 states that a middleman in a network can either increase the number of non-trivial components in the network by removing some critical links, or disconnect some players from the network. In the latter case such disconnected players $j \in N$ are always *marginal* in the sense that $\#L_j(g) = 1$. Remark 5.2 (iii) discusses the case of a so-called *complete star* network, where player i is the center of the star involving all other players, i.e., $g = L_i(g_N) = \{ij \mid j \neq i\}$.

In general it is not true that a player who can refine the partitioning of the player set into components by severing links need be a middleman. Indeed, consider a player i in a network g such that $\#\Gamma(g) < \#\Gamma(g \setminus h)$ for some $h \subset L_i(g)$. While this player could be a middleman, she might also be a marginal player in the network g. In the latter case it is not appropriate to label this player as a "middleman", since she does not play a critical role in communication among other players in the network.

This is illustrated by referring to the trivial two player network $g_1 = \{12\}$ on the player set $N = \{1, 2, 3\}$. Note first that $\Gamma(g_1) = \{\{1, 2\}, \{3\}\}$.[13] Observe that 12 is a critical link in g_1, but neither player 1 nor player 2 are middlemen. On the other hand, in the network $g_2 = \{12, 13\}$, player 1 is a middleman. This conforms with the definition of a middleman.

We now introduce some further notation to describe the removal of a critical link set by some middleman in the network. Let $g \in \mathbb{G}^N$ be some network and let $h \in C(g)$ be one of its components. Let $i \in M(h)$ be a middleman in h and let $h^* \subset L_i(h)$ be a critical link set for middleman i. Now we denote by $C(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\}$ the components obtained from h by deleting the critical link set h^*. It should be clear that one of these components might be empty. In particular, this is the case when $N_0(h \setminus h^*) \neq \varnothing$. Furthermore, we denote by $\widehat{h} \in C(h \setminus h^*)$ as the component of h that contains player i. So $i \in N(\widehat{h})$. Note that $\widehat{h}$ might be the empty set. In that case player i herself has become an isolated node in the disintegrated network after removal of h^*, i.e., $i \in N_0(h \setminus h^*)$. The latter is exactly the situation covered in Remark 5.2(iii).

Definition 5.3 *A pair* $(g, v) \in \mathbb{G}^N \times \mathbb{V}^N$ *is* ***middleman secure*** *if for every component* $h \in C(g)$*, every middleman* $i \in M(h)$*, and every critical link set* $h^* \subset L_i(h)$ *for middleman* i *we*

[13]It should be clear that this case is not covered by Remark 5.2, since it explicitly assumes $\#\Gamma(g_1) = 1$. Instead this case has to be referred back to the general definition of a middleman position.

have that

$$v(h) \geqslant \sum_{i=1}^{m} v(h_i) \text{ implies that } \frac{v(h)}{n(h)} \geqslant \frac{v(\widehat{h})}{n(\widehat{h})}, \tag{11}$$

where $C(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\}$ *and* $\widehat{h} \in C(h \setminus h^*)$ *such that* $i \in N(\widehat{h})$.

We first show that middleman security implies critical link monotonicity.

Proposition 5.4 *Let* $v \in \mathbb{V}_+^N$ *be nonnegative in the sense that* $v(g) \geqslant 0$ *for all* $g \in \mathbb{G}^N$. *If* (g, v) *satisfies middleman security, then* (g, v) *satisfies critical link monotonicity as well.*

Proof. Consider any component $h \in C(g)$ of the network g and a critical link $ij \in h$. Denote by h_1 and h_2 the two components in the reduced network $h - ij$ produced by severing ij where $i \in N(h_1)$ and $j \in N(h_2)$. We have to consider three cases:

Case A: $h_1 = h_2 = \varnothing$.

In this case h consists of a single link, namely $h = \{ij\}$. Hence, $n(h) = 2$, $n(h_1) = n(h_2) = 1$, and $v(h_1) = v(h_2) = 0$. Therefore, $v(h) \geqslant 0 = v(h_1) + v(h_2)$ implies that

$$\frac{v(h)}{n(h)} = \frac{v(h)}{2} \geqslant 0 = \max\left[v(h_1), v(h_2)\right] = \max\left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right].$$

This is equivalent to critical link monotonicity.

Case B: $h_1 \neq \varnothing$ and $h_2 = \varnothing$.

