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## A STRONG APPROXIMATION OF THE SHORTT PROCESS

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*A shortt of a one-dimensional probability distribution is defined to be an interval which has at least probability  $t$  and minimal length. The length of a shortt,  $U(t)$ , and its obvious estimator,  $U_n(t)$ , are significant measures of scale of a probability distribution and the corresponding random sample, respectively. The shortt process is defined to be  $\sqrt{n}(U_n(t) - U(t))/U'(t)$ , similarly to the definition of the quantile process. It is known that this process converges weakly, under natural regularity conditions, to a Brownian bridge. In this note a strong approximation of the shortt process by a Kiefer process is established, which yields the weak convergence as a corollary. Applications of the result to the global and local strong limiting behaviour of the shortt process are also presented.*

*Key words: Kiefer process, length of shortt, oscillation modulus, robust scale estimation, strong approximation, upper and lower class.*

*AMS 1991 Subject Classification: Primary 62G20, 62G30, 62G35; secondary 60F15, 60F17.*

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### 1. Introduction and the Main Result

Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with common distribution function  $F$ . A famous robust scale estimator based on the first  $n$  of these random variables is the length of a shortest closed interval containing at least fraction  $t$  (shortt) of these  $n$  observations. So the length of a shortt is defined to be

$$U_n(t) = \inf\{b - a : F_n(b) - F_n(a-) \geq t\}, \quad 0 < t < 1,$$

where  $F_n$  denotes the right-continuous empirical distribution function (df). Similarly we define the theoretical counterpart of the length of a shortt (i.e., our parameter of interest) by

$$U(t) = \inf\{b - a : F(b) - F(a-) \geq t\}, \quad 0 < t < 1.$$

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The estimator  $U_n(1/2)$  can be shown to be a least median of squares estimator (cf. Croux and Rousseeuw [2]) with asymptotic breakdown point 50% (see Rousseeuw and Leroy [11]) and it is approximately min-max bias robust (see Martin and Zamar [9]). A functional central limit theorem (CLT) for the shortt process

$$\delta_n = \sqrt{n} \frac{U_n - U}{U'}$$

has been established in Grübel [7]. In Einmahl and Mason [5] a functional CLT for a generalized quantile process is derived, which contains Grübel's result as well as the functional CLT for the classic quantile process as special cases.

It is the purpose of this note to refine the functional CLT for  $\delta_n$  by establishing a strong approximation with a Kiefer process of the sequence of shortt processes  $\{\delta_n\}$ . We also briefly present applications to the strong limiting behaviour of the shortt process. In order to present our results we introduce more notation and some assumptions. Throughout let  $f = F'$  be continuous, strictly increasing on  $(\alpha, x_0]$ , strictly decreasing on  $[x_0, \beta)$  and 0 elsewhere, for certain  $-\infty \leq \alpha < x_0 < \beta \leq \infty$ . In addition, let  $f'$  be continuous on  $\mathbb{R}$  with  $|f'(x)| > \varepsilon$  on  $x \in [x_1, x_2] \cup [x_3, x_4]$ , with  $\alpha \leq x_1 < x_2 < x_0 < x_3 < x_4 \leq \beta$  for some  $\varepsilon > 0$ . Also let  $0 < \eta < 1/2$  be such that the endpoints of the interval determining  $U(\eta)$  and  $U(1 - \eta)$  are contained in  $[x_1, x_2] \cup [x_3, x_4]$ . Set, as above,

$$\delta_n(t) = \sqrt{n}(U_n(t) - U(t))/U'(t) = \sqrt{nh}(U(t))(U_n(t) - U(t)), \quad 0 < t < 1,$$

where  $h = H'$  with  $H = U^{-1}$ ; cf. the definition of the classical quantile process or see Einmahl and Mason [5]. Let  $K$  be a Kiefer process, i.e.,

$$K(n, t) = W(n, t) - tW(n, 1), \quad n \in \mathbb{N}, \quad 0 \leq t \leq 1,$$

where  $W$  is a two-parameter Wiener process, see, e.g., Csörgő and Révész [3], Section 1.15. Observe that  $n^{-1/2}K(n, t)$  is a Brownian bridge for any  $n \in \mathbb{N}$ .

