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Einmahl, J.H.J.

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Eindhoven University of Technology

Department of Mathematics and Computing Science
Probability Theory, Statistics, Operations Research and Systems Theory Group
Secretary, Main Building, Room 9.10, Tel. (040)-2473130
P.O. Box 513, 5600 MB Eindhoven, Netherlands

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EXTENSION TO HIGHER DIMENSIONS OF THE JAESCHKE-EICKER RESULT ON THE STANDARDIZED EMPIRICAL PROCESS

John H.J. Einmahl

Dept. of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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ABSTRACT

The asymptotic distribution of the sup-norm of the heavily weighted empirical process is established in the multidimensional case. This theorem extends in particular the famous result in Jaeschke (1975, 1979) to higher dimensions. There is a striking difference between the behaviour for higher dimensions and that for dimension one, especially the limiting distribution is now a simple transformation of a standard exponential random variable.

1. INTRODUCTION AND MAIN RESULTS

Let $X_1, ..., X_n$ be independent random vectors, each uniformly distributed on $[0,1]^d, d \in \mathbb{N}$, and define the multivariate empirical distribution function by

$$F_n(t) = \frac{1}{n}\#\{1 \le i \le n : X_i \le t\}, \quad t \in [0,1]^d,$$

where '$\le$' denotes the componentwise ordering. Note that $F(t) = P(X_i \le t) = \Pi_{j=1}^d t_j, t = (t_1, \cdots, t_d) \in [0,1]^d$. The multivariate empirical process is written as

$$\alpha_n(t) = n^{\frac{1}{2}}(F_n(t) - F(t)), \quad t \in [0,1]^d.$$

Observe that $\mathrm{Var}\,(\alpha_n(t)) = F(t)(1 - F(t))$ and hence it is natural to consider the standardized empirical process

$$\frac{\alpha_n(t)}{(F(t)(1-F(t)))^{\frac{1}{2}}}, \quad t \in [0,1]^d, \tag{1.1}$$

a process having constant variance equal to 1, or, more generally,

$$\alpha_{n,\nu}(t) = \frac{\alpha_n(t)}{(F(t)(1-F(t)))^{\nu}}, \quad t \in [0,1]^d, \quad \nu \in [0,1]. \tag{1.2}$$

(We will not consider the case $\nu > 1$, since then $\sup_{t \in [0,1]^d} |\alpha_{n,\nu}(t)| = \infty$ *a.s.*) It is well-known that for $\nu \in [0, \frac{1}{2})$, and *not* for $\nu \in [\frac{1}{2}, 1]$, $\alpha_{n,\nu}$ converges weakly to $B/(F(1-F))^{\nu}$, where B is a continuous mean zero Gaussian process with $EB(s)B(t) = F(s \wedge t) - F(s)F(t)$, where $s \wedge t$ has to be understood componentwise. Here, however, we are interested in the values of $\nu \in [\frac{1}{2}, 1]$, leading to so-called heavily weighted empirical processes, with special interest in the case $\nu = \frac{1}{2}$. To be more precise, we will consider the weak limiting behaviour of

$$\sup_{t \in [0,1]^d} |\alpha_{n,\nu}(t)|, \quad \nu \in [\tfrac{1}{2}, 1]. \tag{1.3}$$

Of course, for $\nu \in [0, \frac{1}{2})$, we immediately have from the weak convergence of $\alpha_{n,\nu}$ as a process that

$$\sup_{t \in [0,1]^d} |\alpha_{n,\nu}(t)| \to_d \sup_{t \in [0,1]^d} \frac{|B(t)|}{(F(t)(1-F(t)))^{\nu}}, \quad \text{as } n \to \infty.$$

For $d = 1$, the weak limiting behaviour of the properly normalized statistic in (1.3) is well-known. It is the purpose of this paper to extend the one dimensional results to higher dimensions; surprisingly the behaviour turns out to be completely different then. Let us briefly describe the one dimensional results now. When $d = 1$ the behaviour of the cases $\nu = \frac{1}{2}$ and $\nu \in (\frac{1}{2}, 1]$ is substantially different. First we consider the standardized empirical process, i.e. the case $\nu = \frac{1}{2}$. Set $a_n = (2 \operatorname{loglog} n)^{-\frac{1}{2}}$ and $b_n = (2 \operatorname{loglog} n)^{\frac{1}{2}} + \{\frac{1}{2} \operatorname{logloglog} n - \frac{1}{2} \log \frac{\pi}{4}\}/(2 \operatorname{loglog} n)^{\frac{1}{2}}$. Then, as $n \to \infty$,

