Limit theorems for a general weighted process under random censoring

Einmahl, J.H.J.; Koning, A.J.

Published in:
Canadian Journal of Statistics

Publication date:
1992

The Canadian Journal of Statistics
Vol. 20, No. 1, 1992, Pages 77–89
La Revue Canadienne de Statistique

Limit theorems for a general weighted process under random censoring

John H.J. EINMAHL and Alex J. KONING*

University of Limburg

Key words and phrases: Limit theorems, random censoring, weighted processes.
AMS 1985 subject classifications: Primary 60F05, 60F15, 60F17; secondary 62E20, 62G05.

ABSTRACT

Necessary and sufficient conditions for weak and strong convergence are derived for the weighted version of a general process under random censoring. To be more explicit, this means that for this process complete analogues are obtained of the Chibisov-O'Reilly theorem, the Lai-Wellner Glivenko-Cantelli theorem, and the James law of the iterated logarithm for the empirical process. The process contains as special cases the so-called basic martingale, the empirical cumulative hazard process, and the product-limit process. As a tool we derive a Kiefer-process-type approximation of our process, which may be of independent interest.

RÉSUMÉ

On donne des conditions nécessaires et suffisantes assurant la convergence faible et forte d'une version pondérée d'un processus général sujet à une censure aléatoire. Plus explicitement, on obtient pour ce processus des analogues du théorème de Chibisov-O'Reilly, du théorème de Glivenko-Cantelli tel que formulé par Lai et Wellner ainsi que de la loi de James du logarithme itéré pour le processus empirique. Le processus considéré admet comme cas particuliers: la martingale dite de base, le processus de hasard cumulatif empirique et un processus lié à celui de Kaplan-Meier. Une approximation de type Kiefer est établie pour le processus considéré, celle-ci pouvant en soi présenter un intérêt.

1. INTRODUCTION

The classical random censorship model deals with i.i.d. random vectors (Z_1, δ_1), $(Z_2, \delta_2), \ldots$, obtained from two independent sequences of i.i.d. random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ defined on a probability space $(\Omega, \mathbf{A}, P)$ and taking values in $(0, \infty)$, in the following way:

$$Z_i = X_i \wedge Y_i, \qquad \delta_i = 1_{\{X_i \leq Y_i\}}. \tag{1.1}$$

Denote the cumulative distribution functions of X_i, Y_i, and Z_i by F, G, and $H = 1 - (1-F)(1-G)$ respectively, and let $R \in (0, \infty)$ be an arbitrary but fixed number such that $H(R) < 1$; assume F to be continuous and strictly increasing and G continuous on $[0, R]$. Dominant roles in the model are played by H and H^1, defined by

$$H^1(t) = P(Z_i \leq t,\ \delta_i = 1) = \int_0^t \{1 - G(s)\}\, dF(s), \qquad t \in (0, \infty), \tag{1.2}$$

*Currently at the Erasmus University, Rotterdam.

and their empirical counterparts

$$H_n(t) = n^{-1} \sum_{i=1}^{n} 1_{[0,t]}(Z_i), \tag{1.3}$$

$$H_n^1(t) = n^{-1} \sum_{i=1}^{n} \delta_i 1_{[0,t]}(Z_i). \tag{1.4}$$

Note that in general H^1 is defective, i.e., $\lim_{t\to\infty} H^1(t) < 1$.

The statistical analysis of the random-censorship model has benefited much from the approach in which nH_n^1 is considered as a counting process; see e.g. Aalen (1976) for details. Denoting the left-continuous version of a right-continuous function with left limits f by f_-, the compensator of nH_n^1 is given by

$$n \int_0^t \{1 - H_{n-}(s)\} \, d\Lambda(s),$$

where $\Lambda(t) = -\log\{1 - F(t)\}$ is the cumulative hazard function belonging to X_i. The so-called basic martingale

$$M_n(t) := n^{\frac{1}{2}} \left(H_n^1(t) - \int_0^t \{1 - H_{n-}(s)\} \, d\Lambda(s) \right), \tag{1.5}$$

the rescaled difference between nH_n^1 and its compensator, plays a fundamental role, since many interesting processes may be expressed as stochastic integrals with respect to M_n, i.e., as processes of the form

$$Q_n(t) = \int_0^t L_n(s) \, dM_n(s), \qquad t \in [0, R].$$

This process Q_n is the subject of our study. However, we will not rely on martingale methods for proving our results.

It is the purpose of this paper to obtain necessary and sufficient conditions for weak and strong convergence of weighted versions of Q_n, i.e., we present the central limit theorem, the Glivenko-Cantelli theorem, and the functional law of the iterated logarithm for this weighted process. These results are the complete analogues of those by Chibisov (1964) and O'Reilly (1974), by Lai (1974) and Wellner (1977), and by James (1975), respectively, for the ordinary empirical process. We obtain as special cases of the aforementioned theorems these results for the unweighted process Q_n itself.

The remainder of this paper is organized in the following way. In Section 2 the main results are presented. In Section 3 some special choices of L_n are treated and (these specializations of) our results are compared with those in the literature. Section 4 contains some technical tools, including a Kiefer-process-type approximation of Q_n. Finally, in Section 5 the proofs of Theorems 2.1–2.3 are detailed.

