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Bahadur–Kiefer Theorems for the Product-Limit Process

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In the random censorship from the right model, strong and weak limit theorems for Bahadur–Kiefer type processes based on the product-limit estimator are established. The main theorm is sharp and may be considered as a final result as far as this type of research is concerned. As a consequence of this theorem a sharp uniform Bahadur representation for product-limit quantiles is obtained. © 1990 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. rv's with distribution function (df) F and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. rv's with df G. Both sequences are assumed to be independent. In the random censorship from the right model X_i may be censored on the right by Y_i so that the pair (Z_i, δ_i), $i = 1, 2, \ldots$ is observed, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = 1_{\{X_i \leqslant Y_i\}}$. The df H of the Z_i (which are also independent) is then given by $H = 1 - (1 - F)(1 - G)$.

As in most applications all rv's are assumed to be positive. Moreover, we assume throughout that the following condition is satisfied:

(A) F is differentiable on $(0, \infty)$ with continuous and positive derivative f and G is continuous on $(0, \infty)$.

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The product-limit (PL) estimator F_n (at stage n) introduced by Kaplan and Meier [9] comes out as the maximum likelihood estimator of F:

$$1-F_n(x)=\prod_{Z_{(i)}\leqslant x}(1-\delta_{(i)}/(n-i+1)), \qquad x\geqslant 0,$$

where $0<Z_{(1)}\leqslant\cdots\leqslant Z_{(n)}$ are the order statistics of the Z_i, $1\leqslant i\leqslant n$, and $\delta_{(i)}$ are the corresponding δ's. The associated PL process will be given by

$$\alpha_n(x)=n^{1/2}(F_n(x)-F(x)), \qquad x\geqslant 0.$$

The quantile function (or inverse) Q of F is naturally estimated by

$$Q_n(t)=\inf\{x: F_n(x)\geqslant t\}, \qquad t\in(0,1).$$

The PL quantile process is then given by

$$\beta_n(t)=n^{1/2}f(Q(t))(Q_n(t)-Q(t)), \qquad t\in(0,1).$$

In this paper we will study the so-called Bahadur–Kiefer process associated with the above PL and PL quantile process defined by

$$R_n(t)=\alpha_n(Q(t))+\beta_n(t), \qquad t\in(0,1). \tag{1.1}$$

In the uncensored case this process was introduced by Bahadur [2] and further investigated by Kiefer [10, 11]. A discussion of the literature on the subject for the censored as well as the uncensored case is postponed until the end of this section.

Write

$$T_G=\inf\{x: G(x)=1\}$$

and let Λ be a Gaussian process defined on $[0, F(T_G))$, with mean zero and covariance function

$$E(\Lambda(s)\,\Lambda(t))=(1-s)(1-t)\,h(s\wedge t), \qquad 0\leqslant s,\ t<F(T_G),$$

where

$$h(s)=\int_0^s(1-u)^{-2}\,(1-G(Q(u)))^{-1}\,du, \qquad 0\leqslant s<F(T_G).$$

Moreover, let us define the Gaussian process $\bar{\Lambda}_G$ by

$$\bar{\Lambda}_G(s)=\Lambda(s)/(1-G(Q(s))), \qquad 0\leqslant s<F(T_G).$$

; same spirit we write

$$\bar{\beta}_{n,G} = \beta_n/(1 - G \circ Q).$$

r first result gives the weak convergence of the finite dimensional
ɔutions of R_n.

ɜOREM 1. *Let condition* (A) *be satisfied and let* $0 < \theta < F(T_G)$.
ıse that for any $0 < s < Q(\theta)$

$$\lim_{\Delta \downarrow 0} \sup_{t : |t-s| \leqslant \Delta} |f(t) - f(s)|/\Delta^{1/2} = 0.$$

$\in \mathbb{N}$ *and* $0 < s_1 < \cdots < s_k < \theta$ *be fixed. Then as* $n \to \infty$,

$$n^{1/4}(R_n(s_1), ..., R_n(s_k)) \xrightarrow{d} (Z_1 |\bar{\Lambda}_G(s_1)|^{1/2}, ..., Z_k |\bar{\Lambda}_G(s_k)|^{1/2}),$$

$Z_1, ..., Z_k$ *are independent* $N(0, 1)$ *rv's independent of* $\bar{\Lambda}_G$.

; second result is an almost sure analogue of Theorem 1.

EOREM 2. *Under the conditions of Theorem* 1, *we have for* $s \in (0, \theta)$
almost surely,

$$\sup_{\infty} n^{1/4}(\log \log n)^{-3/4} |R_n(s)| \leqslant 2^{3/4}(1-s)^{1/2} h^{1/4}(s)(1 - G(Q(s)))^{-1/2}.$$

; now present our main result, which is so powerful that it has a lot
eresting results as a corollary. For its presentation we use the notation
$= \sup_{t \in [a,b]} |\varphi(t)|$, when φ is a real valued function on $[a, b]$.

EOREM 3. *Let condition* (A) *be satisfied and let* $0 < \theta < F(T_G)$.
ɔse there exists a $C \in (0, \infty)$ *such that*

$$\limsup_{\Delta \downarrow 0} \sup_{\substack{s,t : |t-s| \leqslant \Delta \\ t \leqslant Q(\theta)}} |f(t) - f(s)|/\Delta^{1/2} < C \tag{1.2}$$

et f be right-continuous at 0. *In case* $\lim_{x \downarrow 0} f(x) = 0$, *suppose that, in
ion, for some* $a \in (0, \infty)$,

$$\lim_{x \downarrow 0} F(x) |f'(x)| (f(x))^{-2} = a. \tag{1.3}$$

we have

$$\lim_{n \to \infty} n^{1/4}(\log n)^{-1/2} \|R_n\|_0^\theta / (\|\bar{\beta}_{n,G}\|_0^\theta)^{1/2} = 1 \qquad a.s.$$

Combination of Theorem 3 with the results in Aly, Csörgő, and Horváth [1] yields:

COROLLARY 1. *Under the conditions of Theorem* 3 *we have*

$$n^{1/4}(\log n)^{-1/2}\,\|R_n\|_0^\theta \xrightarrow{d} (\|\bar{\Lambda}_G\|_0^\theta)^{1/2} \qquad as \quad n\to\infty; \tag{1.4}$$