**Theorem 1.1.** *Under the above assumptions there exists a Kiefer process  $K$  such that*

$$\sup_{\eta \leq t \leq 1-\eta} |\delta_n(t) - n^{-1/2}K(n, t)| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad a.s.$$

**Remark 1.** We only present the approximation with a Kiefer process since our method gives the same results for the approximation with a sequence of Brownian bridges. Note that the functional CLT for  $\delta_n$  in Grübel [7] immediately follows from Theorem 1.1.

**Remark 2.** It is not clear if the rate in Theorem 1.1 is optimal. However, in the recent paper Deheuvels [4] it is shown that in the Kiefer process approximation of the classical quantile process the power of  $1/n$  in the approximation cannot be higher than  $1/4$ . Therefore we expect that the power of  $1/n$  in Theorem 1.1 cannot be improved beyond  $1/4$ , since the shortt process is a generalized quantile process.

In Section 2 we present two applications of Theorem 1.1. The proof of Theorem 1.1 is given in Section 3.

**2. Applications**

We first present an analogue of Chung's [1] upper and lower class result for  $D_n = \sup_{\eta \leq t \leq 1-\eta} |\delta_n(t)|$ .

**Theorem 2.1.** *Let  $\lambda_n \uparrow \infty$ . Then*

$$P(D_n > \lambda_n \text{ i.o.}) = \begin{cases} 0 \\ 1 \end{cases}$$

according as

$$\sum_{i=1}^n \frac{\lambda_n^2}{n} e^{-2\lambda_n^2} \begin{cases} < \\ = \end{cases} \infty.$$

*Proof.* We present a brief sketch of the proof. It is well known that Theorem 2.1 holds with  $D_n$  replaced by  $\sup_{0 < t < 1} |n^{-1/2} K(n, t)|$ . Now by the law of the iterated logarithm (LIL) for  $\sup_{t \in (0, \eta) \cup (1-\eta, 1)} |n^{-1/2} K(n, t)|$  it can be easily shown, using  $\lambda_n > \sqrt{\frac{1}{2} \log \log n}$  eventually, that Theorem 2.1 holds with  $D_n$  replaced by  $\sup_{\eta \leq t \leq 1-\eta} |n^{-1/2} K(n, t)|$ . Finally, it follows rather easily from Theorem 1.1 that we can replace the latter expression by  $D_n$  itself, i.e., we have obtained Theorem 2.1.  $\square$

Note that Theorem 2.1 immediately yields the LIL for  $D_n$ . In fact the LIL for  $D_n$  and even the functional LIL for  $\{\delta_n(t), \eta \leq t \leq 1 - \eta\}$  (cf. Finkelstein [6]) follow directly from Theorem 1.1 in conjunction with the corresponding results for the Kiefer process.

Theorem 1.1 also easily yields a result on the strong local behaviour of  $\delta_n$ , provided the bandwidth tends to 0 slowly enough. Define the oscillation modulus of  $\delta_n$  by

$$\omega_{\delta_n}(a) = \sup_{\eta \leq s \leq t \leq 1-\eta, t-s \leq a} |\delta_n(t) - \delta_n(s-)|, \quad 0 < a \leq 1 - 2\eta.$$

**Theorem 2.2.** *Let  $\{a_n\}_{n=1}^\infty$  be a sequence of numbers satisfying*

$$a_n \downarrow 0, \quad na_n \uparrow, \quad \frac{\log(1/a_n)}{\log \log n} \rightarrow c \in [0, \infty] \quad \text{and} \quad a_n(n/\log n)^{1/3} \rightarrow \infty,$$

then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n(\log(1/a_n) + \log \log n))^{1/2}} &= 1 \quad \text{a.s.}, \\ \liminf_{n \rightarrow \infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n(\log(1/a_n) + \log \log n))^{1/2}} &= \left(\frac{c}{c+1}\right)^{1/2} \quad \text{a.s.} \end{aligned}$$