2. MAIN RESULTS

Throughout we assume that the function L_n satisfies the following conditions:

(a) L_n is a random element in (the left-continuous right-limits version of) $D[0, R]$.

(b) the total variation $V(L_n)$ of L_n satisfies $\limsup_{n\to\infty} V(L_n) < c$ a.s., for some $c < \infty$.

(c) L_n converges a.s. on $[0,R]$ in the sup norm to a positive and continuous deterministic function L.

(d) for some fixed $0 < \alpha < 1$ there exists for every $n \in \mathbf{N}$ a partition $0 = x_{0,n} < x_{1,n} < \cdots < x_{m(n),n} = R$ of $[0,R]$ such that

$$H^1(x_{i,n}) - H^1(x_{i-1,n}) \leq n^{-\alpha},\ i = 1,\ldots,m(n), \qquad n \in \mathbf{N},$$

and

$$\sum_{i=1}^{m(n)} |\{L_n(x_{i,n}) - L(x_{i,n}\} - \{L_n(x_{i-1,n}) - L(x_{i-1,n})\}| \underset{\text{a.s.}}{\longrightarrow} 0. \tag{2.1}$$

It can be shown that (b), (c) imply $V(L) < c$. Moreover, for the verification of (d) it suffices to show that for some $0 < \alpha < 1$

$$n^\alpha \sup_{0 \leq t \leq R} |L_n(t) - L(t)| \underset{\text{a.s.}}{\longrightarrow} 0. \tag{2.2}$$

Writing

$$D(t) = \int_0^t L^2(s)\, dH^1(s), \tag{2.3}$$

the weighted version of Q_n is defined by

$$Q_n(t)/q(D(t)), \quad t \in [0, R] \qquad \text{(throughout, } 0/0 := 0), \tag{2.4}$$

where the so-called weight function q is an element of the class

$$\mathbf{Q} = \{q : [0, \infty] \longrightarrow [0, \infty] : q \text{ nondecreasing and } q > 0 \text{ on } (0, \infty)\}. \tag{2.5}$$

Moreover we need the subclasses

$$\mathbf{Q}_0 = \{q \in \mathbf{Q} : q \text{ continuous}\} \tag{2.6}$$

and

$$\mathbf{Q}_{00} = \{q \in Q : q/I^{\frac{1}{2}} \text{ nonincreasing on } (0, \delta] \text{ for some } \delta > 0\}, \tag{2.7}$$

where I is the identity function.

We present our first result, the central limit theorem, in a "convergence in probability" setting, i.e., we consider the convergence in probability to zero of the weighted differences between Q_n and its limiting process. This limiting process turns out to be the process $W \circ D$ (see Section 4), where W denotes a standard Wiener process on $[0,\infty)$. This shows that the weighting in (2.4) is appropriate.

THEOREM 2.1. *Let $q \in \mathbf{Q}_0$. If*

$$\int_0^1 \frac{1}{t} \exp\left(-\frac{\lambda q^2(t)}{t}\right) dt < \infty \qquad \text{for all} \quad \lambda > 0, \tag{2.8}$$

then there exists a sequence $W_1, W_2, \ldots$ of standard Wiener processes such that

$$\sup_{0 \leq t \leq R} \frac{|Q_n(t) - W_n(D(t))|}{q((D(t))} \underset{P}{\longrightarrow} 0 \qquad as \quad n \longrightarrow \infty. \tag{2.9}$$

Conversely, (2.9) *holding true for some sequence* $W_1, W_2, \ldots$ *of standard Wiener processes implies* (2.8).

Our second result is a Glivenko-Cantelli-type theorem.

THEOREM 2.2. *Let* $q \in \mathbf{Q}$. *If* $\int_0^1 \{1/q(t)\}\, dt < \infty$, *then*

$$\lim_{n\to\infty} \sup_{0\le t\le R} \frac{|n^{-\frac{1}{2}} Q_n(t)|}{q(D(t))} = 0 \quad \text{a.s.} \tag{2.10}$$

Conversely, if $\int_0^1 \{1/q(t)\}\, dt = \infty$, *then*

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{|n^{-\frac{1}{2}} Q_n(t)|}{q(D(t))} = \infty \quad \text{a.s.} \tag{2.11}$$

For the functional law of the iterated logarithm we need some more notation. Let $B[0,a]$, $0 < a < \infty$, denote the space of bounded real-valued functions defined on $[0,a]$ with the sup norm, and let $\mathbf{S}$ denote the set of absolutely continuous functions $f \in B[0, D(R)]$ such that

$$f(0) = 0 \quad \text{and} \quad \int_0^{D(R)} \{f'(s)\}^2\, ds \le 1.$$

THEOREM 2.3. *Let* $q \in \mathbf{Q}_{00}$. *If*

$$\int_0^{e^{-e}} \frac{1}{q^2(t) \log\log(1/t)}\, dt < \infty, \tag{2.12}$$

then almost surely the sequence $\{Q_n/[(q\circ D)(2\log\log n)^{\frac{1}{2}}]\}_{n=1}^{\infty}$ *is relatively compact in* $B[0,R]$ *with set of limit points equal to* $\{(f\circ D)/(q\circ D) : f \in \mathbf{S}\}$. *Conversely, if the integral in* (2.12) *is infinite, then*