$$(\sup_{t \in [0,1]} |\alpha_{n,\frac{1}{2}}(t)| - b_n)/a_n \to_d \Gamma, \tag{1.4}$$

where Γ is a standard Gumbel random variable, i.e. $P(\Gamma \leq x) = \exp(-\exp(-x)), x \in \mathbb{R}$. This famous result in empirical process theory is established in Jaeschke (1975, 1979); closely related results (with F in the denominator in (1.1) replaced by F_n) can be found in Eicker (1979). Now we consider $\nu \in (\frac{1}{2}, 1]$. Let N and $\tilde{N}$ be two independent homogeneous unit intensity Poisson processes on $[0, \infty)$, then it is shown in Mason (1983) that, as $n \to \infty$,

$$n^{\frac{1}{2}-\nu} \sup_{t \in [0,1]} |\alpha_{n,\nu}(t)| \to_d \sup_{t>0} \frac{|N(t) - t| \vee |\tilde{N}(t) - t|}{t^\nu}. \tag{1.5}$$

Without making this more precise, we mention that the behaviour of the left hand side of (1.5) is determined by the extreme tails of the (uniform) distribution, whereas in (1.4) the moderate tails are responsible for the behaviour of its left hand side.

Now we turn to our main result, i.e. the proper extensions of (1.4) and (1.5) to dimension $d \geq 2$. For this purpose let Y be a mean one exponential random variable ($P(Y \leq x) = 1 - e^{-x}, \quad x \geq 0$) and set $L = ((d-1)!Y)^{-\nu}$.

THEOREM 1. Let $d \geq 2$ and $\nu \in [\frac{1}{2}, 1]$, then, as $n \to \infty$,

$$\frac{\sup_{t \in [0,1]^d} |\alpha_{n,\nu}(t)|}{n^{\nu-\frac{1}{2}}(\log n)^{\nu(d-1)}} \to_d L. \tag{1.6}$$

Remark 1. It is striking that in the multivariate case the behaviour differs substantially from that in the one dimensional situation. Observe also that in (1.6) the cases $\nu = \frac{1}{2}$ and $\nu \in (\frac{1}{2}, 1]$ are treated 'simultaneously', in contrast to the behaviour in dimension one. It will become clear from the proof that the behaviour of the left hand side of (1.6) is determined by the left tail ($\{t : F(t)$ 'small'$\}$) of the distribution, to be more precise, we will show that the 'multivariate minimum' (the X_i with $F(X_i) = \min_{1 \le j \le n} F(X_j)$) plays the dominant role.

Remark 2. There is already a related result for $d = 2$ and $\nu \in (\frac{1}{2}, 1)$ in the literature (Csörgő and Horváth (1990)). This result yields

$$\frac{n^{\frac{1}{2}-\nu} \sup_{t\in[0,1]^d} |\alpha_n(t)|}{(F(t))^\nu} \to_d \sup_{t_1,t_2>0} \frac{|N(t_1,t_2) - t_1 t_2|}{(t_1 t_2)^\nu} \quad (n \to \infty), \tag{1.7}$$

where N is a homogeneous unit intensity Poisson process on $[0,\infty)^2$. However, it can be readily verified that the right hand side of (1.7) is equal to infinity with probability one. This means that the normalization ($n^{\frac{1}{2}-\nu}$) is not appropriate, which is also confirmed by Theorem 1.

Remark 3. It is standard to generalize the one dimensional results (1.4) and (1.5) to X_i's having a continuous distribution function, instead of being uniformly-$[0, 1]$ distributed. Similarly for $d \ge 2$, it readily follows from the fact that the uniform-$[0,1]^d$ distribution has independent marginals, that Theorem 1 remains true when this distribution is replaced by a probability distribution on $\mathbb{R}^d$ with distribution function $F(t) = \Pi_{j=1}^d F_j(t_j)$, $t = (t_1, ..., t_d) \in \mathbb{R}^d$, where the F_j, $1 \le j \le d$, are continuous one dimensional distribution functions on $\mathbb{R}$ (of course, in (1.6), $[0,1]^d$ has to be replaced by $\mathbb{R}^d$ then). The class of these distributions is still restricted, since the marginals remain independent. On the other hand, when $d \ge 2$, some serious restriction on the distribution of the X_i is needed, since it easily follows that when, e.g., $d = 2$ and the X_i are uniformly distributed on the line segment from (1,0) to (0,1), then $\sup_{t\in[0,1]^2} |\alpha_{n,\nu}(t)| = \infty$ a.s. for all $\nu \in (0, 1]$, i.e. trying to extend Theorem 1 to this distribution is senseless.