Here, $n(h) \geqslant 3$, $n(h_1) = n(h) - 1 \geqslant 2$, $n(h_2) = 1$, and $v(h_2) = 0$. This case corresponds to disconnecting exactly one marginal player j from the network g by middleman i. In other words, player i is a middleman with the critical link set being $\{ij\}$. Suppose that $v(h) \geqslant v(h_1) + v(h_2) = v(h_1)$. Then from the middleman security condition applied to middleman i and critical link set $\{ij\}$ it follows that

$$\frac{v(h)}{n(h)} \geqslant \frac{v(h_1)}{n(h_1)} = \max\left[\frac{v(h_1)}{n(h_1)}, 0\right] = \max\left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right],$$

since by nonnegativity $v(h_1) \geqslant 0$. Clearly this case satisfies critical link monotonicity as well.

Case C: $h_1 \neq \varnothing$ and $h_2 \neq \varnothing$.

Here both players i and j could be middlemen. Considering player i as the middleman with critical link set $\{ij\}$, middleman security for i implies that

$$v(h) \geqslant v(h_1) + v(h_2) \quad \Longrightarrow \quad \frac{v(h)}{n(h)} \geqslant \frac{v(h_1)}{n(h_1)} \tag{12}$$

Similarly, considering player j as the middleman with critical link set $\{ij\}$, middleman security for j implies that

$$v(h) \geqslant v(h_1) + v(h_2) \quad \Longrightarrow \quad \frac{v(h)}{n(h)} \geqslant \frac{v(h_2)}{n(h_2)}. \tag{13}$$

Hence, from (12) and (13) it follows that

$$v(h) \geqslant v(h_1) + v(h_2) \quad \Longrightarrow \quad \frac{v(h)}{n(h)} \geqslant \max\left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right]$$

which is equivalent to the condition of critical link monotonicity.

This completes the proof of the assertion. □

Note that the construction in Example 4.7 does not satisfy middleman security. Consider the critical link set $h^* = \{ik, il\}$ for middleman i in the network $g = \{ij, ik, il\}$. Severing all links in h^* results in one non-null component $h_1 = \widehat{h} = \{ij\}$ and two disconnected players k and l. Now, $v(g) = 13 > 10 = v(\widehat{h}) + v(\varnothing) + v(\varnothing)$ but $\frac{v(g)}{n(g)} = 3\frac{1}{4} < 5 = \frac{v(\widehat{h})}{n(\widehat{h})}$. Observe that in a middlemen secure network, a middleman prefers not to create disconnected components by deleting a critical link set h^*. Thus for such networks an efficient network is also $SLDP$ for the component wise egalitarian rule.

Proposition 5.5 *If $g \in \mathbb{G}^N$ is efficient relative to a component additive $v \in \mathbb{V}^N$, then g is strong link deletion proof with respect to the component-wise egalitarian allocation rule Y^{ce} if and only if (g, v) is middleman secure.*

Proof. Without loss of generality we restrict ourselves to a network $g \in \mathbb{G}^N$ that consists of a single component, i.e., $\#\Gamma(g) = 1$, and such that $g \neq \varnothing$.

Only if: Suppose g is efficient relative to v as well as strong deletion proof for Y^{ce} relative to v. Then for any critical link set $h^* \subset L_i(g)$ for middleman $i \in M(g)$, it must hold that i does not wish to sever the links in that set. With the notation employed above, this requires that

$$\frac{v(g)}{n(g)} \geqslant \frac{v(\widehat{h})}{n(\widehat{h})} \tag{14}$$

This evidently implies that middleman security holds for (g, v).

If: Suppose that g is efficient relative to v and that (g, v) is middleman secure.
Severing a non-critical link set by any player will only change the value of the component without changing the number of players in that component. By efficiency of g and component additivity of v, this value is already at a maximum and hence there can be no net gain.

Suppose that some middleman $i \in M(g)$ in g severs a critical link set h^* from $L_i(g)$. This results into the component set $C(g \setminus h^*) = \{h_1, \ldots, h_m\}$. This has no benefit for the middleman i because by efficiency of g and component additivity, we have that

$$v(g) \geqslant \sum_{k=1}^{m} v(h_k)$$

which by middleman security implies that (14) has to hold. This confirms that g is in fact strong link deletion proof.
This completes the proof of the assertion. □

The next result is a straightforward corollary of Proposition 5.5.

Corollary 5.6 *If $g \in \mathbb{G}^N$ is efficient relative to a nonnegative and component additive $v \in \mathbb{V}^N_+$, then g is strongly pairwise stable for the component-wise egalitarian allocation rule Y^{ce} if and only if (g, v) is middleman secure.*

Proof. From Proposition 5.5 we know that middleman security implies that g is strong link deletion proof for Y^{ce}. Using Proposition 5.4 we know that middleman security implies critical link monotonicity. From Claim 4.6, we know that if a network g satisfies critical link monotonicity, it is pairwise stable as well and, therefore, link addition proof. Hence, g has to be strongly pairwise stable. □

Corollary 5.6 demonstrates that middlemen exert crucial *positional power* in the allocation process of network benefits. When they have no incentive to disconnect the network, component-wise egalitarianism resolves the conflict between stability and efficiency. As shown in the previous discussion, the presence of middlemen is crucial for the allocation of network benefits such that the efficient networks are strongly pairwise stable.