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{|Q_n(t)|}{q(D(t))(2\log\log n)^{\frac{1}{2}}} = \infty \qquad \text{a.s.}$$

It should be noted that the respective conditions on q in Theorems 2.1–2.3 coincide exactly with those in the corresponding theorems for the ordinary empirical process. Furthermore, observe that the weight function q is bounded from below on $[R,\infty]$ by $q(R)$, implying that the behaviour of the weighted process is determined on the interval $[0,R]$. Therefore the assumption $H(R) < 1$ is essentially not restrictive in the study of the process $Q_n/q\circ D$. Finally note that the assumption that F is strictly increasing is too severe. It is only used for technical convenience and can be replaced by simply $F(R) > 0$, which is in turn only needed for the necessity parts of the theorems.

3. EXAMPLES

By giving the corresponding L_n, we now introduce some special examples of the process Q_n. For each of these examples conditions (a)–(d) hold.

EXAMPLE 1. The basic martingale M_n, i.e., $L_n = 1$. Note that $D = H^1$.

EXAMPLE 2. The empirical cumulative hazard process

$$B_n := n^{\frac{1}{2}}(\Lambda_n - \Lambda) \tag{3.1}$$

with

$$\Lambda_n(t) := \int_0^t \{1 - H_{n-}(s)\}^{-1}\, dH_n^1(s). \tag{3.2}$$

Here

$$L_n(t) = \{1 - H_{n-}(t)\}^{-1} 1_{\{H_{n-}(t)<1\}};$$

see Shorack and Wellner (1986, p. 301). Observe that $L = 1/(1-H)$ and hence that the asymptotic variance D of B_n is given by

$$D(t) = \int_0^t \{1 - H(s)\}^{-2}\, dH^1(s).$$

EXAMPLE 3. The product-limit process divided by $1 - F$, i.e.

$$\frac{X_n}{1-F} := \frac{n^{\frac{1}{2}}(\hat{F}_n - F)}{1-F}, \tag{3.3}$$

where $\hat{F}_n$ is the important Kaplan-Meier (1958) product-limit estimator of F. In the uncensored case, i.e. $G \equiv 0$, $\hat{F}_n$ coincides with the empirical distribution function. Now

$$L_n(t) = \frac{1 - \hat{F}_{n-}(t)}{1 - F(t)} \frac{1}{1 - H_{n-}(t)} 1_{\{H_{n-}(t)<1\}};$$

see Shorack and Wellner (1986, p. 301). Observe that again $L = 1/(1-H)$ and

$$D(t) = \int_0^t \{1 - H(s)\}^{-2}\, dH^1(s).$$

Theorem 2.1, specialized to Example 3, can be applied to the construction of asymptotic confidence bands for the quantile function F^{-1}; see Einmahl (1990).

Next, we consider the product-limit process X_n itself, instead of $X_n/(1-F)$, and show that the theorems in Section 2 essentially remain unaltered for this process.

EXAMPLE 3*. The product-limit process X_n. Since $1 - F$ is bounded away from zero on $[0, R]$, it is easily seen that the conditions on q in Theorems 2.1–2.3 remain exactly the same when Q_n is replaced by X_n. The only change which has to be made in Theorem 2.1 is replacing $W_n(D(t))$ by $\{1 - F(t)\} W_n(D(t))$ in (2.9). In Theorem 2.2 no change is required (apart from writing X_n instead of Q_n). To show that this theorem is really a weighted Glivenko-Cantelli theorem, we rewrite (2.10) using $X_n = n^{\frac{1}{2}}(\hat{F}_n - F)$:

$$\lim_{n\to\infty} \sup_{0\le t\le R} \frac{|\hat{F}_n(t) - F(t)|}{q(D(t))} = 0 \quad \text{a.s.}$$

Theorem 2.3 remains true for this example when the set of limit points is replaced by $\{[(1-F)(f \circ D)]/(q \circ D) : f \in \mathbf{S}\}$. This set can be rewritten as

$$\left\{ \frac{(1-F)(1+D)\left(g \circ \dfrac{D}{1+D}\right)}{q \circ D} : g \in \mathbf{F} \right\},$$

where $\mathbf{F}$ is the restriction to $[0, D(R)/\{1+D(R)\}]$ of the set of absolutely continuous functions $g \in B[0,1]$ such that

$$g(0) = g(1) = 0 \quad \text{and} \quad \int_0^t \{g'(s)\}^2\, ds \leq 1.$$

In the uncensored case ($G \equiv 0$), this set should and does reduce to

$$\left\{ \frac{g \circ F}{q \circ \{F/(1-F)\}} : g \in \mathbf{F} \right\},$$

which is essentially just the limit set of the ordinary weighted empirical process (see James 1975).

Other examples of Q_n are encountered in the theory of goodness-of-fit tests (see Koning 1991).