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\,\|R_n\|_0^\theta = 2^{1/4}\left(\left\|\frac{(1-I)\,h^{1/2}}{1-G\circ Q}\right\|_0^\theta\right)^{1/2} \quad a.s., \tag{1.5}$$

where I denotes the identity function;

$$\begin{aligned} &2^{-3/4}\pi^{1/2}(1-\theta)^{1/2}\,h^{1/4}(\theta)\\ &\quad\leqslant \liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\,\|R_n\|_0^\theta\\ &\quad\leqslant 2^{-3/4}\pi^{1/2}h^{1/4}(\theta)(1-G(Q(\theta)))^{-1/2} \qquad a.s. \end{aligned} \tag{1.6}$$

If $\lim_{x\downarrow 0} f(x)>0$, *then* (1.5) *entails that uniformly over all* $s\in(0,\theta)$ *we have*

$$Q_n(s)=Q(s)+\frac{s-F_n(Q(s))}{f(Q(s))}+O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \qquad a.s.$$

Discussion and Bibliography. In the uncensored case Kiefer [10] proved both Theorem 1 for the case $k=1$ and Theorem 2 (with the right constant). Note that in the uncensored case ($G\equiv 0$) the constant on the right in Theorem 2 reduces to $2^{3/4}(s(1-s))^{1/4}$, whereas in that case the actual value of the limsup is equal to $2^{5/4}3^{-3/4}(s(1-s))^{1/4}$. Theorem 1 for arbitrary $k\in\mathbb{N}$ and $G\equiv 0$ is presented in Beirlant *et al.* [3]. An in probability version of Theorem 3 (with $\theta=1$) in the uncensored case is established in Kiefer [11], where the author claims that the statement holds true almost surely; he did not publish a proof, however. Recently, his claim has been proved in Shorack [15, upper bound]) and Deheuvels and Mason [8, lower bound].

In the literature on the random censorship model only the type of problem discussed in Theorem 2 and (1.5) has been considered. A version of the statement in (1.5) can be found in Cheng [6], but with a worse rate. Aly *et al.* [1] derived the exact rate in (1.5), but did not find the right constant. Comparing Theorem 3 with its uncensored analogue (Theorem 1A in Deheuvels and Mason [8]), it is striking that $\bar{\beta}_{n,G}$ instead of β_n shows up in the denominator. Finally, note that the assumptions on F are somewhat milder than the usual Csörgő–Révész conditions, cf. Theorem 4.3

in Aly *et al.* [1]. Hence, for positive random variables, the main result in that paper (Theorem 4.4; a Kiefer process type strong approximation of β_n) is improved as far as the assumptions on F are considered.

2. Proofs

Consider the new set of rv's

$$U_i = F(X_i), \qquad V_i = F(Y_i), \qquad W_i = F(Z_i) = U_i \wedge V_i.$$

Then the U_i are i.i.d. uniform (0, 1) rv's, independent of the V_i; the V_i are also i.i.d. with *df* $G \circ Q$. The PL estimator based on these reduced rv's is then given by

$$\Gamma_n(t) = F_n(Q(t)), \qquad t \in (0, 1),$$

and the corresponding PL process is given by

$$a_n(t) = n^{1/2}(\Gamma_n(t) - t) = \alpha_n(Q(t)), \qquad t \in (0, 1).$$

Moreover, we put

$$q_n(t) = \inf\{s : \Gamma_n(s) \geqslant t\} = F(Q_n(t))$$

and

$$b_n(t) = n^{1/2}(q_n(t) - t), \qquad t \in (0, 1).$$

The corresponding Bahadur–Kiefer process is denoted by

$$r_n(t) = a_n(t) + b_n(t), \qquad t \in (0, 1).$$

We first present a number of lemmas which relate R_n to r_n.

Lemma 1. *Under the conditions of Theorem* 1 *we have for any* $t \in (0, \theta)$ *that as* $n \to \infty$

$$n^{1/4}(R_n(t) - r_n(t)) \xrightarrow{P} 0;$$

$$n^{1/4}(\log\log n)^{-3/4}\,(R_n(t) - r_n(t)) \longrightarrow 0 \qquad a.s.$$

Proof. We only prove the first statement; the second one is proved in an analogous way. Let $0 < \theta < F(T_G)$. As $R_n - r_n = \beta_n - b_n$, it remains to derive that as $n \to \infty$,

$$n^{1/4}(\beta_n(t) - b_n(t)) \xrightarrow{P} 0, \qquad 0 < t < \theta.$$

Remark that

$$\beta_n(t) = n^{1/2} \frac{f(Q(t))}{f(Q(\theta_{t,n}))} (q_n(t) - t) = \frac{f(Q(t))}{f(Q(\theta_{t,n}))} b_n(t),$$

where $|\theta_{t,n} - t| \leqslant n^{-1/2} |b_n(t)|$. As $n \to \infty$, $b_n(t) = O_P(1)$; hence

$$n^{1/4} |\beta_n(t) - b_n(t)| \leqslant n^{1/4} |b_n(t)| \left| \frac{f(Q(t))}{f(Q(\theta_{t,n}))} - 1 \right|$$
$$= O_P(1) \cdot \sup_{v : |t-v| \leqslant n^{-1/2} |b_n(t)|} \frac{|f(Q(t)) - f(Q(v))|}{(n^{-1/2} |b_n(t)|)^{1/2}},$$

which tends to zero in probability by assumption. ∎

LEMMA 2. *Under the conditions of Theorem* 3 *we have, as* $n \to \infty$,

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_0^\theta \to 0 \qquad a.s.$$

Proof. Let $\theta < F(T_G)$. As in the proof of Lemma 1 we find that it suffices to show that

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|b_n\|_0^\theta \left\| \frac{f(Q(I))}{f(Q(\theta_{I,n}))} - 1 \right\|_0^\theta \to 0 \qquad \text{a.s.}$$

First consider the case $\lim_{x \downarrow 0} f(x) > 0$. From Theorem 5.1 in Aly *et al.* [1]

$$\limsup_{n \to \infty} (2 \log \log n)^{-1/2} \|b_n\|_0^\theta = \|(1-I) h^{1/2}\|_0^\theta \qquad \text{a.s.,} \tag{2.1}$$

so that we are finished if we show that under the given conditions

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left\| \frac{f(Q(I))}{f(Q(\theta_{I,n}))} - 1 \right\|_0^\theta \to 0 \qquad \text{a.s.}$$

By (1.2) for some $K_\theta \in (0, \infty)$ we have almost surely

$$\limsup_{n \to \infty} n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left\| \frac{f(Q(I)) - f(Q(\theta_{I,n}))}{f(Q(\theta_{I,n}))} \right\|_0^\theta$$
$$\leqslant CK_\theta \limsup_{n \to \infty} (\log n)^{-1/2} (\log \log n)^{3/4} (\|b_n\|_0^\theta)^{1/2},$$

which equals zero almost surely by application of (2.1).