Remark 2.2 emphasizes that the component-wise egalitarian allocation rule is the unique rule that combines the benign requirement of component balance and the equal treatment of members of the same component in the network. This implies that it is the unique rule that links the payoff of individuals *directly* with the collective value generated by these individuals. In this regard it is the unique allocation rule that points individuals directly to efficiency. In other words, the collective value becomes the individualized payoff for all players, and network formation thus becomes a common interest non-cooperative endeavor.[14]

[14]We refer to Bowles (2004, Chapter 2) for a complete discussion of the properties of this type of non-cooperative game.

6 Networks without middlemen

We first turn to the study of networks that are always middlemen secure, irrespective of the network value function employed. These networks are denoted as middleman-free. Subsequently we investigate whether regular (or symmetric) networks are necessarily middleman-free.

Formally a network $g \in \mathbb{G}^N$ is called **middleman-free** if $M(g) = \varnothing$, i.e., in such networks there are no middleman positions. The next proposition proves that these networks are always middleman secure and, hence, will also satisfy critical link monotonicity.

Proposition 6.1 *A network $g \in \mathbb{G}^N$ is middleman-free if and only if for every network value function $v \in \mathbb{V}^N$ the pair (g, v) is middleman secure.*

Proof. We first consider the case when $n \geqslant 3$ and the network $g \in \mathbb{G}^N$ consists of a single non-trivial component, i.e., $\#\Gamma(g) = 1$. Since $n \geqslant 3$ it is obvious that g has to consist of at least two links.

If:
Suppose to the contrary that g has at least one middleman. We proceed by constructing a network value function v' for which (g, v') is not middleman secure.
Let $i \in M(g)$ be a middleman in g. Note that by definition of a middleman position it has to hold that $\# L_i(g) \geqslant 2$.
Next, consider a critical link set $h^* \subset L_i(g)$ such that $C(g \setminus h^*) = \{h_1, \ldots, h_m\}$ with $i \in N(h_1)$ and $n(h) > n(h_1)$. It is clear that since g consists of at least two links, we can select the critical link set h^* for middleman i in this fashion. This follows from an application of the characterization of a middleman position given in Remark 5.2.
Now select the network value function v' such that $v'(h) = v'(h_1) = 1$ and $v'(h_k) = 0$ for all $k = 2, \ldots, m$. Then we have obviously that

$$v'(h) = 1 = v'(h_1) = \sum_{k=1}^{m} v'(h_k)$$

and

$$\frac{v'(h)}{n(h)} = \frac{1}{n(h)} < \frac{1}{n(h_1)} = \frac{v'(h_1)}{n(h_1)}.$$

This implies that middleman security is not satisfied for the pair (g, v').

Only if:
Suppose that g is middleman-free. Since $M(g) = \varnothing$ it follows immediately that for any network value function $v \in \mathbb{V}^N$ the pair (g, v) has to be middleman secure.

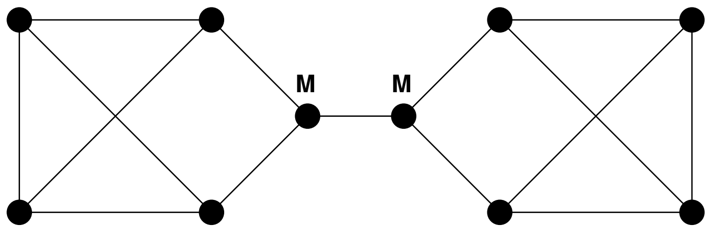

Figure 1: A 3-regular network with one critical link

To show the assertion for any non-empty network, the only remaining case to be investigated is that $g = \{ij\}$ for some $i, j \in N$. This network g is middleman free and as a consequence it is middleman secure as well. Combining this insight with the previously investigated case we have shown the assertion for any non-empty network. □

Combining Corollary 5.6 and Proposition 6.1 we attain the insight that efficient and middleman-free networks are always strongly pairwise stable under component-wise egalitarian payoffs. In particular this has bearing on situations with link monotone value functions, in which the complete network is efficient.