Theorem 2.1 for X_n is virtually identical to Theorem 3.2 in Csörgő, Csörgő, and Horváth (1987). To the best of our knowledge, Theorem 2.1 for the other examples and Theorems 2.2 and 2.3 for all the examples are new. A result related to Theorem 2.1 for X_n is given in Gill (1983) [see also Shorack and Wellner (1986, p. 319)]; an analogous result for B_n can be found on the same page of the latter reference. (In both cases the weighting $q \circ D$ is replaced there by $q \circ \{D/(1+D)\}$. This change, however, is inessential, since $1 + D$ is bounded from above on $[0, R]$.) A weighted central limit theorem, weaker than Theorem 2.1, for a multivariate version of B_n is presented in Ruymgaart (1989). Unweighted ($q \equiv 1$) Glivenko-Cantelli theorems for B_n and X_n can be found in e.g. Shorack and Wellner (1986, p. 304). The unweighted version of Theorem 2.3 for X_n is established in Csörgö and Horváth (1983).

To conclude this section we mention that our basic inequality (4.1) also yields Theorem 4.1 in Aly, Csörgő and Horváth (1985). That theorem is a law of the iterated logarithm à la Csáki (1977, Theorem 3.2) for the normalized product-limit process. Even the basic inequality with $q \equiv 1$ gives a number of suboptimal results for our examples. For example, Theorem 3.2 in Földes and Rejtő (1981) immediately improves to

$$\sup_{0 \leq t \leq R} |X_n(t)| = O(\log\log n)^{1/2}) \qquad \text{a.s.}$$

Also, an exponential inequality like Theorem 3.1 in that paper readily follows.

4. TOOLS

In this section we gather some tools that we need for the proofs of the main results.

Since L is positive and continuous on $[0, R]$, there are positive constants c_1, c_2 such that

$$c_1 < L(t) < c_2 \qquad \text{for all} \quad t \in [0, R]. \tag{4.1}$$

Hence, conditions (b), (c) imply that the sequence $\{\Omega_n\}_{n=1}^{\infty}$ of subsets of Ω defined by

$$\Omega_n = \{\omega \in \Omega : c_1 \leq L_n(t) \leq c_2 \text{ for all } t \in [0, R],\ V(L_n) \leq c\}$$

satisfies

$$P(\Omega_n^c \text{ i.o.}) = 0. \tag{4.2}$$

The following inequality follows from integration by parts and the monotonicity of q and D, after expressing M_n as

$$M_n(t) = U_n^1(t) + \int_0^t U_{n-}(s)\, d\Lambda(s), \tag{4.3}$$

where

$$U_n(t) = n^{\frac{1}{2}}\{H_n(t) - H(t)\}, \qquad U_n^1(t) = n^{\frac{1}{2}}\{H_n^1(t) - H^1(t)\} \tag{4.4}$$

are the empirical processes associated with H_n and H_n^1.

INEQUALITY 4.1. (Basic inequality). *Let $q \in \mathbf{Q}$. There exists a finite constant c_3 such that for every $\omega \in \Omega_n$ and $0 < \delta \le R$*

$$\sup_{0\le t\le\delta} \frac{|Q_n(t)|}{q(D(t))} \le c_3 \left(\sup_{0\le t\le\delta} \frac{|U_n^1(t)|}{q(D(t))} + \sup_{0\le t\le\delta} \frac{D(t)}{q(D(t))} \sup_{0\le t\le\delta} |U_n(t)| \right). \tag{4.5}$$

Next a strong approximation of the process Q_n is presented. In the proof of this result and also in Section 5 we shall make use of the following transformation of the pair (Z_i, δ_i):

$$\tilde{Z}_i = \delta_i H^1(Z_i) + (1-\delta_i)\{H^1(\infty) + H^0(Z_i)\}, \tag{4.6}$$

where

$$H^0(t) = P(Z_i \le t,\ \delta_i = 0) = H(t) - H^1(t). \tag{4.7}$$

It is easily shown that $\tilde{Z}_i$ is uniformly-(0, 1) distributed.

PROPOSITION 4.1. *There exists a two-parameter zero-mean Gaussian process Q on $[0,R]\times[0,\infty)$ with covariance function*

$$\mathcal{E}\, Q(t_1, n_1)Q(t_2, n_2) = (n_1 \wedge n_2)D(t_1 \wedge t_2) \tag{4.8}$$

such that

$$\sup_{0\le t\le R} |Q_n(t) - n^{-\frac{1}{2}} Q(t, n)| \xrightarrow[P]{} 0, \tag{4.9}$$

$$\sup_{0\le t\le R} \frac{|Q_n(t) - n^{-\frac{1}{2}} Q(t, n)|}{(\log\log n)^{\frac{1}{2}}} \xrightarrow[\text{a.s.}]{} 0. \tag{4.10}$$

Proof. First we show the existence of a two-parameter zero-mean Gaussian process M on $[0,R]\times[0,\infty)$ such that

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{n^{\frac{1}{2}}|M_n(t) - n^{-\frac{1}{2}} M(t, n)|}{\log^2 n} \le c_4 \quad \text{a.s.} \tag{4.11}$$

for some finite constant c_4. Let $\tilde{U}_n$ denote the (uniform) empirical process based on $\tilde{Z}_1, \ldots, \tilde{Z}_n$. The approximation theorem of Komlós, Major, and Tusnády (1975) yields the existence of a Kiefer process K with continuous sample paths such that