Now suppose $\lim_{x \downarrow 0} f(x) = 0$. With the same method as above it is immediate that for any $0 < \varepsilon < \theta$,

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_\varepsilon^\theta \to 0 \qquad \text{a.s.} \tag{2.2}$$

As already mentioned in Aly *et al.* [1, proof of Theorem 4.3], the proof of (3.3) in Csörgő and Révész [7] can be mimicked to show that for some $C_1 \in (0, \infty)$ and for "small" $\varepsilon > 0$

$$n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\, \|\beta_n - b_n\|_{\delta(n)}^{\varepsilon} \to 0 \qquad \text{a.s.,} \tag{2.3}$$

where $\delta(n) = C_1 n^{-1}\log\log n$, since for "small" $\varepsilon > 0$,

$$\sup_{0 < x \leqslant Q(2\varepsilon)} F(x)\,|f'(x)|\,(f(x))^{-2} \leqslant 2a.$$

In Aly *et al.* [1] it is also shown that

$$\|b_n\|_0^{\delta(n)} \overset{\text{a.s.}}{=} O(n^{-1/2}\log\log n) \qquad \text{as} \quad n \to \infty. \tag{2.4}$$

As $f \circ Q$ is regularly varying at zero with positive index a, one can construct a non-decreasing function $\tilde{f}_Q$ such that $\tilde{f}_Q \leqslant f \circ Q$ and $\lim_{x \downarrow 0} \tilde{f}_Q(x)/f(Q(x)) = 1$. (See, e.g., Theorem 1.5.3 in Bingham *et al.* [4].) Let $U_{(1)} \leqslant \cdots \leqslant U_{(v_n)}$ denote the order statistics of those observation among $U_1, ..., U_n$, which are uncensored, i.e., for which $\delta_i = 1$. Assume $\Gamma_n(U_{(i-1)}) < t \leqslant \Gamma_n(U_{(i)})$. Then for $t \leqslant U_{(i)}$ and n large enough

$$\begin{aligned} |\beta_n(t)| &\leqslant n^{1/2} \int_t^{U_{(i)}} \frac{f(Q(t))}{f(Q(u))}\, du \\ &\leqslant n^{1/2} \int_t^{U_{(i)}} \frac{f(Q(t))}{\tilde{f}_Q(u)}\, du \\ &\leqslant \sup_{t \in (0, \delta(n)]} \frac{f(Q(t))}{\tilde{f}_Q(t)} \cdot n^{1/2} \int_t^{U_{(i)}} \frac{\tilde{f}_Q(t)}{\tilde{f}_Q(u)}\, du \\ &\leqslant 2b_n(t). \end{aligned} \tag{2.5}$$

The case $t > U_{(i)}$ can be handled along the lines of (3.14) in Csörgő and Révész [7], cf. Aly *et al.* [1, pp. 200–201]. From this remark, (2.5) in combination with (2.4), and (2.4) itself, we have

$$\|\beta_n - b_n\|_0^{\delta(n)} \leqslant \|\beta_n\|_0^{\delta(n)} + \|b_n\|_0^{\delta(n)} \overset{\text{a.s.}}{=} O(n^{-1/2}(\log n)^{2a}).$$

Combining this with (2.2) and (2.3) completes the proof for the case $\lim_{x \downarrow 0} f(x) = 0$ and hence of the lemma. ∎

LEMMA 3. *Under the conditions of Theorem 3 we have, as* $n \to \infty$,

$$n^{1/4}(\log n)^{-1/2}\, \|R_n - r_n\|_0^{\theta} / (\|\beta_{n,G}\|_0^{\theta})^{1/2} \to 0 \qquad a.s.;$$

$$\left\{ \frac{\|r_n\|_0^{\theta}}{(\|\beta_{n,G}\|_0^{\theta})^{1/2}} - \frac{\|r_n\|_0^{\theta}}{(\|b_n/(1-G)\|_0^{\theta})^{1/2}} \right\} \Big/ \left(\frac{\|r_n\|_0^{\theta}}{(\|b_n/(1-G)\|_0^{\theta})^{1/2}} \right) \to 0 \qquad a.s. \tag{2.6}$$

Proof. Combination of Lemma 2 and

$$\liminf_{n\to\infty} (\log\log n)^{1/2} \|b_n\|_0^\theta > 0 \qquad \text{a.s.} \tag{2.7}$$

(see Fact 3 below) yields the first statement in (2.6). To prove the second statement it suffices to show that

$$\{(\|b_n/(1-G)\|_0^\theta)^{1/2} - (\|\bar{\beta}_{n,G}\|_0^\theta)^{1/2}\}/(\|\bar{\beta}_{n,G}\|_0^\theta)^{1/2} \to 0 \qquad \text{a.s.}$$

Using $x^{1/2} - y^{1/2} = (x-y)/(x^{1/2}+y^{1/2})$, $x, y > 0$, (2.7), and again Lemma 2, the proof reduces to showing

$$(\log\log n)^{1/2} \|\beta_n - b_n\|_0^\theta \to 0 \qquad \text{a.s.,}$$

which follows from one more application of Lemma 2. ∎

From Lemma 1 and Lemma 3, it follows that we can confine ourselves to the proofs of Theorems 1–2 and Theorem 3, respectively, in case the X_i's are uniformly (0, 1) distributed and the Y_i's are distributed according to a *df* G (which is now shorthand for $G \circ Q$) with support on (0, 1). Observe that $G \circ Q$ is continuous, since F is strictly increasing. We also adopt the notation introduced at the beginning of this section.

The remainder of this paper is organized as follows. We begin by recording a number of facts, which are required for the proofs. After that, we give a detailed proof of Theorem 3. Finally, short proofs of Theorems 1 and 2 are presented.