Corollary 6.2 *If $g \in \mathbb{G}^N$ is middleman-free as well as efficient relative to a nonnegative and component additive $v \in \mathbb{V}^N_+$, then g is strongly pairwise stable for the component-wise egalitarian allocation rule Y^{ce}.*

We emphasize that in general very large networks with sufficient clustering can be expected to be middleman-free. The reason is that in such networks there are enough redundant links to allow for multiple paths between different individuals preventing any individual from having positional power.

Example 6.3 Regular networks

Regular networks form an interesting class of networks that is also popular in the networks literature.[15] For instance the empty network and the complete network are both regular networks.

Formally a network g is k-**regular** if $\# \Gamma(g) = 1$ and for every player $i \in N$ it holds that $\# N_i(g) = k$. Hence, the network consists of exactly one component and every player is connected to exactly k other players. Using strong pairwise stability Goyal and Joshi (2003) find many instances of regular networks both in case of positive and negative spillovers.

[15] They are also sometimes referred to as symmetric networks presumably since every agent has the same number of links (Goyal and Joshi (2003)). As Figure 1 demonstrates their shape need not necessarily be symmetric.

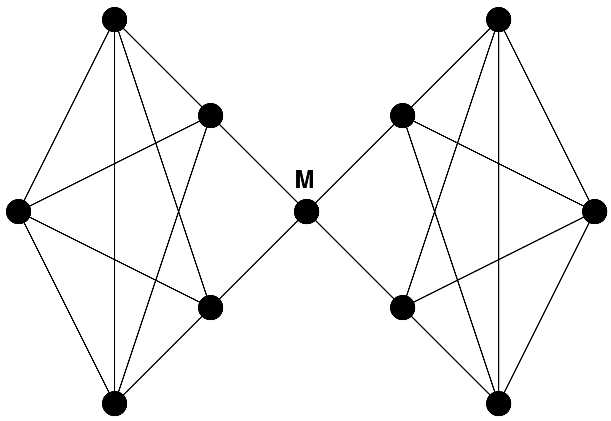

Figure 2: A 4-regular network with a unique middleman position

The class of k-regular networks has a non-empty intersection with the class of middleman-free networks for every $k \geqslant 2$. First, observe that middlemen free graphs are not a subset of regular graphs since regular graphs need not be connected. Next, it is trivial to see that every 2-regular network has essentially a unique topology and can be described as a circle consisting of all n players. For $k \geqslant 3$ any complete network consisting of $n = k + 1$ players is k-regular and middleman free. Similarly, every k-bipartite graph with $k \geqslant 2$ will be middlemen free.

On the other hand, for $k \geqslant 3$ there exist networks with critical links and middlemen. Figure 1 depicts a 3-regular network with a unique critical link and, therefore, two middlemen indicated by "M" in the figure. It should be remarked that it is impossible to construct a 3-regular network that has a unique middleman.

For larger values of k it is possible to construct k-regular networks with a unique middleman. This is illustrated in Figure 2 that depicts a 4-regular network with a unique middleman indicated by "M". □

7 Coda

In this paper we have shown that under the component-wise egalitarian rule there is no tension between strong pairwise stability and efficiency only for middlemen secure networks. Our analysis makes it is clear that middleman positions give occupants widespread control over the functioning of the network. Kalai, Postlewaite, and Roberts (1978) already investigated the consequences of middleman positions on payoffs. They arrived at some surprising insights, that have great affinity with the main result from our analysis.

Our analysis makes clear that further research is needed on the role of middlemen in the allocation of benefits over participants in network situations. This analysis should not be limited to collective benefit problems, but also extend to individualistic payoff situations.

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Appendix: Proof of Proposition 4.2

In order to proof the assertion of Proposition 4.2 we first prove a lemma whose repeated application ensures the result. All subsequent notions are developed within the context of the symmetric connections model with payoff parameter $\delta \in (0,1)$ and cost parameter $c \geqslant 0$.

Consider any network g and the severance of some link ij. Deletion of this link cannot reduce the geodesic distance between player i and any other arbitrary player k. Therefore i's benefits are nonincreasing in the deletion of any arbitrary link. Denote by $\beta_{ik}(g - ij)$ the reduction in gross benefits accruing to player i from player k by deleting link $ij \in g$ through a possible increase in geodesic distance between i and k. Then, $\beta_{ik}(g - ij) \geqslant 0$. The set of k for which $\beta_{ik}(g - ij)$ is positive is rather restricted. Namely $\beta_{ik}(g - ij) > 0$ for all $p_{ik}(g) \in P_{ik}(g)$, $p_{ik}(g) \cap L_i(g) = \{ij\}$. In that case

$$\beta_{ik}(g - ij) = \delta^{t_{ik}(g)} - \delta^{t_{ik}(g-ij)}$$

Define $W_{ij}(g) = \{k \in N \mid \beta_{ik}(g - ij) > 0\}$. Obviously, $j \in W_{ij}(g)$.

