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Bahadur–Kiefer Theorems for the Product-Limit Process

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In the random censorship from the right model, strong and weak limit theorems for Bahadur–Kiefer type processes based on the product-limit estimator are established. The main theorem is sharp and may be considered as a final result as far as this type of research is concerned. As a consequence of this theorem a sharp uniform Bahadur representation for product-limit quantiles is obtained. © 1990 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

Let X_1, X_2, \dots be a sequence of i.i.d. rv's with distribution function (*df*) F and let Y_1, Y_2, \dots be a sequence of i.i.d. rv's with *df* G . Both sequences are assumed to be independent. In the random censorship from the right model X_i may be censored on the right by Y_i so that the pair (Z_i, δ_i) , $i = 1, 2, \dots$ is observed, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = 1_{\{X_i \leq Y_i\}}$. The *df* H of the Z_i (which are also independent) is then given by $H = 1 - (1 - F)(1 - G)$.

As in most applications all rv's are assumed to be positive. Moreover, we assume throughout that the following condition is satisfied:

(A) F is differentiable on $(0, \infty)$ with continuous and positive derivative f and G is continuous on $(0, \infty)$.

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The product-limit (PL) estimator F_n (at stage n) introduced by Kaplan and Meier [9] comes out as the maximum likelihood estimator of F :

$$1 - F_n(x) = \prod_{Z_{(i)} \leq x} (1 - \delta_{(i)}/(n - i + 1)), \quad x \geq 0,$$

where $0 < Z_{(1)} \leq \dots \leq Z_{(n)}$ are the order statistics of the Z_i , $1 \leq i \leq n$, and $\delta_{(i)}$ are the corresponding δ 's. The associated PL process will be given by

$$\alpha_n(x) = n^{1/2}(F_n(x) - F(x)), \quad x \geq 0.$$

The quantile function (or inverse) Q of F is naturally estimated by

$$Q_n(t) = \inf\{x : F_n(x) \geq t\}, \quad t \in (0, 1).$$

The PL quantile process is then given by

$$\beta_n(t) = n^{1/2}f(Q(t))(Q_n(t) - Q(t)), \quad t \in (0, 1).$$

In this paper we will study the so-called Bahadur-Kiefer process associated with the above PL and PL quantile process defined by

$$R_n(t) = \alpha_n(Q(t)) + \beta_n(t), \quad t \in (0, 1). \tag{1.1}$$

In the uncensored case this process was introduced by Bahadur [2] and further investigated by Kiefer [10, 11]. A discussion of the literature on the subject for the censored as well as the uncensored case is postponed until the end of this section.

Write

$$T_G = \inf\{x : G(x) = 1\}$$

and let A be a Gaussian process defined on $[0, F(T_G))$, with mean zero and covariance function

$$E(A(s)A(t)) = (1-s)(1-t)h(s \wedge t), \quad 0 \leq s, t < F(T_G),$$

where

$$h(s) = \int_0^s (1-u)^{-2} (1-G(Q(u)))^{-1} du, \quad 0 \leq s < F(T_G).$$

Moreover, let us define the Gaussian process \bar{A}_G by

$$\bar{A}_G(s) = A(s)/(1-G(Q(s))), \quad 0 \leq s < F(T_G).$$

in the same spirit we write

$$\tilde{\beta}_{n,G} = \beta_n / (1 - G \circ Q).$$

The first result gives the weak convergence of the finite dimensional distributions of R_n .

THEOREM 1. *Let condition (A) be satisfied and let $0 < \theta < F(T_G)$. Assume that for any $0 < s < Q(\theta)$*

$$\lim_{\Delta \downarrow 0} \sup_{t: |t-s| \leq \Delta} |f(t) - f(s)| / \Delta^{1/2} = 0.$$

$k \in \mathbb{N}$ and $0 < s_1 < \dots < s_k < \theta$ be fixed. Then as $n \rightarrow \infty$,

$$n^{1/4} (R_n(s_1), \dots, R_n(s_k)) \xrightarrow{d} (Z_1 |\bar{A}_G(s_1)|^{1/2}, \dots, Z_k |\bar{A}_G(s_k)|^{1/2}),$$

Z_1, \dots, Z_k are independent $N(0, 1)$ rv's independent of \bar{A}_G .