Fact 1 (Burke, Csörgő, and Horváth [5], Major and Rejtő [12]). There exists a two-parameter standard Wiener process W such that, for any $\theta \in (0, T_G)$,

$$\|a_n - n^{-1/2}(1-I)\, W(h(I), n)\|_0^\theta = O(n^{-1/2}(\log n)^2) \qquad \text{a.s.} \tag{2.8}$$

Define a sequence $\{W_n\}_{n=1}^\infty$ of (one-parameter) standard Wiener processes by

$$W_n = n^{-1/2} W(I, n), \tag{2.9}$$

write

$$\Lambda_n = (1-I)\, W_n \circ h, \tag{2.10}$$

and note that for all $n \in \mathbb{N} : \Lambda_n \stackrel{d}{=} \Lambda$, with Λ as in Section 1.

Fact 2 (cf. Shorack [15]). Let W_n be as above, $c \in (0, \infty)$ arbitrary, and $\{k_n\}_{n=1}^\infty$ a sequence of positive numbers such that $k_n \downarrow$, $nk_n \uparrow$, $\log(1/k_n)/\log\log n \to \infty$ and $\log(1/k_n)/(nk_n) \to 0$. Then

$$\limsup_{n\to\infty} \sup_{\substack{|u-v| \leqslant k_n \\ 0 \leqslant u \leqslant c \\ v \geqslant 0}} \frac{|W_n(u) - W_n(v)|}{(2k_n \log(1/k_n))^{1/2}} \leqslant 1 \qquad \text{a.s.} \tag{2.11}$$

and

$$\limsup_{n\to\infty} \sup_{\substack{|u-v|>k_n\\ 0\leqslant u,v\leqslant c}} \frac{|W_n(u)-W_n(v)|}{(2\,|u-v|\log(1/k_n))^{1/2}} \leqslant 1 \qquad \text{a.s.} \tag{2.12}$$

Fact 3 (Aly, Csörgő, and Horváth [1]). We have almost surely

$$\limsup_{n\to\infty} (\log\log n)^{-1/2} \left\| \frac{b_n}{1-G} \right\|_0^\theta = 2^{1/2} \left\| \frac{(1-I)\,h^{1/2}}{1-G} \right\|_0^\theta \tag{2.13}$$

and

$$\pi 8^{-1/2}(1-\theta)\,h^{1/2}(\theta) \leqslant \liminf_{n\to\infty} (\log\log n)^{1/2} \left\| \frac{b_n}{1-G} \right\|_0^\theta \leqslant \pi 8^{-1/2} \frac{h^{1/2}(\theta)}{1-G(\theta)}. \tag{2.14}$$

Proof of Theorem 3. The proof of the upper bound part is an adaptation of that in Shorack [15], whereas the proof of the lower bound part is based on Deheuvels and Mason [8]. We first show that if $0<\theta<T_G$,

$$LS := \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}\, \|r_n\|_0^\theta / (\|b_n/(1-G)\|_0^\theta)^{1/2} \leqslant 1 \qquad \text{a.s.} \tag{2.15}$$

Note that for any $s\in[0,\theta]$,

$$r_n(s) = a_n(s) - a_n(q_n(s)) + n^{1/2}(\Gamma_n(q_n(s))-s).$$

In Sander [13] it is shown that

$$n^{1/2}\,\|\Gamma_n\circ q_n - I\|_0^\theta = O(n^{-1/2}) \qquad \text{a.s.}$$

Hence, (2.8) and (2.14) entail that

$$LS = \limsup_{n\to\infty} \frac{n^{1/4}\,\|A_n - A_n\circ q_n\|_0^\theta}{(\log n\, \|b_n/(1-G)\|_0^\theta)^{1/2}} \qquad \text{a.s.}$$

Let

$$k_n = \pi(1-\theta)\,h^{1/2}(\theta)/(8n\log\log n)^{1/2},$$

$$I_n = \{(s,t): s\geqslant 0,\ 0\leqslant t\leqslant\theta,\ |h(t)-h(s)| \leqslant \|h\circ q_n - h\|_0^\theta,$$
$$(1-t)^2\,|h(t)-h(s)| \leqslant \|(1-I)^2\,(h\circ q_n - h)\|_0^\theta\},$$

$$J_n = \{(s,t)\in I_n: |h(t)-h(s)| \leqslant k_n\},$$

$$K_n = \{(s,t)\in I_n: |h(t)-h(s)| > k_n\}.$$

Then almost surely

$$LS \leqslant \limsup_{n\to\infty} \sup_{(s,t)\in I_n} \frac{|A_n(t)-A_n(s)|}{(\log n \, \|(q_n-I)/(1-G)\|_0^\theta)^{1/2}}$$

$$\leqslant \limsup_{n\to\infty} \sup_{(s,t)\in I_n} \frac{|t-s| \, |W_n(h(s))|}{(\log n \, \|(q_n-I)/(1-G)\|_0^\theta)^{1/2}}$$

$$+ \limsup_{n\to\infty} \sup_{(s,t)\in I_n} \frac{(1-t) \, |W_n(h(t))-W_n(h(s))|}{(\log n \, \|(q_n-I)/(1-G)\|_0^\theta)^{1/2}}$$

$$=: LS_1 + \limsup_{n\to\infty} \sup_{(s,t)\in I_n} A_n(s,t) =: LS_1 + LS_2.$$

It is well known that for arbitrary $0<c<\infty$,

$$\|W_n\|_0^c = O((\log\log n)^{1/2}) \qquad \text{a.s.} \tag{2.17}$$

From (2.13) we obtain

$$\|h\circ q_n - h\|_0^\theta \leqslant \|q_n - I\|_0^\theta \, \|h'\|_0^{\theta\vee q_n(\theta)} \overset{\text{a.s.}}{=} O(n^{-1/2}(\log\log n)^{1/2}). \tag{2.18}$$

Hence from (2.17), (2.18), and (2.14) we have a.s. as $n\to\infty$,

$$\sup_{(s,t)\in I_n} \frac{|t-s| \, |W_n(h(s))|}{(\log n \, \|(q_n-I)/(1-G)\|_0^\theta)^{1/2}} = O(n^{-1/4}(\log n)^{-1/2} \, (\log\log n)^{5/4}),$$

implying that $LS_1 = 0$ a.s. Furthermore,

$$LS_2 = \limsup_{n\to\infty} \sup_{(s,t)\in J_n} A_n(s,t) \vee \limsup_{n\to\infty} \sup_{(s,t)\in K_n} A_n(s,t)$$

$$=: LS_3 \vee LS_4.$$

First, by (2.14) we have a.s.

$$LS_3 \leqslant \limsup_{n\to\infty} \sup_{(s,t)\in J_n} \frac{(1-t) \, |W_n(h(t))-W_n(h(s))|}{(k_n \log n)^{1/2}}$$

$$\leqslant \limsup_{n\to\infty} \sup_{\substack{|u-v|\leqslant k_n \\ 0\leqslant u\leqslant h(\theta) \\ v\geqslant 0}} \frac{|W_n(u)-W_n(v)|}{(2k_n \log(1/k_n))^{1/2}}$$

as $\log(1/k_n)/\log n \to 1/2$ as $n\to\infty$. Hence $LS_3 \leqslant 1$ a.s., because of (2.11).