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{n^{\frac{1}{2}}|\tilde{U}_n(t) - n^{-\frac{1}{2}} K(t, n)|}{\log^2 n} \le c_5 \quad \text{a.s.}$$

for some finite constant c_5. Now define processes B^1, B^0, and B by

$$B^1(t, n) = K\big(H^1(t), n\big),$$
$$B^0(t, n) = K\big(H^1(\infty) + H^0(t), n\big) - K\big(H^1(\infty), n\big),$$
$$[B(t, n) = B^1(t, n) + B^0(t, n),$$

(see also Burke, Csörgő, and Horváth 1981), and the process M by

$$M(t, n) = B^1(t, n) + \int_0^t B(s, n)\, d\Lambda(s)$$

[compare with (4.3)]. It follows that

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{n^{\frac{1}{2}}|U_n^1(t) - n^{-\frac{1}{2}}B^1(t, n)|}{\log^2 n} \le c_5 \qquad \text{a.s.},$$

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{n^{\frac{1}{2}}|U_{n-}(t) - n^{-\frac{1}{2}}B(t, n)|}{\log^2 n} \le 3c_5 \qquad \text{a.s.},$$

and hence (4.11). Furthermore, M is a zero-mean Gaussian process. Covariance calculations yield

$$\mathcal{E} M(t_1, n_1)M(t_2, n_2) = (n_1 \wedge n_2)H^1(t_1 \wedge t_2).$$

Now we introduce the process

$$Q(t, n) = L(t)M(t, n) - \int_0^t M(s, n)\, dL(s).$$

It is easily proved that Q is a zero-mean Gaussian process with covariance function (4.8). It remains to show (4.9) and (4.10).

We will approximate $M(t, n)$ for given n by a pure jump process $S_n(t)$ defined by

$$S_n(t) = n^{-\frac{1}{2}}M(x_{i-1,n}, n) \qquad \text{for every } t \in [x_{i-1,n}, x_{i,n}), \qquad 1 \le i \le m(n),$$
$$S_n(R) = n^{-\frac{1}{2}}M(x_{m(n)-1,n}, n),$$

where $x_{0,n}, x_{1,n}, \ldots, x_{m(n),n}$ is the partition of condition (d). By Lemma 1.1.1 in Csörgő and Révész (1981) and condition (d) we have

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{n^{\alpha/2}|S_n(t) - n^{-\frac{1}{2}}M(t, n)|}{(\log n)^{\frac{1}{2}}} \le c_6 \qquad \text{a.s.} \tag{4.12}$$

for some finite constant c_6. Also, since

$$\sup_{0\le t\le R} |S_n(t)| \le \sup_{0\le t\le R} n^{-\frac{1}{2}}|M(t, n)|, \tag{4.13}$$

where the right-hand side is bounded from above by the supremum of an appropriate standard Wiener process on $[0,1]$, the left-hand side of (4.13) is $O_p(1)$. Furthermore, combining (4.13) with Theorem 1.41.1 in Csörgő and Révész (1981) yields

$$\limsup_{n\to\infty} \sup_{0\le t\le R} \frac{|S_n(t)|}{(\log\log n)^{\frac{1}{2}}} \le 2^{\frac{1}{2}} \qquad \text{a.s.} \tag{4.14}$$

Letting $V(f)$ denote the total variation of f, it follows that on Ω_n

$$\begin{aligned}\sup_{0\le t\le R} &|Q_n(t) - n^{-\frac{1}{2}}Q(t,n)| \\ &\le \Big(\sup_{0\le t\le R} |M_n(t) - n^{-\frac{1}{2}}M(t,n)|\Big)\Big(\sup_{0\le t\le R} |L_n(t)| + V(L_n)\Big) \\ &\quad + \Big(\sup_{0\le t\le R} |n^{-\frac{1}{2}}M(t,n) - S_n(t)|\Big)\Big(\sup_{0\le t\le R} |L_n(t) - L(t)| + V(L_n - L)\Big) \\ &\quad + \Big(\sup_{0\le t\le R} |S_n(t)|\Big)\Big(\sup_{0\le t\le R} |L_n(t) - L(t)| \\ &\qquad\qquad + \sum_{i=1}^{m(n)} |\{L_n(x_{i,n}) - L(x_{i,n})\} - \{L_n(x_{i-1,n}) - L(x_{i-1,n})\}|\Big) \\ &=: \Delta_{1n} + \Delta_{2n} + \Delta_{3n}.\end{aligned}$$

Now Δ_{1n} and Δ_{2n} tend to zero almost surely as consequences of (4.2), (4.11) and (2.1), (4.2), (4.12), respectively, whereas it can be shown that

$$\Delta_{3n} \xrightarrow[P]{} 0,$$

$$\frac{\Delta_{3n}}{(\log\log n)^{\frac{1}{2}}} \xrightarrow[\text{a.s.}]{} 0$$

as n tends to ∞, because of (2.1), the boundedness in probability of $\sup_{0\le t\le R} |S_n(t)|$, and (4.14). Q.E.D.

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Assume $q \in \mathbf{Q}_0$ satisfies (2.8). Define for each $n \in \mathbf{N}$ the standard Wiener process of (2.9) by

$$W_n(t) = n^{-\frac{1}{2}}Q\big(D^{-1}(t), n\big), \tag{5.1}$$

where Q is the Gaussian process of Proposition 4.1.