When  $c = \infty$ ,  $c/(c+1)$  should be read as 1 and hence we then have

$$\lim_{n \rightarrow \infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n \log(1/a_n))^{1/2}} = 1 \quad \text{a.s.}$$

*Proof.* These results are well known for the Kiefer process (or for the uniform empirical process); see, e.g., Chapter 14 of Shorack and Wellner [12]. Hence, using  $a_n(n/\log n)^{1/3} \rightarrow \infty$ , Theorem 2.2 immediately follows from Theorem 1.1.  $\square$

**3. Proof of Theorem 1.1**

In order to prove Theorem 1.1 we first need a few auxiliary results. One of these results, Proposition 3.1, may be of independent interest. We begin with introducing more notation. We write  $P$  and  $P_n$  for the probability measures corresponding to  $F$  and  $F_n$  respectively. Also for a function  $z: \mathbb{R} \rightarrow \mathbb{R}$  and a closed interval  $I = [a, b] \subset \mathbb{R}$  we set  $z(I) = z(b) - z(a-)$ , provided  $z(a-)$  exists. The class of intervals  $I \subset \mathbb{R}$  is denoted with  $\mathbb{I}$  and the length of such an  $I$  is written as  $|I|$ . The empirical process of  $X_1, \dots, X_n$  is denoted by

$$\alpha_n = \sqrt{n}(P_n - P),$$

and we will briefly write  $\alpha_n(x)$  for  $\alpha_n((-\infty, x])$ ,  $x \in \mathbb{R}$ .

Next we state for convenient reference later on some results from the literature. Recall that the Komlós–Major–Tusnády [18] Kiefer process approximation of the empirical process yields the existence of a Kiefer process  $\tilde{K}$  such that

$$(1) \quad \sup_{x \in \mathbb{R}} \left| \alpha_n(x) - \frac{\tilde{K}(n, F(x))}{\sqrt{n}} \right| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \quad \text{a.s.}$$

Write  $B_{P,n}(x) = \tilde{K}(n, F(x))/\sqrt{n}$ . Then (1) immediately yields

$$(2) \quad \sup_{I \in \mathbb{I}} |\alpha_n(I) - B_{P,n}(I)| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \quad \text{a.s.}$$

Let  $\alpha_n$  be the uniform empirical process (based on i.i.d. uniform-(0,1) random variables) and define, for  $a \in (0, 1)$ , its oscillation modulus and its Lipschitz- $\frac{1}{2}$  modulus, respectively, by

$$\omega_n(a) = \sup_{|I| \leq a, I \subset [0,1]} |\alpha_n(I)|, \quad \tilde{\omega}_n(a) = \sup_{|I| \geq a, I \subset [0,1]} |\alpha_n(I)|/|I|^{1/2}.$$

It is well known (see, e.g., Mason, Shorack, and Wellner [10]), that for  $a_n = n^{-c}$ ,  $c \in (0, 1)$ ,

$$(3) \quad \lim_{n \rightarrow \infty} \frac{\omega_n(a_n)}{(2a_n \log(1/a_n))^{1/2}} = 1 \quad \text{a.s.}$$

and

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\tilde{\omega}_n(a_n)}{(2 \log(1/a_n))^{1/2}} = 1 \quad \text{a.s.}$$

We will need a somewhat similar result for the Kiefer process (see, e.g., Theorem 1.15.2 in Csörgő and Révész [3]). Let  $\{a_n\}_{n=1}^\infty$  be a sequence of positive numbers for which  $\log(1/a_n)/\log \log n \rightarrow \infty$ , then

$$(5) \quad \lim_{n \rightarrow \infty} \sup_{|I| \leq a_n, I \subset [0,1]} \frac{n^{-1/2} |\tilde{K}(n, I)|}{(2a_n \log(1/a_n))^{1/2}} = 1 \quad \text{a.s.}$$

Let  $I_x$  be the unique interval with  $|I_x| = x$ ,  $U(\eta) \leq x \leq U(1-\eta)$ , and maximal probability  $P$ . Denote by  $I_{n,x}$  a sample analogue of  $I_x$ , i.e., an interval with length  $x$  and maximal empirical measure  $P_n$ . Denote the left endpoints of  $I_x$  and  $I_{n,x}$  by  $\ell_x$  and  $L_{n,x}$ , respectively. The next lemma on the distance between  $\ell_x$  and  $L_{n,x}$  is the key ingredient for the proof of Theorem 1.1.