Now we present an essential generalization of Theorem 1, namely to the case where the marginals may be dependent.

THEOREM 2. Let the X_i have a distribution on $[0,1]^d$, with a continuous density f with respect to Lebesgue measure on $[0,1]^d, d \geq 2$, satisfying for some m and $M: 0 < m \leq f(t) \leq M < \infty$, $t \in [0,1]^d$. Then, as $n \to \infty$, (1.6) remains true for $\nu \in [\frac{1}{2}, 1]$.

Remark 4. Theorem 2 shows that for a wide class of distributions the rate of convergence of $\sup_{t\in[0,1]^d}|\alpha_{n,\nu}(t)|$ and the limiting law are the same. Observe that this class, in turn, can be extended to distributions on $\mathbb{R}^d$ as indicated in Remark 3. It is also shown in Remark 3, however, that (1.6) is definitely not satisfied for all distributions. Finally, note that the rate in (1.6) is also not the 'slowest' possible one, even not within the class of continuous distribution functions, since, e.g., again when $d = 2$, the uniform distribution on the line segment from (0,0) to (1,1) leads to an essentially one dimensional situation, i.e. the rates in (1.4) and (1.5) apply.

2. PROOFS

For the proof of Theorem 1 the following three results on probabilities, concerning the empirical distribution function of independent uniform-$[0,1]^d$ random vectors, which all can be found in Einmahl (1987, pages 19, 38 and 26 respectively), will be required.

FACT 1. Let $d \in \mathbb{N}$ and $\nu \in [\frac{1}{2}, 1]$. Then for any $\delta \in (0,1)$, $0 < \alpha \leq \beta \leq \frac{1}{2}(1-\delta)$ and $\lambda \geq 0$

$$P\Big(\sup_{\alpha \leq F(t) \leq \beta} n^{\frac{1}{2}-\nu}|\alpha_n(t)|/(F(t))^{\nu} \geq \lambda\Big)$$

$$\leq C \int_{(1-\delta)\alpha}^{\beta/(1-\delta)} \frac{(\log(1/x))^{d-1}}{x} \exp\Big(-\tfrac{1}{2}(1-\delta)\lambda^2 (nx)^{2\nu-1}\psi\Big(\frac{\lambda}{(n\alpha)^{1-\nu}}\Big)\Big)dx,$$

where $C = C(d,\delta) \in (0,\infty)$ and $\psi(y) = 2y^{-2}\{(1+y)\log(1+y) - y\}$, $y > 0$.

FACT 2. Let $d \in \mathbb{N}, m \in \{1, \cdots, n\}$ and $0 < \alpha \leq 1/e$. Then

$$P(\sup_{F(t)\le\alpha} nF_n(t) \ge m) \le c_1\binom{n}{m}(c_2\alpha)^m(\log(1/\alpha))^{d-1},$$

where $c_1 = c_1(d), c_2 = c_2(d)$ and $c_1, c_2 \ge 1$. Of course $c_1(1)$ and $c_2(1)$ can both be taken equal to 1.

FACT 3. Let $d \in \mathbb{N}$ and $\gamma \in (0,\infty)$. Then, as $n \to \infty$,

$$P(\sup_{F(t)\le\frac{(d-1)!}{\gamma n(\log n)^{d-1}}} nF_n(t) = 0) \to e^{-1/\gamma}.$$

PROOF OF THEOREM 1. From the weak convergence of α_n it is immediate that for all $0 < a < b < 1$

$$\frac{1}{n^{\nu-\frac{1}{2}}(\log n)^{\nu(d-1)}} \sup_{a\le F(t)\le b} |\alpha_{n,\nu}(t)| \to_P 0 \quad (n \to \infty). \tag{2.1}$$

This shows that in the 'middle' no contribution to the limit is made. Hence it suffices to consider the left tail and the right tail separately. However, it can be shown as in Theorem 3.2 in Einmahl (1987), that 'large d-dimensional points' behave (modulo multiplicative constants) as 'small (or large) 1-dimensional points'. By using (1.4) and (1.5), this statement leads to the fact that (2.1) remains true with $b = 1$. So, taking $a = \frac{1}{4}$, it suffices to show, that as $n \to \infty$,

$$\frac{1}{n^{\nu-\frac{1}{2}}(\log n)^{\nu(d-1)}} \sup_{F(t)\le\frac{1}{4}} |\alpha_{n,\nu}(t)| \to_d L. \tag{2.2}$$