The second result is an almost sure analogue of Theorem 1.

THEOREM 2. *Under the conditions of Theorem 1, we have for $s \in (0, \theta)$ almost surely,*

$$\sup_{\infty} n^{1/4} (\log \log n)^{-3/4} |R_n(s)| \leq 2^{3/4} (1-s)^{1/2} h^{1/4}(s) (1 - G(Q(s)))^{-1/2}.$$

We now present our main result, which is so powerful that it has a lot of interesting results as a corollary. For its presentation we use the notation $\|\varphi\|_{[a,b]} = \sup_{t \in [a,b]} |\varphi(t)|$, when φ is a real valued function on $[a, b]$.

THEOREM 3. *Let condition (A) be satisfied and let $0 < \theta < F(T_G)$. Assume there exists a $C \in (0, \infty)$ such that*

$$\limsup_{\Delta \downarrow 0} \sup_{\substack{s, t: |t-s| \leq \Delta \\ t \leq Q(\theta)}} |f(t) - f(s)| / \Delta^{1/2} < C \quad (1.2)$$

and f be right-continuous at 0. In case $\lim_{x \downarrow 0} f(x) = 0$, suppose that, in addition, for some $a \in (0, \infty)$,

$$\lim_{x \downarrow 0} F(x) |f'(x)| (f(x))^{-2} = a. \quad (1.3)$$

we have

$$\lim_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \|R_n\|_0^\theta / (\|\tilde{\beta}_{n,G}\|_0^\theta)^{1/2} = 1 \quad a.s.$$

Combination of Theorem 3 with the results in Aly, Csörgő, and Horváth [1] yields:

COROLLARY 1. *Under the conditions of Theorem 3 we have*

$$n^{1/4}(\log n)^{-1/2} \|R_n\|_0^\theta \xrightarrow{d} (\|\bar{A}_G\|_0^\theta)^{1/2} \quad \text{as } n \rightarrow \infty; \quad (1.4)$$

$$\limsup_{n \rightarrow \infty} n^{1/4}(\log n)^{-1/2} (\log \log n)^{-1/4} \|R_n\|_0^\theta = 2^{1/4} \left(\left\| \frac{(1-I)h^{1/2}}{1-G \circ Q} \right\|_0^\theta \right)^{1/2} \quad \text{a.s.}, \quad (1.5)$$

where I denotes the identity function;

$$\begin{aligned} & 2^{-3/4}\pi^{1/2}(1-\theta)^{1/2} h^{1/4}(\theta) \\ & \leq \liminf_{n \rightarrow \infty} n^{1/4}(\log n)^{-1/2} (\log \log n)^{1/4} \|R_n\|_0^\theta \\ & \leq 2^{-3/4}\pi^{1/2}h^{1/4}(\theta)(1-G(Q(\theta)))^{-1/2} \quad \text{a.s.} \end{aligned} \quad (1.6)$$

If $\lim_{x \downarrow 0} f(x) > 0$, then (1.5) entails that uniformly over all $s \in (0, \theta)$ we have

$$Q_n(s) = Q(s) + \frac{s - F_n(Q(s))}{f(Q(s))} + O(n^{-3/4}(\log n)^{1/2} (\log \log n)^{1/4}) \quad \text{a.s.}$$

Discussion and Bibliography. In the uncensored case Kiefer [10] proved both Theorem 1 for the case $k = 1$ and Theorem 2 (with the right constant). Note that in the uncensored case ($G \equiv 0$) the constant on the right in Theorem 2 reduces to $2^{3/4}(s(1-s))^{1/4}$, whereas in that case the actual value of the limsup is equal to $2^{5/4}3^{-3/4}(s(1-s))^{1/4}$. Theorem 1 for arbitrary $k \in \mathbb{N}$ and $G \equiv 0$ is presented in Beirlant *et al.* [3]. An in probability version of Theorem 3 (with $\theta = 1$) in the uncensored case is established in Kiefer [11], where the author claims that the statement holds true almost surely; he did not publish a proof, however. Recently, his claim has been proved in Shorack [15, upper bound]) and Deheuvels and Mason [8, lower bound].