**Theorem 3.1.** *There exists a  $C \in (0, \infty)$  such that with probability one*

$$\sup_{U(\eta) \leq x \leq U(1-\eta)} |\ell_x - L_{n,x}| \leq C \left( \frac{\log n}{n} \right)^{1/3}$$

eventually.

*Proof.* Define  $I_{1,n} = I_{n,x} \setminus I_x$  and  $I_{2,n} = I_x \setminus I_{n,x}$ . Then  $|I_{1,n}| = |I_{2,n}| = |\ell_x - L_{n,x}| =: B_n$ . Observe that

$$\begin{aligned} P(I_{n,x}) - P(I_x) &= P(I_{1,n}) - P(I_{2,n}), \\ P_n(I_{n,x}) - P_n(I_x) &= P_n(I_{1,n}) - P_n(I_{2,n}). \end{aligned}$$

Write  $b = f'(\ell_x)$  and  $d = -f'(\ell_x + x)$ . Then by a Taylor series expansion and the Glivenko–Cantelli theorem

$$P(I_x) - P(I_{n,x}) = \frac{1}{2} B_n^2 (b + d) + o(B_n^2) \quad \text{a.s.}$$

Hence with probability one

$$(6) \quad \begin{aligned} \frac{1}{2} B_n^2 (b + d) + o(B_n^2) &= P(I_x) - P(I_{n,x}) \\ &\leq P_n(I_{n,x}) - P_n(I_x) - (P(I_{n,x}) - P(I_x)) \\ &= P_n(I_{1,n}) - P_n(I_{2,n}) - (P(I_{1,n}) - P(I_{2,n})) \\ &= \frac{1}{n^{1/2}} \{ \alpha_n(I_{1,n}) - \alpha_n(I_{2,n}) \}. \end{aligned}$$

We have

$$(7) \quad \begin{aligned} |\alpha_n(I_{j,n})| &\leq |\alpha_n(I_{j,n})| \mathbf{1}_{\{|I_{j,n}| < n^{-1/2}\}} \\ &\vee \frac{|\alpha_n(I_{j,n})|}{|I_{j,n}|^{1/2}} \mathbf{1}_{\{|I_{j,n}| \geq n^{-1/2}\}} |I_{j,n}|^{1/2}, \quad j = 1, 2. \end{aligned}$$

Using the results in (3) and (4) and the probability integral transform it readily follows that the right-hand side of (7) is almost surely bounded by

$\tilde{C}(n^{-1/2} \vee B_n)^{1/2}(\log n)^{1/2}$  eventually, for some  $\tilde{C} \in (0, \infty)$ . Combining this with (6), (7), and the fact that  $b$  and  $d$  are bounded away from 0, Lemma 3.1 follows.  $\square$

Note that

$$H(x) = \sup_{I \in \mathbb{I}, |I| \leq x} P(I).$$

Set

$$H_n(x) = \sup_{I \in \mathbb{I}, |I| \leq x} P_n(I)$$

and

$$\gamma_n(x) = \sqrt{n}(H_n(x) - H(x)), \quad x \geq 0.$$

**Proposition 3.1.** *There exists a Kiefer process  $K$  such that*

$$\sup_{U(\eta) \leq x \leq U(1-\eta)} |\gamma_n(x) + n^{-1/2}K(n, H(x))| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \text{ a.s.}$$

*Proof.* We will take  $K$  as follows:

$$K(n, H(x)) = -\left\{ \tilde{K}(n, F(\ell_x + x)) - \tilde{K}(n, F(\ell_x)) \right\} = -n^{1/2}B_{P,n}(I_x).$$