Write $a_n = 1/(n(\log n)^{\frac{1}{2}(d-1)})$ and note that

$$\sup_{a_n\le F(t)\le\frac{1}{4}} |\alpha_{n,\nu}(t)| \le \frac{4}{3} \sup_{a_n\le F(t)\le\frac{1}{4}} |\alpha_n(t)|/(F(t))^{\nu}.$$

We hence have for arbitrary $\varepsilon > 0$ by Fact 1, with $\delta = \frac{1}{2}$,

$$P(\sup_{a_n\le F(t)\le\frac{1}{4}} n^{\frac{1}{2}-\nu}|\alpha_{n,\nu}(t)| \ge \varepsilon(\log n)^{\nu(d-1)})$$

$$\leq C \int_{\frac{1}{2}a_n}^{\frac{1}{2}} \frac{(\log(1/x))^{d-1}}{x}\, dx \tag{2.3}$$

$$\cdot\, exp\,(-\tfrac{1}{4}(\tfrac{3}{4})^2\varepsilon^2(\log n)^{2\nu(d-1)}(\tfrac{1}{2}na_n)^{2\nu-1}\psi(\tfrac{3}{4}\varepsilon(\log n)^{\nu(d-1)}(na_n)^{\nu-1})).$$

Using $x\psi(x) \to \infty$ $(x \to \infty)$, the right hand side of (2.3) is for large n bounded from above by

$$C(\log n)^d \exp(-(\log n)^{\frac{1}{4}\nu(d-1)}),$$

which tends to 0, as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, it is now sufficient to prove (2.2) with $\frac{1}{4}$ replaced by a_n. But since on $\{t \in [0,1]^d : F(t) \leq a_n\}$ we have $(1-a_n)^\nu \leq (1-F(t))^\nu \leq 1$ and $\lim_{n\to\infty}(1-a_n)^\nu = 1$, it finally remains to show that

$$L_n = \frac{1}{n^{\nu-\frac{1}{2}}(\log n)^{\nu(d-1)}} \sup_{F(t)\leq a_n} \frac{|\alpha_n(t)|}{(F(t))^\nu} \to_d L \quad (n \to \infty). \tag{2.4}$$

Now Fact 2 yields

$$P(\sup_{F(t)\leq a_n} nF_n(t) \geq 2) \leq c_1\binom{n}{2}(c_2a_n)^2(\log(1/a_n))^{d-1}$$

$$= O((\log n)^{-\frac{1}{2}(d-1)}) \to 0 \quad (n \to \infty), \tag{2.5}$$

and Fact 3 yields

$$P(\sup_{F(t)\leq a_n} nF_n(t) = 0) \to 0 \quad (n \to \infty). \tag{2.6}$$

Statements (2.5) and (2.6) play the crucial role in the proof of (2.4). They imply, that, with arbitrary high probability (n large) there are observations in the region $\{t \in [0,1]^d : F(t) \leq a_n\}$, but in that region $nF_n(t)$ is at most one. Set $M_n = \sup_{F(t)\leq a_n} nF_n(t)$. We have by (2.5) and (2.6)

$$\mathbb{1}_{\{1\}}(M_n) \to_P 1 \quad (n \to \infty). \tag{2.7}$$

Write

$$L_n' = \frac{n^{1-\nu}}{(\log n)^{\nu(d-1)}} \frac{|\frac{1}{n} - F(X)_{1:n}|}{(F(X)_{1:n})^\nu},$$

where $F(X)_{1:n} = \min_{1 \leq i \leq n} F(X_i)$. Then it follows that for $n \geq 3$

$$L_n = L_n' \mathbb{1}_{\{1\}}(M_n) + L_n \mathbb{1}_{\{0,2,3,\cdots,n\}}(M_n). \tag{2.8}$$

Also, using (2.7),

$$L_n' \mathbb{1}_{\{1\}}(M_n) / \frac{1}{(\log n)^{\nu(d-1)} n^\nu (F(X)_{1:n})^\nu} \to_P 1 \quad (n \to \infty). \tag{2.9}$$