Next, since G is continuous $\|(q_n-I)/(1-G)\|_0^\theta \sim \|(h\circ q_n - h)(1-I)^2\|_0^\theta$ a.s. as $n\to\infty$, so that

$$LS_4 = \limsup_{n\to\infty} \sup_{(s,t)\in K_n} \frac{(1-t)\,|W_n(h(t)) - W_n(h(s))|}{(\log n\, \|(h\circ q_n - h)(1-I)^2\|_0^\theta)^{1/2}}$$

$$\leqslant \limsup_{n\to\infty} \sup_{(s,t)\in K_n} \frac{|W_n(h(t)) - W_n(h(s))|}{|h(t)-h(s)|^{1/2}\,(\log n)^{1/2}}$$

$$\leqslant \limsup_{n\to\infty} \sup_{\substack{|u-v|>k_n\\ 0\leqslant u,v\leqslant 2h(\theta)}} \frac{|W_n(u)-W_n(v)|}{|u-v|^{1/2}\,(2\log(1/k_n))^{1/2}} \qquad \text{a.s.}$$

Applying (2.12) yields $LS_4 \leqslant 1$ a.s. Hence the proof of (2.15) is completed.

Now it remains to show that if $0<\theta<T_G$,

$$LI := \liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}\, \|r_n\|_0^\theta / (\|b_n/(1-G)\|_0^\theta)^{1/2} \geqslant 1 \qquad \text{a.s.} \qquad (2.19)$$

Using similar steps as in the upper bound part of this proof we find that if suffices to show that

$$\liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2} \frac{\|(1-I)(W_n\circ h - W_n\circ h\circ q_n)\|_0^\theta}{(\|b_n/(1-G)\|_0^\theta)^{1/2}} \geqslant 1 \qquad \text{a.s.}$$

Let

$$h(q_n(t)) = h(t) + n^{-1/2} b_n(t)\, h'(a(n,t)), \qquad (2.20)$$

where $|a(n,t)-t| \leqslant n^{-1/2}\,|b_n(t)|$. Then, with h^i denoting the inverse of h,

$$LI = \liminf_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}} \times \frac{\|(1-h^i)(W_n - W_n\circ(I + n^{-1/2}(b_n\circ h^i)\, h'(a(n,h^i))))\|_0^{h(\theta)}}{(\|(b_n\circ h^i)\, h'(a(n,h^i))(1-h^i)^2\|_0^{h(\theta)})^{1/2}} \qquad \text{a.s.}$$

Write for $v\in[0,n]$,

$$\begin{aligned} \psi_{\theta,n}(v) &= h^i(vh(\theta)/n),\\ \pi_n(v) &= 1-\psi_{\theta,n}(v), \qquad (2.21)\\ f_n(v) &= (n^{1/2}/h(\theta))\, b_n(\psi_{\theta,n}(v))\, h'(a(n,\psi_{\theta,n}(v))), \end{aligned}$$

and observe that for any standard Wiener process W_n, the process $\bar{W}_n$ defined by

$$\bar{W}_n(v) = (n/h(\theta))^{1/2}\, W_n(vh(\theta)/n), \qquad v\geqslant 0,$$

is again a standard Wiener process. So by changing variables $(v=(n/h(\theta))t)$, we obtain

$$LI=\liminf_{n\to\infty}\frac{\|\pi_n(\bar{W}_n-\bar{W}_n\circ(I+f_n))\|_0^n}{(\log n)^{1/2}\,(\|\pi_n^2 f_n\|_0^n)^{1/2}}\qquad \text{a.s.} \tag{2.22}$$

Now to show that the right side of (2.22) is not smaller than one a.s., we can make use of the following proposition, which constitutes a generalization of Proposition 1 in Deheuvels and Mason [8]. In our proposition we will abuse notation by again using sequences of functions $\{\pi_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ and a sequence $\{\bar{W}_n\}_{n=1}^{\infty}$ of Wiener processes. These sequences are defined below and are not related to the above sequences with the same names. However, we will apply the Proposition with π_n, f_n, and $\bar{W}_n$ as above. Let $\{\pi_n\}_{n=1}^{\infty}$ be a sequence of decreasing functions satisfying:

(π1) there exists some $c>0$, such that

$$c\leqslant\|\pi_n\|_0^n\leqslant 1\qquad \text{for all}\quad n\geqslant 1,$$

(π2) $\limsup n\,\|\pi_n'\|_0^n<\infty$.

For any $\gamma>1$, $a>1$, $\eta>0$, $v\geqslant 1$ we denote by $\mathfrak{F}_\pi(\gamma, a, \eta, v)$ the subclass of all sequences $\{f_n\}_{n=1}^{\infty}$ of real-valued functions defined on $[0,\infty)$ such that

(F_π1) for all $n\geqslant 3$,

$$\gamma^{-1}n^{1/2}/\log^2 n\leqslant\|\pi_n^2 f_n\|_0^n\leqslant\gamma n^{1/2}\log^2 n,$$

(F_π2) for all $n\geqslant v$,

$$M_n(\pi_n^2 f_n):=\max\{\inf_{s\in I_n}\pi_n^2(s)\,f_n(s),\ \inf_{s\in I_n}(-\pi_n^2(s)\,f_n(s))\}$$
$$\geqslant a^{-1}\,\|\pi_n^2 f_n\|_0^n,$$

for some closed interval $I_n\subset[0,n]$ of length $\eta n e^{-(\log\log n)^2}$,

(F3) for all $n\geqslant 1$, $0\leqslant s+f_n(s)$ for $s\in[0,n]$.

Let $\mathfrak{F}_\pi=\bigcap_{a>1}(\bigcup_{\gamma>0}\bigcup_{\eta>0}\bigcup_{v\geqslant 1}\mathfrak{F}_\pi(\gamma,a,\eta,v))$. (Here γ, a, η are assumed to be rational and v an integer.)