Note that  $K(n, t)$ ,  $n \in \mathbb{N}$ ,  $\eta \leq t \leq 1 - \eta$ , is a Kiefer process. Now we have

$$\begin{aligned} (8) \quad & \sup_{U(\eta) \leq x \leq U(1-\eta)} |\gamma_n(x) + n^{-1/2}K(n, H(x))| \\ &= \sup_{U(\eta) \leq x \leq U(1-\eta)} \left| \sqrt{n} \left( \sup_{|I| \leq x} P_n(I) - H(x) \right) - B_{P,n}(I_x) \right| \\ &= \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sqrt{n} \left( \sup_{|I|=x} P_n(I) - H(x) \right) - B_{P,n}(I_x) \right\} \\ & \quad \vee \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ B_{P,n}(I_x) - \sqrt{n} \left( \sup_{|I|=x} P_n(I) - H(x) \right) \right\}. \end{aligned}$$

The second term in the right-hand side of (8) is bounded from above by

$$\begin{aligned} & \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ B_{P,n}(I_x) - \sqrt{n}(P_n(I_x) - P(I_x)) \right\} \\ &= \sup_{U(\eta) \leq x \leq U(1-\eta)} (B_{P,n}(I_x) - \alpha_n(I_x)) \\ &\leq \sup_{I \in \mathbb{I}} |B_{P,n}(I) - \alpha_n(I)| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \text{ a.s.,} \end{aligned}$$

where the last “equality” follows from (2).

Using Lemma 3.1, we see that with probability one the first term on the right in (8) can be bounded from above for large  $n$  by

$$\begin{aligned}
& \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sqrt{n} \left( \sup_{|I|=x, |\ell_x - L_{n,x}| \leq C(\log n/n)^{1/3}} P_n(I) - H(x) \right) - B_{P,n}(I_x) \right\} \\
& \leq \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sqrt{n} \sup_{|I|=x, |\ell_x - L_{n,x}| \leq C(\log n/n)^{1/3}} (P_n(I) - P(I)) - B_{P,n}(I_x) \right\} \\
& \leq \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sup_{|I|=x, |\ell_x - L_{n,x}| \leq C(\log n/n)^{1/3}} |\alpha_n(I) - B_{P,n}(I)| \right\} \\
& \quad + \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sup_{|I|=x, |\ell_x - L_{n,x}| \leq C(\log n/n)^{1/3}} B_{P,n}(I) - B_{P,n}(I_x) \right\} \\
& \leq \sup_{I \in \mathcal{I}} |\alpha_n(I) - B_{P,n}(I)| \\
& \quad + \sup_{U(\eta) \leq x \leq U(1-\eta)} \left\{ \sup_{|I|=x, |\ell_x - L_{n,x}| \leq C(\log n/n)^{1/3}} B_{P,n}(I) - B_{P,n}(I_x) \right\} \\
& = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) + O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad \text{a.s.},
\end{aligned}$$

where again (2) is applied to bound the first term; the bound on the second term follows from (5).

The shortt process  $\delta_n$  can be viewed as a quantile process version of the empirical-type process  $\gamma_n$ . Therefore we will obtain Theorem 1.1 by “inverting” Proposition 3.1. We will begin with proving a version of Theorem 1.1 for the “uniformized” version  $\bar{\delta}_n$  of  $\delta_n$ . Set

$$\bar{P}_n(t) = \sup_{|I| \leq U(t)} P_n(I), \quad 0 < t < 1, \quad \bar{P}_n(0) = 0, \quad \bar{P}_n(1) = 1,$$

and

$$\bar{P}_n^{-1}(s) = \inf \{t: \bar{P}_n(t) \geq s, 0 \leq t \leq 1\}, \quad 0 \leq s \leq 1.$$

Then

$$\gamma_n(x) = \sqrt{n}(\bar{P}_n(H(x)) - H(x));$$

set

$$\bar{\delta}_n(t) = \sqrt{n}(\bar{P}_n^{-1}(t) - t), \quad 0 \leq t \leq 1.$$

**Proposition 3.2.** *For the Kiefer process of Proposition 3.1 we have*

$$\sup_{\eta \leq t \leq 1-\eta} |\bar{\delta}_n(t) - n^{-1/2}K(n, t)| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad \text{a.s.}$$