From (2.7)-(2.9) and

$$P(\frac{1}{(\log n)^{\nu(d-1)} n^\nu (F(X)_{1:n})^\nu} \leq x) = \{P(F(X_1) \geq \frac{1}{x^{1/\nu} n (\log n)^{d-1}})\}^n,$$

it follows that for a proof of (2.4), it remains to show that for $x > 0$, as $n \to \infty$,

$$\{P(F(X_1) \geq \frac{1}{x^{1/\nu} n (\log n)^{d-1}})\}^n \to P(L \leq x). \tag{2.10}$$

Using, e.g., that $-\log F(X_1)$ has a gamma distribution with density $((d-1)!)^{-1} x^{d-1} e^{-x} 1_{(0,\infty)}(x)$, a straightforward but tedious calculation shows that the left hand side of (2.10) converges to

$$\exp(-\frac{1}{(d-1)! x^{1/\nu}}), \text{ as } n \to \infty. \tag{2.11}$$

Observing that

$$P(L \leq x) = P(\frac{1}{((d-1)! Y)^\nu} \leq x) = P(Y \geq \frac{1}{(d-1)! x^{1/\nu}})$$

$$= \exp(-\frac{1}{(d-1)! x^{1/\nu}}),$$

now completes the proof of (2.4) and hence that of Theorem 1. □

For the proof of Theorem 2 we will need the following 'local' generalization of the one dimensional probability integral transform, which may be of independent interest.

PROPOSITION 1. Under the condition on f in Theorem 2 we have

$$\lim_{s \downarrow 0} \frac{(d-1)!P(F(X_1) \leq s)}{s(\log(1/s))^{d-1}} = 1. \tag{2.12}$$

PROOF. The proof is essentially easy, but the details are somewhat technical. Therefore we will only give a sketch of the proof. When $F(t) = \Pi_{j=1}^{d} t_j$, i.e. the uniform distribution on $[0,1]^d$, then (2.12) easily follows, e.g., by using that $-\log F(X_1)$ has a gamma density (see the end of the proof of Theorem 1). If $F(t) = c\Pi_{j=1}^{d} t_j$, $c \in (0,\infty)$, for small $F(t)$, then (2.12) also easily follows by using the fact that it holds for $c = 1$, which we just observed.

Now we turn to the general case. Let $\delta > 0$ be arbitrary. Then, because of continuity of f, we can find an $\eta \in (0,1)$ such that on $t \in [0,\eta]^d$, $|f(t) - f(\bar{0})| \leq \varepsilon f(\bar{0})$ $(\bar{0} = (0,\cdots,0))$.
Hence we have for small $\eta > 0$

$$\begin{aligned} &P(F(X_1) \leq s) \\ &= P(F(X_1) \leq s,\ X_1 \in [0,\eta]^d) + P(F(X_1) \leq s,\ X_1 \notin [0,\eta]^d) \qquad (2.13) \\ &\leq P(F'(X'') \leq s) + P(F(X_1) \leq s,\ X_1 \notin [0,\eta]^d), \end{aligned}$$

where F' is a distribution function on $[0,1]^d$, such that for F' small, $F'(t) = (1-\varepsilon)f(\bar{0})\Pi_{j=1}^{d} t_j$ and X'' is for small values of its distribution function, distributed according to $F''(t) = (1+\varepsilon)f(\bar{0})\Pi_{j=1}^{d} t_j$. Write $h(s) = ((d-1)!)^{-1}s(\log(1/s))^{d-1}$, then it easily follows by applying the just treated case with $c = (1+\varepsilon)f(\bar{0})$, that, as $s \downarrow 0$,

$$P(F'(X'') \leq s) \leq (1 + o(1))\frac{1+\varepsilon}{1-\varepsilon}\, h(s)\,. \tag{2.14}$$

Also, by using that $m \leq f \leq M$, as $s \downarrow 0$,

$$P(F(X_1 \leq s, \quad X_1 \notin [0,\eta]^d) = o(h(s)). \tag{2.15}$$

Hence by combining (2.13)-(2.15) we see that since $\varepsilon > 0$ is arbitrary that

$$\limsup_{s \downarrow 0} P(F(X_1) \leq s)/h(s) \leq 1.$$

It similarly follows that the 'liminf' is not smaller than 1. □

PROOF OF THEOREM 2. The proof follows along the same lines as that of Theorem 1, using obvious modifications of Facts 1-3 (see section 6.2 in Einmahl (1987) for details). The only problem arises in the very last part of the proof, i.e. proving that the left hand side of (2.10) converges to (2.11). This, however, follows immediately from Proposition 1. □

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