PROPOSITION. *With the above notation and* $\{\bar{W}_n\}_{n=1}^{\infty}$ *being* any *sequence of standard Wiener process on* $[0,\infty)$ *sitting on a joint probability space, we have with probability one for all sequences* $\{f_n\}_{n=1}^{\infty}\in\mathfrak{F}_\pi$,

$$\liminf_{n\to\infty}R_n(\pi_n, f_n)\geqslant 1,$$

where $R_n(\pi_n, f_n)=\{\|\pi_n^2 f_n\|_0^n\log n\}^{-1/2}\,\|\pi_n(\bar{W}_n\circ(I+f_n)-\bar{W}_n)\|_0^n$.

Proof. Choose any $\{f_n\}_{n=1}^{\infty} \in \mathfrak{F}_\pi(\gamma, a, \eta, \nu)$, where $\gamma > 1$, $a > 1$, $\eta > 0$ are rationals and ν is a positive integer. Define

$$h_n(k) = \gamma^{-1} a^k n^{1/2}/\log^2 n \quad \text{for} \quad k = -3, -2, -1, 0, ..., k(n) := [\log_a (c^{-2}\gamma^2 \log^4 n)] + 1$$

and

$$I_n(m) = [m\, \delta n \exp(-(\log\log n)^2), (m+1)\, \delta n \exp(-(\log\log n)^2)] \quad \text{for} \quad m = 0, 1, ..., m(n) := [\delta^{-1} \exp((\log\log n)^2)] + 1,$$

with $\delta = \eta/6$. Let $I_n = [\lambda_n, \rho_n]$. By ($F_\pi 1$) and ($\pi 1$), for all $n \geqslant 3$ we can find an $0 \leqslant l_n \leqslant k(n)$ such that

$$h_n(l_n - 1) \leqslant \|\pi_n^2 f_n\|_0^n / \pi_n^2(\rho_n) \leqslant h_n(l_n). \tag{2.23}$$

Hence by ($F_\pi 2$), for all $n \geqslant \max(\nu, 3)$,

$$\pi_n^2(\rho_n)\, h_n(l_n - 2) \leqslant a^{-1} \|\pi_n^2 f_n\|_0^n \leqslant M_n(\pi_n^2 f_n) \leqslant \|\pi_n^2 f_n\|_0^n \leqslant \pi_n^2(\rho_n)\, h_n(l_n). \tag{2.24}$$

Now

$$R_n(\pi_n, f_n) \geqslant \{\sup_{s \in I_n} |\bar{W}_n(s + f_n(s)) - W_n(s)|\}/((\log n)\, h_n(l_n))^{1/2} =: A_n. \tag{2.25}$$

Furthermore by ($\pi 1$) and ($\pi 2$) there exist $K > 0$, $\nu_1 > 1$ such that for $n \geqslant \nu_1$,

$$\begin{aligned} \pi_n(\lambda_n)/\pi_n(\rho_n) &= 1 + \{\pi_n(\lambda_n) - \pi_n(\rho_n)\}/\pi_n(\rho_n) \\ &\leqslant 1 + |c^{-1}(\rho_n - \lambda_n)\, \pi_n'(\theta_n)| \\ &\leqslant 1 + K\eta n e^{-(\log\log n)^2}/n \leqslant a^{1/2}, \end{aligned} \tag{2.26}$$

with $\lambda_n \leqslant \theta_n \leqslant \rho_n$. Also we may choose an $1 \leqslant m \leqslant m(n)$ such that

$$I_n(m-1) \cup I_n(m) \subset I_n. \tag{2.27}$$

Suppose first that $M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (\pi_n^2(s)\, f_n(s))$. Then by (2.23), (2.24), and (2.26) we have for all $n \geqslant \nu_1$ and $s \in I_n$,

$$\begin{aligned} h_n(l_n - 3) &\leqslant h_n(l_n - 2)\, \pi_n^2(\rho_n)/\pi_n^2(\lambda_n) \\ &\leqslant M_n(\pi_n^2 f_n)/\pi_n^2(\lambda_n) \leqslant f_n(s) \\ &\leqslant \|\pi_n^2 f_n\|_0^n / \pi_n^2(\rho_n) \leqslant h_n(l_n). \end{aligned} \tag{2.28}$$

So for $s \in I_n$ and n large enough,

$$|f_n(s) - h_n(l_n)| \leqslant (1 - a^{-3})\, h_n(l_n)$$

and, thus,

$$\sup_{s \in I_n} |\bar{W}_n(s + f_n(s)) - \bar{W}_n(s)| \geqslant \sup_{s \in I_n(m)} |\bar{W}_n(s + h_n(l_n)) - \bar{W}_n(s)| - \sup_{0 \leqslant s \leqslant 2n} \sup_{0 \leqslant t \leqslant \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)|,$$

where $\tau = 1 - a^{-3}$. Hence in this case,

$$A_n \geqslant \Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau),$$

where

$$\Delta_n(a, \gamma, \delta) = \min_{-3 \leqslant k \leqslant k(n)} \min_{0 \leqslant m \leqslant m(n)} \sup_{s \in I_n(m)} |\bar{W}_n(s + h_n(k)) - \bar{W}_n(s)| / (h_n(k) \log n)^{1/2}$$

and

$$D_n(a, \gamma, \tau) = \max_{-3 \leqslant k \leqslant k(n)} \sup_{0 \leqslant s \leqslant 2n} \sup_{0 \leqslant t \leqslant \tau h_n(k)} |\bar{W}_n(s + t) - \bar{W}_n(s)| / (h_n(k) \log n)^{1/2}.$$

Next, suppose $M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (-\pi_n^2(s)\, f_n(s))$. Then similarly as in the preceding case one shows that for $s \in I_n$ and n large enough,

$$|f_n(s) + h_n(l_n - 3)| \leqslant \tau h_n(l_n)$$

and $0 \leqslant s + f_n(s) \leqslant s - h_n(l_n - 3)$. Thus,

$$\sup_{u \in I_n} |\bar{W}_n(u + f_n(u)) - \bar{W}_n(u)| \geqslant \sup_{u \in I_n} |\bar{W}_n(u - h_n(l_n - 3)) - \bar{W}_n(u)| - \sup_{0 \leqslant s \leqslant 2n} \sup_{0 \leqslant t \leqslant \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)|.$$