*Proof.* Since  $H_n$  and hence  $\bar{P}_n$  make only jumps of size  $1/n$  almost surely, we obtain that

$$(9) \quad \sup_{\eta \leq t \leq 1-\eta} \left| \bar{\delta}_n(t) + \sqrt{n}(\bar{P}_n(\bar{P}_n^{-1}(t)) - \bar{P}_n^{-1}(t)) \right| = \frac{1}{\sqrt{n}} \quad \text{a.s.}$$



So by Proposition 3.1 and the LIL for the Kiefer process it follows that

$$(10) \quad \sup_{\eta \leq t \leq 1-\eta} |\bar{P}^{-1}(t) - t| = O\left(\sqrt{\frac{\log \log n}{n}}\right) \quad \text{a.s.}$$

Using (10) in conjunction with (5), again Proposition 3.1, and (9), we have

$$\begin{aligned} & \sup_{\eta \leq t \leq 1-\eta} |\bar{\delta}_n(t) - n^{-1/2}K(n, t)| \\ & \leq \sup_{\eta \leq t \leq 1-\eta} \left| \sqrt{n}(\bar{P}_n(\bar{P}_n^{-1}(t)) - \bar{P}_n^{-1}(t)) + n^{-1/2}K(n, \bar{P}_n^{-1}(t)) \right| \\ & \quad + \sup_{\eta \leq t \leq 1-\eta} n^{-1/2} \left| K(n, \bar{P}_n^{-1}(t)) - K(n, t) \right| + \frac{1}{\sqrt{n}} \\ & = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) + O\left(\frac{(\log \log n)^{1/4}(\log n)^{1/2}}{n^{1/4}}\right) + \frac{1}{\sqrt{n}} \\ & = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad \text{a.s.} \end{aligned}$$

Note that Proposition 3.1 has to be used actually on an interval slightly larger than  $[U(\eta), U(1 - \eta)]$ . This causes no problems however, since  $f'$  is continuous.  $\square$

Now we have in hands the necessary ingredients to give a short proof of our main result.

*Proof of Theorem 1.1.* It is shown in Einmahl and Mason [5] that almost surely

$$U_n(t) = U(P_n^{-1}(t)), \quad \eta \leq t \leq 1 - \eta.$$

So by the mean value theorem, we have for some  $t_n$  between  $t$  and  $\bar{P}_n^{-1}(t)$ , that

$$(11) \quad \begin{aligned} \delta_n(t) - n^{-1/2}K(n, t) &= \frac{h(U(t))}{h(U(t_n))} (\bar{\delta}_n(t) - n^{-1/2}K(n, t)) \\ & \quad + \left( \frac{h(U(t))}{h(U(t_n))} - 1 \right) n^{-1/2}K(n, t) \quad \text{a.s.} \end{aligned}$$

The following facts, which can be shown by elementary analysis, will be needed: for  $U(\eta) \leq x \leq U(1 - \eta)$  we have

$$h(x) = f(\ell_x) = f(\ell_x + x), \quad h'(x) = \frac{f'(\ell_x)f'(\ell_x + x)}{f'(\ell_x) - f'(\ell_x + x)} < 0.$$

Hence we have that  $h$  and  $|h'|$  are bounded away from 0 and  $\infty$  on  $[U(\eta), U(1 - \eta)]$ ; in fact this even holds on a slightly larger interval. Observe that by the mean value theorem  $h(U(t)) - h(U(t_n)) = (t - t_n)h'(U(\tilde{t}_n))/h(U(\tilde{t}_n))$ , with  $\tilde{t}_n$  between  $t$  and  $t_n$ . So Proposition 3.2, (10), (11) and the LIL for the Kiefer process yield

$$\begin{aligned} & \sup_{\eta \leq t \leq 1-\eta} |\delta_n(t) - n^{-1/2}K(n, t)| \\ & = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) + O\left(\sqrt{\frac{\log \log n}{n}}\right) O(\log \log n)^{1/2} \\ & = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad \text{a.s.} \quad \square \end{aligned}$$

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