In the literature on the random censorship model only the type of problem discussed in Theorem 2 and (1.5) has been considered. A version of the statement in (1.5) can be found in Cheng [6], but with a worse rate. Aly *et al.* [1] derived the exact rate in (1.5), but did not find the right constant. Comparing Theorem 3 with its uncensored analogue (Theorem 1A in Deheuvels and Mason [8]), it is striking that $\beta_{n,G}$ instead of β_n shows up in the denominator. Finally, note that the assumptions on F are somewhat milder than the usual Csörgő-Révész conditions, cf. Theorem 4.3

in Aly *et al.* [1]. Hence, for positive random variables, the main result in that paper (Theorem 4.4; a Kiefer process type strong approximation of β_n) is improved as far as the assumptions on F are considered.

2. PROOFS

Consider the new set of rv's

$$U_i = F(X_i), \quad V_i = F(Y_i), \quad W_i = F(Z_i) = U_i \wedge V_i.$$

Then the U_i are i.i.d. uniform $(0, 1)$ rv's, independent of the V_i ; the V_i are also i.i.d. with *df* $G \circ Q$. The PL estimator based on these reduced rv's is then given by

$$\Gamma_n(t) = F_n(Q(t)), \quad t \in (0, 1),$$

and the corresponding PL process is given by

$$a_n(t) = n^{1/2}(\Gamma_n(t) - t) = \alpha_n(Q(t)), \quad t \in (0, 1).$$

Moreover, we put

$$q_n(t) = \inf\{s: \Gamma_n(s) \geq t\} = F(Q_n(t))$$

and

$$b_n(t) = n^{1/2}(q_n(t) - t), \quad t \in (0, 1).$$

The corresponding Bahadur–Kiefer process is denoted by

$$r_n(t) = a_n(t) + b_n(t), \quad t \in (0, 1).$$

We first present a number of lemmas which relate R_n to r_n .

LEMMA 1. *Under the conditions of Theorem 1 we have for any $t \in (0, \theta)$ that as $n \rightarrow \infty$*

$$\begin{aligned} n^{1/4}(R_n(t) - r_n(t)) &\xrightarrow{P} 0; \\ n^{1/4}(\log \log n)^{-3/4} (R_n(t) - r_n(t)) &\longrightarrow 0 \quad a.s. \end{aligned}$$

Proof. We only prove the first statement; the second one is proved in an analogous way. Let $0 < \theta < F(T_G)$. As $R_n - r_n = \beta_n - b_n$, it remains to derive that as $n \rightarrow \infty$,

$$n^{1/4}(\beta_n(t) - b_n(t)) \xrightarrow{P} 0, \quad 0 < t < \theta.$$

Remark that

$$\beta_n(t) = n^{1/2} \frac{f(Q(t))}{f(Q(\theta_{t,n}))} (q_n(t) - t) = \frac{f(Q(t))}{f(Q(\theta_{t,n}))} b_n(t),$$

where $|\theta_{t,n} - t| \leq n^{-1/2} |b_n(t)|$. As $n \rightarrow \infty$, $b_n(t) = O_p(1)$; hence

$$\begin{aligned} n^{1/4} |\beta_n(t) - b_n(t)| &\leq n^{1/4} |b_n(t)| \left| \frac{f(Q(t))}{f(Q(\theta_{t,n}))} - 1 \right| \\ &= O_p(1) \cdot \sup_{v: |t-v| \leq n^{-1/2} |b_n(t)|} \frac{|f(Q(t)) - f(Q(v))|}{(n^{-1/2} |b_n(t)|)^{1/2}}, \end{aligned}$$

which tends to zero in probability by assumption. ■

LEMMA 2. Under the conditions of Theorem 3 we have, as $n \rightarrow \infty$,

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_0^\theta \rightarrow 0 \quad \text{a.s.}$$

Proof. Let $\theta < F(T_G)$. As in the proof of Lemma 1 we find that it suffices to show that

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|b_n\|_0^\theta \left\| \frac{f(Q(I))}{f(Q(\theta_{I,n}))} - 1 \right\|_0^\theta \rightarrow 0 \quad \text{a.s.}$$

First consider the case $\lim_{x \downarrow 0} f(x) > 0$. From Theorem 5.1 in Aly *et al.* [1]

$$\limsup_{n \rightarrow \infty} (2 \log \log n)^{-1/2} \|b_n\|_0^\theta = \|(1 - I) h^{1/2}\|_0^\theta \quad \text{a.s.}, \quad (2.1)$$

so that we are finished if we show that under the given conditions

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left\| \frac{f(Q(I))}{f(Q(\theta_{I,n}))} - 1 \right\|_0^\theta \rightarrow 0 \quad \text{a.s.}$$