Following Jackson and Wolinsky (1996), let $u_i(g - ij)$ denote the gain to agent i by deleting link ij. Then,

$$u_i(g - ij) = c - \sum_{k \in W_{ij}(g)} \beta_{ik}(g - ij)$$

In general we use $u_i(g \setminus h^*)$ to denote the gain to agent i by deleting a star $h^* \subset g$.[16]

Lemma 1 *For any network g such that $ii_1, ii_2 \in g$, $u_i(g - ii_2 - ii_1) \leqslant u_i(g - ii_2)$.*

Proof. First recall that any path between i and $k \in N$ cannot include more than one member of $L_i(g)$. Given that any path between $k \in W_{ii_2}(g)$ must by definition include ii_2, it cannot possibly include ii_1. Hence, elimination of ii_1 cannot disconnect any such path. Hence,

$$W_{ii_2}(g) = W_{ii_2}(g - ii_1) = W \tag{15}$$

This also means that the geodesic distance between i and k, where $k \in W$, in g and $g - ii_1$ are the same. Hence, for all $k \in W$,

$$t_{ik}(g) = t_{ik}(g - ii_1) \tag{16}$$

Now,

$$\begin{aligned} u_i(g - ii_2) &= c - \sum_{k \in W_{ii_2}(g)} \beta_{ik}(g - ii_2) \\ &= c - \sum_{k \in W} \left[\delta^{t_{ik}(g)} - \delta^{t_{ik}(g-ii_2)}\right] \end{aligned}$$

[16] Obviously i has to be the center of the star h.

Also,

$$\begin{aligned} u_i(g - ii_1 - ii_2) &= c - \sum_{k \in W_{ii_2}(g - ii_1)} \beta_{ik}(g - ii_1 - ii_2) = c - \sum_{k \in W} \beta_{ik}(g - ii_1 - ii_2) \\ &= c - \sum_{k \in W} \left[\delta^{t_{ik}(g - ii_1)} - \delta^{t_{ik}(g \setminus \{ii_1, ii_2\})}\right] = c - \sum_{k \in W} \left[\delta^{t_{ik}(g)} - \delta^{t_{ik}(g \setminus \{ii_1, ii_2\})}\right] \end{aligned}$$

Hence, in order to find out which one is greater, we have to compare $t_{ik}(g - ii_2)$ and $t_{ik}(g \setminus \{ii_1, ii_2\})$ for all $k \in W$. Given, $g \setminus \{ii_1, ii_2\} \subset g - ii_2$, $t_{ik}(g \setminus \{ii_1, ii_2\}) \geqslant t_{ik}(g - ii_2)$. Also, given $0 < \delta < 1$ and $t_{ik}(g \setminus \{ii_1, ii_2\}) \geqslant t_{ik}(g - ii_2)$, $\delta^{t_{ik}(g \setminus \{ii_1, ii_2\})} \leqslant \delta^{t_{ik}(g - ii_2)}$ for all $k \in W$. Hence, the assertion of the lemma has been proved. □

Proof of Proposition 4.2:
Proof. Let g be a pairwise stable network in the symmetric connections model. To prove the assertion we only have to show that the network is strong link deletion proof. Consider any player i contemplating deletion of a set of m links $ii_1, ii_2, \ldots, ii_m$ where $i_1, i_2, \ldots, i_m \in N_i(g)$. Let $h^* = \{ii_1, ii_2, \ldots, ii_m\}$. Then, one way to represent the resulting gain is as follows:

$$u_i(g \setminus h^*) = u_i(g - ii_1) + u_i(g \setminus \{ii_1, ii_2\}) + \cdots + u_i(g \setminus \{ii_1, ii_2, \ldots, ii_m\}) \tag{17}$$

Since, g is strong deletion proof, $u_i(g - ii_l) \leqslant 0$ for all $l = 1, 2, \ldots, m$. Applying Lemma 1 we get

$$u_i(g \setminus \{ii_1, ii_2\} \leqslant u_i(g - ii_2) \leqslant 0$$

Repeated application of the lemma gives us

$$u_i(g \setminus \{ii_1, ii_2, ii_3\}) \leqslant u_i(g \setminus \{ii_1, ii_3\} \leqslant u_i(g - ii_3) \leqslant 0$$

Proceeding thus each term on the right hand side of the third inequality is non-positive. Hence $u_i(g \setminus h^*)$ being a sum of non-positive terms is non-positive as well. Consequently, g is strong link deletion proof.
This completes the proof of Proposition 4.2. □