For any $0 < \delta < R$ we have

$$\begin{aligned}\sup_{0\le t\le R} \frac{|Q_n(t) - W_n(D(t))|}{q(D(t))} &\le \sup_{0\le t\le \delta} \frac{|Q_n(t)|}{q(D(t))} + \sup_{0\le t\le \delta} \frac{|W_n(D(t))|}{q(D(t))} \\ &\quad + \sup_{0\le t\le R} \frac{|Q_n(t) - W_n(D(t))|}{q(D(\delta))} \\ &=: \Delta_{4n} + \Delta_{5n} + \Delta_{6n}.\end{aligned} \tag{5.2}$$

Hence we must show that for any $\epsilon > 0$ and each $k = 4, 5, 6$ there exist $\delta = \delta(\epsilon)$ and $N = N(\epsilon) \in \mathbf{N}$ such that

$$P(\Delta_{kn} \ge \epsilon) \le \epsilon \qquad \text{for} \quad n \ge N. \tag{5.3}$$

From the basic inequality and (4.2), it follows that for a proof of (5.3) with $k = 4$ it is sufficient to show that

$$P\left(\sup_{0\le t\le\delta}\frac{|U_n^1(t)|}{q(D(t))}\ge\epsilon\right)\le\epsilon \tag{5.4}$$

and

$$P\left(\sup_{0\le t\le\delta}\frac{D(t)}{q(D(t))}\sup_{0\le t\le\delta}|U_n(t)|\ge\epsilon\right)\le\epsilon. \tag{5.5}$$

First consider (5.4). Note that it follows from (2.5) and (4.1) that for $0 < t \le R$

$$q(D(t))\ge q(c_1^2H^1(t)).$$

Define $q_1(\sigma) = q(c_1^2\sigma)$, $\sigma > 0$, and observe that q_1 satisfies (2.8). Now we have

$$\sup_{0\le t\le\delta}\frac{|U_n^1(t)|}{q(D(t))}\le\sup_{0\le t\le\delta}\frac{|U_n^1(t)|}{q_1(H^1(t))}\le\sup_{0\le t\le\delta'}\frac{|\tilde{U}_n(t)|}{q_1(t)},$$

with $\tilde{U}_n$ as in the proof of Proposition 4.1 and with $\delta' = H^1(\delta)$. In Shorack and Wellner (1986, pp. 462–463) it is shown that

$$P\left(\sup_{0\le t\le\delta'}\frac{|\tilde{U}_n(t)|}{q_1(t)}\ge\epsilon\right)\le\epsilon$$

for small enough δ' and $n \ge N$. This completes the proof of (5.4).

Next consider (5.5). Since $\sup_{0\le t\le\delta}|U_n(t)|$ is bounded in probability for every $\delta > 0$, it is enough to show that

$$\lim_{\delta\downarrow 0}\sup_{0\le t\le\delta}\frac{D(t)}{q(D(t))}=0,$$

which is a consequence of the fact that $q(\sigma)/\sigma^{\frac{1}{2}} \to \infty$ as $\sigma \downarrow 0$ (see e.g. Shorack and Wellner 1986, p. 462).

It is well known that for $q \in \mathrm{Q}_0$ satisfying (2.8) and δ small enough,

$$P\left(\sup_{0\le t\le D(\delta)}\frac{|W_n(t)|}{q(t)}\ge\epsilon\right)\le\epsilon,$$

and hence that (5.3) holds for $k = 5$. Finally, for $k = 6$, (5.3) follows from Proposition 4.1 and (5.1).

Now assume that for $q \in \mathrm{Q}_0$ (2.9) holds true. Let $\epsilon > 0$ be arbitrary, and write $c_7 = (c_1/c_2)^2\{1-H(R)\}$. Using (1.5) and the fact that $D(t)$ can be written as $\int_0^t L^2(s)\{1-H(s)\}\,d\Lambda(s)$, it is easily seen that on Ω_n for t with $0 \le \Lambda(t) \le c_7\epsilon^2/n$

$$Q_n(t)\ge -n^{\frac{1}{2}}c_2\Lambda(t)\ge -c_2\epsilon\{c_7\Lambda(t)\}^{\frac{1}{2}}\ge -\epsilon\{D(t)\}^{\frac{1}{2}}. \tag{5.6}$$

Hence on Ω_n

$$\sup_{0\le t\le R}\frac{|Q_n(t)-W_n(D(t))|}{q(D(t))}\ge\sup\left\{\frac{-\epsilon\{D(t)\}^{\frac{1}{2}}-W_n(D(t))}{q(D(t))}:0\le\Lambda(t)\le\frac{c_7\epsilon^2}{n}\right\}.$$

Thus, using (4.2), as $n \to \infty$,

$$P\left\{-W_n(D(t)) \le \epsilon\left[q(D(t)) + \{D(t)\}^{\frac{1}{2}}\right] : 0 \le \Lambda(t) \le \frac{c_7\epsilon^2}{n}\right\} \to 1,$$

and hence obviously

$$P\left\{-W_1(D(t)) \le \epsilon\left[q(D(t)) + \{D(t)\}^{\frac{1}{2}}\right] : 0 \le \Lambda(t) \le \frac{c_7\epsilon^2}{n}\right\} \to 1.$$

This implies that $\lim_{s\downarrow 0} |W_1(s)|/\{q(s) + s^{\frac{1}{2}}\} = 0$ a.s., which yields that q satisfies (2.8) (cf. Shorack and Wellner 1986, pp. 464–465). Q.E.D.