Note that there exists $\nu_2 \geqslant \max(\nu, 3)$ such that for all $n \geqslant \nu_2$,

$$h_n(l_n) \leqslant h_n(k(n)) \leqslant \tfrac{1}{2} \delta n \exp(-(\log \log n)^2). \tag{2.29}$$

Hence we have $\{u = s + h_n(l_n - 3) : s \in I_n(m-1)\} \subset I_n(m-1) \cup I_n(m) \subset I_n$, so that in the present case,

$$\begin{aligned} A_n &\geqslant (h_n(l_n) \log n)^{-1/2} \Big\{ \sup_{s \in I_n(m-1)} |\bar{W}_n(s + h_n(l_n - 3)) - \bar{W}_n(s)| \\ &\quad - \sup_{0 \leqslant s \leqslant 2n} \sup_{0 \leqslant t \leqslant \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)| \Big\} \\ &\geqslant (h_n(l_n - 3)/h_n(l_n))^{1/2} (\Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau)). \end{aligned}$$

Hence, in both cases possible we have

$$A_n \geqslant a^{-3/2} \Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau).$$

Now from (slight modifications of) Lemmas 1 and 2 in Deheuvels and Mason [8] we obtain with probability one uniformly over all sequences $\{f_n\}_{n=1}^{\infty} \in \mathfrak{F}_\pi(\gamma, a, \eta, v)$,

$$\liminf_{n \to \infty} R_n(\pi_n, f_n) \geqslant a^{-3/2} - 2(1-a^{-3})^{1/2}. \tag{2.30}$$

Observing that the right side of (2.30) can be chosen arbitrary close to one for a suitable choice of $a > 1$ completes the proof. ▌

Let us now finish the proof of Theorem 3. Observe that (π1) and (π2) are easily checked for π_n as defined in (2.21). So it suffices to verify the conditions ($F_\pi 1$), ($F_\pi 2$), and (F3) in case

$$\begin{aligned} f_n(s) &= (n^{1/2}/h(\theta))\, b_n(\psi_{\theta,n}(s))\, h'(a(n, \psi_{\theta,n}(s))) \\ &= -s + (n/h(\theta))\, h(q_n(\psi_{\theta,n}(s))), \end{aligned}$$

where the last equality follows from (2.20). So (F3) is immediate. To check ($F_\pi 1$) remark that for any $0 < \tilde{\theta} < T_G$,

$$1 \leqslant h'(u) \leqslant (1-\tilde{\theta})^{-2}\,(1-G(\tilde{\theta}))^{-1}, \qquad 0 \leqslant u \leqslant \tilde{\theta}.$$

Using condition (π1) and Fact 3 we see that $\{f_n\}_{n=1}^{n}$ satisfies ($F_\pi 1$) almost surely for γ large enough.

Finally, we show that $\{f_n\}_{n=1}^{\infty}$ satisfies ($F_\pi 2$) almost surely. Let $\kappa_n = \eta n \exp(-(\log\log n)^2)$. For any $s, t \in [0, n]$,

$$\begin{aligned} |\pi_n^2(t)\, f_n(t) - \pi_n^2(s)\, f_n(s)| &\leqslant |(\pi_n^2(t) - \pi_n^2(s))\, f_n(t)| + |\pi_n^2(s)(f_n(t) - f_n(s))| \\ &=: d_1(s,t) + d_2(s,t). \end{aligned}$$

First,

$$\begin{aligned} d_1(s,t) &\leqslant 2\, \|\pi_n\|_0^n\, \|f_n\|_0^n\, |\pi_n(t) - \pi_n(s)| \\ &\leqslant 2c^{-2}\, \|\pi_n^2 f_n\|_0^n\, \|\pi_n'\|_0^n\, |t-s|, \end{aligned}$$

where for the last inequality (π1) is used twice. Hence, uniformly over all intervals I_n of length κ_n, we have a.s. as $n \to \infty$ that

$$\sup_{s,t \in I_n} d_1(s,t) = O((\kappa_n/n)\, \|\pi_n^2 f_n\|_0^n) = o(\|\pi_n^2 f_n\|_0^n).$$

Next by ($\pi 1$) and standard manipulations

$$\begin{aligned} d_2(s,t) &\leqslant |f_n(t)-f_n(s)| \\ &\leqslant (n^{1/2}/h(\theta))\, \|b_n\|_0^\theta\, |h'(a(n,\psi_{\theta,n}(t)))-h'(a(n,\psi_{\theta,n}(s)))| \\ &\quad + (n^{1/2}/h(\theta))\, \|h'\|^{\theta \vee q_n(\theta)}\, |b_n(\psi_{\theta,n}(t))-b_n(\psi_{\theta,n}(s))| \\ &=: d_3(s,t)+d_4(s,t). \end{aligned}$$

As $h' \geqslant 1$ on $[0, \theta]$, it follows that $\psi_{\theta,n}$ is a Lipschitz function: for all $n \geqslant 1$ and all $s, t \in [0, n]$,

$$|\psi_{\theta,n}(t)-\psi_{\theta,n}(s)| \leqslant (h(\theta)/n)\, |t-s|. \tag{2.31}$$

Moreover, h' is uniformly continuous on $[0, \tilde{\theta}]$ for $0<\tilde{\theta}<T_G$, since G is assumed to be continuous. Hence, also using (2.13), we have uniformly over all intervals I_n of length κ_n that a.s. as $n \to \infty$,

$$\sup_{s,t \in I_n} d_3(s,t) = o(n^{1/2}\, \|b_n\|_0^\theta) = o(\|\pi_n^2 f_n\|_0^n),$$

where the last "equality" follows from the fact that $\|\pi_n^2 f_n\|_0^n \geqslant c^2 n^{1/2}\, \|b_n\|_0^\theta / h(\theta)$.