By (1.2) for some $K_\theta \in (0, \infty)$ we have almost surely

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log \log n)^{3/4} \left\| \frac{f(Q(I)) - f(Q(\theta_{I,n}))}{f(Q(\theta_{I,n}))} \right\|_0^\theta \\ \leq CK_\theta \limsup_{n \rightarrow \infty} (\log n)^{-1/2} (\log \log n)^{3/4} (\|b_n\|_0^\theta)^{1/2}, \end{aligned}$$

which equals zero almost surely by application of (2.1).

Now suppose $\lim_{x \downarrow 0} f(x) = 0$. With the same method as above it is immediate that for any $0 < \varepsilon < \theta$,

$$n^{1/4} (\log n)^{-1/2} (\log \log n)^{1/4} \|R_n - r_n\|_\varepsilon^\theta \rightarrow 0 \quad \text{a.s.} \quad (2.2)$$

As already mentioned in Aly *et al.* [1, proof of Theorem 4.3], the proof of (3.3) in Csörgő and Révész [7] can be mimicked to show that for some $C_1 \in (0, \infty)$ and for "small" $\varepsilon > 0$

$$n^{1/4}(\log n)^{-1/2} (\log \log n)^{1/4} \|\beta_n - b_n\|_{\delta(n)}^\varepsilon \rightarrow 0 \quad \text{a.s.}, \quad (2.3)$$

where $\delta(n) = C_1 n^{-1} \log \log n$, since for "small" $\varepsilon > 0$,

$$\sup_{0 < x \leq Q(2\varepsilon)} F(x) |f'(x)| (f(x))^{-2} \leq 2a.$$

In Aly *et al.* [1] it is also shown that

$$\|b_n\|_0^{\delta(n)} \stackrel{\text{a.s.}}{=} O(n^{-1/2} \log \log n) \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

As $f \circ Q$ is regularly varying at zero with positive index a , one can construct a non-decreasing function \tilde{f}_Q such that $\tilde{f}_Q \leq f \circ Q$ and $\lim_{x \downarrow 0} \tilde{f}_Q(x)/f(Q(x)) = 1$. (See, e.g., Theorem 1.5.3 in Bingham *et al.* [4].) Let $U_{(1)} \leq \dots \leq U_{(r_n)}$ denote the order statistics of those observation among U_1, \dots, U_n , which are uncensored, i.e., for which $\delta_i = 1$. Assume $\Gamma_n(U_{(i-1)}) < t \leq \Gamma_n(U_{(i)})$. Then for $t \leq U_{(i)}$ and n large enough

$$\begin{aligned} |\beta_n(t)| &\leq n^{1/2} \int_t^{U_{(i)}} \frac{f(Q(t))}{f(Q(u))} du \\ &\leq n^{1/2} \int_t^{U_{(i)}} \frac{f(Q(t))}{\tilde{f}_Q(u)} du \\ &\leq \sup_{t \in (0, \delta(n)]} \frac{f(Q(t))}{\tilde{f}_Q(t)} \cdot n^{1/2} \int_t^{U_{(i)}} \frac{\tilde{f}_Q(t)}{\tilde{f}_Q(u)} du \\ &\leq 2b_n(t). \end{aligned} \quad (2.5)$$

The case $t > U_{(i)}$ can be handled along the lines of (3.14) in Csörgő and Révész [7], cf. Aly *et al.* [1, pp. 200–201]. From this remark, (2.5) in combination with (2.4), and (2.4) itself, we have

$$\|\beta_n - b_n\|_0^{\delta(n)} \leq \|\beta_n\|_0^{\delta(n)} + \|b_n\|_0^{\delta(n)} \stackrel{\text{a.s.}}{=} O(n^{-1/2} (\log n)^{2a}).$$

Combining this with (2.2) and (2.3) completes the proof for the case $\lim_{x \downarrow 0} f(x) = 0$ and hence of the lemma. ■