Proof of Theorem 2.2. Assume $\int_0^1 \{1/q(t)\}\,dt < \infty$. For any $0 < \delta < D(R)$ we have

$$\sup_{0\le t\le R} \frac{D(t)}{q(D(t))} \le \sup_{0\le t\le \delta} \frac{t}{q(t)} + \frac{D(R)}{q(\delta)}. \tag{5.7}$$

Since for any $0 < \lambda < 1$

$$-\frac{\lambda t}{q(t)}\log\lambda = \frac{\lambda t}{q(t)}\int_{\lambda t}^{t} \frac{1}{s}\,ds \le \int_{\lambda t}^{t} \frac{1}{q(s)}\,ds \to 0 \qquad \text{as} \quad t \downarrow 0,$$

it follows that $t/q(t) \to 0$ as $t \downarrow 0$, and hence the left-hand side of (5.7) is finite.

The basic inequality yields for $\omega \in \Omega_n$

$$\sup_{0\le t\le R} \frac{|n^{-\frac{1}{2}}Q_n(t)|}{q(D(t))} \le c_3\left(\sup_{0\le t\le R} \frac{|H_n^1(t) - H^1(t)|}{q(D(t))} + \sup_{0\le t\le R} \frac{D(t)}{q(D(t))} \sup_{0\le t\le R} |H_n(t) - H(t)|\right).$$

Because

$$\sup_{0\le t\le R} |H_n(t) - H(t)| \xrightarrow[\text{a.s.}]{} 0 \qquad \text{for} \quad n \to \infty$$

by the Glivenko-Cantelli theorem and because of (4.2), we now only have to prove

$$\sup_{0\le t\le R} \frac{|H_n^1(t) - H^1(t)|}{q(D(t))} \xrightarrow[\text{a.s.}]{} 0 \qquad \text{as} \quad n \to \infty, \tag{5.8}$$

in order to verify (2.10).

Write again $q_1(\sigma) = q(c_1^2\sigma)$ and observe that $\int_0^1 \{1/q_1(t)\}\,dt < \infty$. Now

$$\sup_{0\le t\le R} \frac{|H_n^1(t) - H^1(t)|}{q(D(t))} \le \sup_{0\le t\le R} \frac{|H_n^1(t) - H^1(t)|}{q_1(H^1(t))} \le \sup_{0\le t\le H^1(R)} \frac{|\tilde{H}_n(t) - t|}{q_1(t)},$$

where $\tilde{H}_n$ is the empirical distribution function based on the random variables $\tilde{Z}_1, \ldots, \tilde{Z}_n$ defined in (4.6). Applying Theorem 10.2.1 in Shorack and Wellner (1986) yields (5.8), and hence (2.10).

Now assume $\int_0^1 \{1/q(t)\}\,dt = \infty$. Using (1.5) and noting that $\int_0^t L_n(s)\{1 - H_{n-}(s)\}\,d\Lambda(s)$ is a process with continuous sample paths, we see that the jumps of the process $n^{-\frac{1}{2}}Q_n(t)$ have the same location and size as the jumps of the process $\int_0^t L_n(s)\,dH_n^1(s)$. It is easily seen that on Ω_n for $t = Z_n$ this last process makes a jump of at least $c_1\delta_n/n$. Hence for $0 < \delta < H^1(R)$

$$\sup_{0\le t\le R} \frac{n^{-\frac{1}{2}}|Q_n(t)|}{q(D(t))} \ge \sup_{0\le t\le R} \frac{n^{-\frac{1}{2}}|Q_n(t)|}{q_2(H^1(t))} \ge \frac{c_1\delta_n 1_{(0,R]}(Z_n)}{2nq_2(H^1(Z_n))} \ge \frac{c_1 1_{(0,\delta]}(\tilde{Z}_n)}{2nq_2(\tilde{Z}_n)}, \tag{5.9}$$

where $q_2(\sigma) = q(c_2^2\sigma)$. Since for all $M > 0$

$$1 + \sum_{n=1}^{\infty} P\left(\frac{1_{(0,\delta]}(\tilde{Z}_n)}{nq_2(Z_n)} \ge M\right) = \sum_{n=0}^{\infty} P\left(\frac{1_{(0,\delta]}(\tilde{Z}_1)}{Mq_2(\tilde{Z}_1)} \ge n\right)$$
$$\ge \mathcal{E}\,\frac{1_{(0,\delta]}(\tilde{Z}_1)}{Mq_2(\tilde{Z}_1)} = \frac{1}{M}\int_0^\delta \frac{1}{q_2(t)}\,dt = \infty,$$

the Borel-Cantelli lemma in combination with (5.9) implies (2.11). This completes the proof. Q.E.D.

Proof of Theorem 2.3. This theorem is a rather easy consequence of Lemma 5.1 and 5.2 below. The proof of Lemma 5.2 is omitted, since it closely resembles the proof of Theorem 2.2, with the formal change that the application of Theorem 10.2.1 in Shorack and Wellner (1986) is replaced by an application of Lemma 3.3 in James (1975). Q.E.D.