Observe that for any $s, t \in [0, n]$,

$$\begin{aligned} d_4(s,t) &\leqslant (n^{1/2}/h(\theta))\, \|h'\|_0^{\theta \vee q_n(\theta)}\, \|r_n\|_0^\theta \\ &\quad + n^{1/2}\, \|h'\|_0^{\theta \vee q_n(\theta)}\, |a_n(\psi_{\theta,n}(t))-a_n(\psi_{\theta,n}(s))| \\ &=: d_5 + d_6(s,t). \end{aligned}$$

From (2.13), the upper bound part of this proof and the fact that h' is bounded from above on $[0, \tilde{\theta}]$ for $0<\tilde{\theta}<T_G$, we see that a.s. as $n \to \infty$,

$$d_5 = O(n^{1/4}(\log n)^{1/2}\, (\log\log n)^{1/4}) = o(\|\pi_n^2 f_n\|_0^n),$$

where for the last "equality" ($F_\pi 1$) is applied. From (2.31) we get with the help of Schäfer [14, Corollary 3.2] or Aly *et al.* [1, Theorem 2.1] that a.s. as $n \to \infty$,

$$\begin{aligned} &\sup_{\substack{I_n : |I_n| \leqslant \kappa_n \\ I_n \subset [0,n]}} \sup_{s,t \in I_n} d_6(s,t) \\ &\quad \leqslant n^{1/2}\, \|h'\|_0^{\theta \vee q_n(\theta)} \sup_{\substack{J_n : |J_n| \leqslant h(\theta)\kappa_n/n \\ J_n \subset [0,\theta]}} \sup_{u,v \in J_n} |a_n(u)-a_n(v)| \\ &\quad = O(n^{1/2}(\log n)^{1/2}\, (\kappa_n/n)^{1/2}). \end{aligned}$$

As $(\kappa_n \log n)^{1/2} = o(\|\pi_n^2 f_n\|_0^n)$, we can conclude that uniformly over all intervals $I_n \subset [0, n]$ of length κ_n we have a.s. as $n \to \infty$,

$$\sup_{s,t \in I_n} |\pi_n^2(s) f_n(s) - \pi_n^2(t) f_n(t)| = o(\|\pi_n^2 f_n\|_0^n).$$

Hence $(F_\pi 2)$ holds almost surely ($a > 1$ arbitrary), finishing the proof. ∎

Proof of Theorem 1. The derivation of the limit finite dimensional distributions of r_n follows the lines of the proof of Theorem 3 in Beirlant *et al.* [3]. We only sketch the proof.

First with the help of approximation results (cf. Fact 1) one shows that it is possible to construct a sequence $\{W_n\}_{n=1}^{\infty}$ of standard Wiener processes extended to $(-\infty, \infty)$, in such a way that for $0 < s < \theta$, as $n \to \infty$,

$$n^{1/4} \left| r_n(s) - (1-s)\left\{ W_n(h(s)) - W_n\left(h(s) - \frac{n^{-1/2} W_n(h(s))}{(1-s)(1-G(s))} \right) \right\} \right| = o_P(1).$$

For any choice of $k \geqslant 1$ and $0 < s_1 < \cdots < s_k < \theta$ fixed, let

$$W_n^{(i)}(x_i) = n^{1/4}\{ W_n(h(s_i) + n^{-1/2} x_i) - W_n(h(s_i))\}, \qquad x_i \in \mathbb{R}, \qquad i = 1, ..., k,$$

and let

$$V_n := -(W_n \circ h)/((1-I)(1-G)).$$

Using Lemma 2.2 in Beirlant *et al.* [3] one shows that as $n \to \infty$,

$$(W_n^{(1)}, ..., W_n^{(k)}, V_n) \xrightarrow{d} (W^{(1)}, ..., W^{(k)}, V),$$

where $W^{(1)}, ..., W^{(k)}$ are independent two-sided Wiener processes independent of $V =^d V_n$. To this end one only needs to check that

$$\begin{aligned} n^{1/4}\{ &\mathrm{Cov}[W_n(h(s) + n^{-1/2}x), -W_n(h(t))/((1-t)(1-G(t)))] \\ &- \mathrm{Cov}[W_n(h(s)), -W_n(h(t))/((1-t)(1-G(t)))]\} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

for any $s \in (0, \theta)$ and $x \in \mathbb{R}$. From this weak convergence result one deduces that as $n \to \infty$,

$$\begin{aligned} &(-(1-s_1)\, W_n^{(1)}(V_n(s_1)), ..., -(1-s_k)\, W_n^{(k)}(V_n(s_k))) \\ &\xrightarrow{d} (-(1-s_1)\, W^{(1)}(V(s_1)), ..., -(1-s_k)\, W^{(k)}(V(s_k))). \qquad (2.32) \end{aligned}$$

Since the right side of (2.32) is equal in distribution to

$$\begin{aligned} &((1-s_1)\, W^{(1)}(\bar{\Lambda}_G(s_1)/(1-s_1)^2), ..., (1-s_k)\, W^{(k)}(\bar{\Lambda}_G(s_k)/(1-s_k)^2)) \\ &\stackrel{d}{=} (Z_1\, |\bar{\Lambda}_G(s_1)|^{1/2}, ..., Z_k\, |\bar{\Lambda}_G(s_k)|^{1/2}), \end{aligned}$$

the result follows. ∎

Proof of Theorem 2. This proof can be given along similar lines as that of the upper bound part of Theorem 3. However, it is simpler because no supremum ($0 \leqslant s \leqslant \theta$) and no denominator $((\|\beta_{n,G}\|_0^\theta)^{1/2})$ is involved. Here follows a short proof.

Writing $l_n = n^{1/4}/(\log\log n)^{3/4}$ we have for arbitrary $\varepsilon > 0$, almost surely, the following string of (in)equalities:

$$\begin{aligned}
&\limsup_{n\to\infty} l_n\, |r_n(s)| \\
&\quad = \limsup_{n\to\infty} l_n\, |\Lambda_n(s) - \Lambda_n(q(s))| \\
&\quad \leqslant \limsup_{n\to\infty} l_n \sup_{t:|h(s)-h(t)| \leqslant |h(s)-h(q_n(s))|} |(1-s)\, W_n(h(s)) - (1-t)\, W_n(h(t))| \\
&\quad \leqslant \limsup_{n\to\infty} l_n(1-s) \sup_{t:|h(s)-h(t)| \leqslant \frac{((1+\varepsilon)\, 2h(s) \log\log n)^{1/2}}{n^{1/2}(1-s)(1-G(s))}} |W_n(h(s)) - W_n(h(t))| =: L(s).
\end{aligned}$$

It is easily shown that

$$L(s) \leqslant (1+\varepsilon)^{1/2}\, 2^{3/4}\, h^{1/4}(s)(1-s)^{1/2}\, (1-G(s))^{-1/2} \qquad \text{a.s.}$$

Noting that $\varepsilon > 0$ is arbitrary, yields the desired result. ∎

Note added in proof. After completion of our paper, Paul Deheuvels informed us that he and Ming Gu did research on this subject too.

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