LEMMA 3. Under the conditions of Theorem 3 we have, as $n \rightarrow \infty$,

$$\begin{aligned} n^{1/4}(\log n)^{-1/2} \|R_n - r_n\|_0'' / (\|\tilde{\beta}_{n,G}\|_0'')^{1/2} &\rightarrow 0 \quad \text{a.s.}; \\ \left\{ \frac{\|r_n\|_0''}{(\|\tilde{\beta}_{n,G}\|_0'')^{1/2}} - \frac{\|r_n\|_0''}{(\|b_n/(1-G)\|_0'')^{1/2}} \right\} \cdot \left(\frac{\|r_n\|_0''}{(\|b_n/(1-G)\|_0'')^{1/2}} \right) &\rightarrow 0 \quad \text{a.s.} \end{aligned} \quad (2.6)$$

Proof. Combination of Lemma 2 and

$$\liminf_{n \rightarrow \infty} (\log \log n)^{1/2} \|b_n\|_0^\theta > 0 \quad \text{a.s.} \quad (2.7)$$

(see Fact 3 below) yields the first statement in (2.6). To prove the second statement it suffices to show that

$$\{(\|b_n/(1-G)\|_0^\theta)^{1/2} - (\|\beta_{n,G}\|_0^\theta)^{1/2}\}/(\|\beta_{n,G}\|_0^\theta)^{1/2} \rightarrow 0 \quad \text{a.s.}$$

Using $x^{1/2} - y^{1/2} = (x - y)/(x^{1/2} + y^{1/2})$, $x, y > 0$, (2.7), and again Lemma 2, the proof reduces to showing

$$(\log \log n)^{1/2} \|\beta_n - b_n\|_0^\theta \rightarrow 0 \quad \text{a.s.,}$$

which follows from one more application of Lemma 2. ■

From Lemma 1 and Lemma 3, it follows that we can confine ourselves to the proofs of Theorems 1–2 and Theorem 3, respectively, in case the X_i 's are uniformly $(0, 1)$ distributed and the Y_i 's are distributed according to a dfG (which is now shorthand for $G \circ Q$) with support on $(0, 1)$. Observe that $G \circ Q$ is continuous, since F is strictly increasing. We also adopt the notation introduced at the beginning of this section.

The remainder of this paper is organized as follows. We begin by recording a number of facts, which are required for the proofs. After that, we give a detailed proof of Theorem 3. Finally, short proofs of Theorems 1 and 2 are presented.

Fact 1 (Burke, Csörgö, and Horváth [5], Major and Rejtő [12]). There exists a two-parameter standard Wiener process W such that, for any $\theta \in (0, T_G)$,

$$\|a_n - n^{-1/2}(1-I)W(h(I), n)\|_0^\theta = O(n^{-1/2}(\log n)^2) \quad \text{a.s.} \quad (2.8)$$

Define a sequence $\{W_n\}_{n=1}^\infty$ of (one-parameter) standard Wiener processes by

$$W_n = n^{-1/2}W(I, n), \quad (2.9)$$

write

$$A_n = (1-I)W_n \circ h, \quad (2.10)$$

and note that for all $n \in \mathbb{N}$: $A_n \stackrel{d}{=} A$, with A as in Section 1.

Fact 2 (cf. Shorack [15]). Let W_n be as above, $c \in (0, \infty)$ arbitrary, and $\{k_n\}_{n=1}^\infty$ a sequence of positive numbers such that $k_n \downarrow$, $nk_n \uparrow$, $\log(1/k_n)/\log \log n \rightarrow \infty$ and $\log(1/k_n)/(nk_n) \rightarrow 0$. Then

$$\limsup_{n \rightarrow \infty} \sup_{\substack{|u-v| \leq k_n \\ 0 \leq u \leq c \\ v \geq 0}} \frac{|W_n(u) - W_n(v)|}{(2k_n \log(1/k_n))^{1/2}} \leq 1 \quad \text{a.s.} \quad (2.11)$$

and

$$\limsup_{n \rightarrow \infty} \sup_{\substack{|u-v| > k_n \\ 0 \leq u, v \leq c}} \frac{|W_n(u) - W_n(v)|}{(2|u-v| \log(1/k_n))^{1/2}} \leq 1 \quad \text{a.s.} \quad (2.12)$$