LEMMA 5.1. *The sequence* $\{Q_n/(2\log\log n)^{\frac{1}{2}}\}_{n=1}^{\infty}$ *is almost surely relatively compact in* $B[0,R]$ *with set of limit points equal to* $\{f\circ D : f \in \mathbf{S}\}$.

Proof. Proposition 4.1 implies

$$\sup_{0\le t\le R}\left|\frac{Q_n(t)}{(2\log\log n)^{\frac{1}{2}}} - \frac{Q(t,n)}{(2n\log\log n)^{\frac{1}{2}}}\right| \xrightarrow[\text{a.s.}]{} 0 \qquad \text{as } n\to\infty, \tag{5.10}$$

and hence the sequence $\{Q_n/(2\log\log n)^{\frac{1}{2}}\}_{n=1}^{\infty}$ has almost surely the same asymptotic behaviour as the sequence $\{Q(\cdot,n)/(2\log\log n)^{\frac{1}{2}}\}_{n=1}^{\infty}$, which is relatively compact in $B[0,R]$ with set of limit points equal to $\{f\circ D : f \in S\}$, as easily follows from Theorem 1.14.1 in Csörgő and Révész (1981). Q.E.D.

LEMMA 5.2. *Let* $q \in \mathbf{Q}_{00}$. *If* (2.12) *holds, then for every* $\epsilon > 0$ *there exists* $0 < \eta \le R$ *such that*

$$\limsup_{n\to\infty} \sup_{0\le t\le\eta} \frac{|Q_n(t)|}{q(D(t))(\log\log n)^{\frac{1}{2}}} < \epsilon \qquad a.s. \tag{5.11}$$

If the integral in (2.12) *is infinite, then for every* $0 < \eta \le R$

$$\limsup_{n\to\infty} \sup_{0\le t\le\eta} \frac{|Q_n(t)|}{q(D(t))(\log\log n)^{\frac{1}{2}}} = \infty \qquad a.s. \tag{5.12}$$

REFERENCES

Aalen, O.O. (1976). *Statistical Theory for a Family of Counting Processes*. Institute of Mathematical Statistics, Univ. of Copenhagen.

Aly, E.-E.A.A., Csörgő, M., and Horváth, L. (1985). Strong approximations of the quantile process of the product-limit estimator. *J. Multivariate Anal.*, 16, 185–210.

Burke, M.D., Csörgő, S., and Horváth, L. (1981). Strong approximations of some biometric estimates under random censorship. *Z. Wahrsch. verw. Geb.*, 56, 87–112.

Chibisov, D.M. (1964). Some theorems on the limiting behavior of empirical distribution functions. *Select. Transl. Math. Statist. Probab.*, 6, 147–156.

Csáki, E. (1977). The law of the iterated logarithm for normalized empirical distribution function. *Z. Wahrsch. verw. Geb.*, 38, 147–167.

Csörgő, M., Csörgő, S., and Horváth, L. (1987). Estimation of total time on test transforms and Lorenz curves under random censorship. *Statistics*, 18, 77–97.

Csörgő, S., and Horváth, L. (1983). The rate of strong uniform consistency for the product-limit estimator. *Z. Wahrsch. verw. Geb.*, 62, 411–426.

Csörgő, M., and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York.

Einmahl, J.H.J. (1990). Confidence bands for the quantile function under random censoring. Medical informatics and Statistics Report 31, University of Limburg, Maastricht.

Földes, A., and Rejtő, L. (1981). Strong uniform consistency for nonparametric survival curve estimators from randomly censored data. *Ann. Statist.*, 9, 122–129.

Gill, R.D. (1983). Large sample behaviour of the product-limit estimator on the whole line. *Ann. Statist.*, 11, 49–58.

James, B.R. (1975). A functional law of the iterated logarithm for weighted empirical distributions. *Ann. Probab.*, 3, 762–772.

Kaplan, E.L., and Meier, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.*, 53, 457–481.

Komlós, J., Major, P., and Tusnády, G. (1975). An approximation of partial sums of independent rv's and the sample df, I. *Z. Wahrsch. verw. Geb.*, 32, 111–131.

Koning, A.J. (1991). Stochastic integrals and goodness-of-fit tests. Ph.D. Thesis, University of Twente.

Lai, T.L. (1974). Convergence rates in the strong law of large numbers for random variables taking values in Banach spaces. *Bull. Inst. Math. Acad. Sinica*, 2, 67–85.

O'Reilly, N.E. (1974). On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probab.*, 2, 642–651.

Ruymgaart, F.H. (1989). Some properties of bivariate empirical hazard processes under random censoring. *J. Multivariate Anal.*, 28, 271–281.

Shorack, G.R., and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

Wellner, J.A. (1977). A Glivenko-Cantelli theorem and strong laws of large numbers for functions of order statistics. *Ann. Statist.*, 5, 473–480. Correction: *Ann. Statist.*, 6, 1394 (1978).

Received 30 May 1990
Revised 10 April 1991
Accepted 18 April 1991

Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Erasmus University
P.O. Box 1738
3000 DR Rotterdam
Rotterdam
The Netherlands