Fact 3 (Aly, Csörgő, and Horváth [1]). We have almost surely

$$\limsup_{n \rightarrow \infty} (\log \log n)^{-1/2} \left\| \frac{b_n}{1-G} \right\|_0^\theta = 2^{1/2} \left\| \frac{(1-I)h^{1/2}}{1-G} \right\|_0^\theta \quad (2.13)$$

and

$$\begin{aligned} \pi 8^{-1/2} (1-\theta) h^{1/2}(\theta) &\leq \liminf_{n \rightarrow \infty} (\log \log n)^{1/2} \left\| \frac{b_n}{1-G} \right\|_0^\theta \\ &\leq \pi 8^{-1/2} \frac{h^{1/2}(\theta)}{1-G(\theta)}. \end{aligned} \quad (2.14)$$

Proof of Theorem 3. The proof of the upper bound part is an adaptation of that in Shorack [15], whereas the proof of the lower bound part is based on Deheuvels and Mason [8]. We first show that if $0 < \theta < T_G$,

$$LS := \limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \|r_n\|_0^\theta / (\|b_n/(1-G)\|_0^\theta)^{1/2} \leq 1 \quad \text{a.s.} \quad (2.15)$$

Note that for any $s \in [0, \theta]$,

$$r_n(s) = a_n(s) - a_n(q_n(s)) + n^{1/2}(\Gamma_n(q_n(s)) - s).$$

In Sander [13] it is shown that

$$n^{1/2} \|\Gamma_n \circ q_n - I\|_0^\theta = O(n^{-1/2}) \quad \text{a.s.}$$

Hence, (2.8) and (2.14) entail that

$$LS = \limsup_{n \rightarrow \infty} \frac{n^{1/4} \|A_n - A_n \circ q_n\|_0^\theta}{(\log n \|b_n/(1-G)\|_0^\theta)^{1/2}} \quad \text{a.s.}$$

Let

$$\begin{aligned} k_n &= \pi(1-\theta) h^{1/2}(\theta) / (8n \log \log n)^{1/2}, \\ I_n &= \{(s, t) : s \geq 0, 0 \leq t \leq \theta, |h(t) - h(s)| \leq \|h \circ q_n - h\|_0^\theta, \\ &\quad (1-t)^2 |h(t) - h(s)| \leq \|(1-I)^2 (h \circ q_n - h)\|_0^\theta\}, \\ J_n &= \{(s, t) \in I_n : |h(t) - h(s)| \leq k_n\}, \\ K_n &= \{(s, t) \in I_n : |h(t) - h(s)| > k_n\}. \end{aligned}$$

Then almost surely

$$\begin{aligned} LS &\leq \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} \frac{|A_n(t) - A_n(s)|}{(\log n \|(q_n - I)/(1 - G)\|_0^\theta)^{1/2}} \\ &\leq \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} \frac{|t - s| |W_n(h(s))|}{(\log n \|(q_n - I)/(1 - G)\|_0^\theta)^{1/2}} \\ &\quad + \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} \frac{(1 - t) |W_n(h(t)) - W_n(h(s))|}{(\log n \|(q_n - I)/(1 - G)\|_0^\theta)^{1/2}} \\ &=: LS_1 + \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} A_n(s, t) =: LS_1 + LS_2. \end{aligned}$$

It is well known that for arbitrary $0 < c < \infty$,

$$\|W_n\|_0^c = O((\log \log n)^{1/2}) \quad \text{a.s.} \tag{2.17}$$

From (2.13) we obtain

$$\|h \circ q_n - h\|_0^\theta \leq \|q_n - I\|_0^\theta \|h'\|_0^{\theta \vee q_n(\theta)} \stackrel{\text{a.s.}}{=} O(n^{-1/2}(\log \log n)^{1/2}). \tag{2.18}$$

Hence from (2.17), (2.18), and (2.14) we have a.s. as $n \rightarrow \infty$,

$$\sup_{(s,t) \in J_n} \frac{|t - s| |W_n(h(s))|}{(\log n \|(q_n - I)/(1 - G)\|_0^\theta)^{1/2}} = O(n^{-1/4}(\log n)^{-1/2} (\log \log n)^{5/4}),$$

implying that $LS_1 = 0$ a.s. Furthermore,

$$\begin{aligned} LS_2 &= \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} A_n(s, t) \vee \limsup_{n \rightarrow \infty} \sup_{(s,t) \in K_n} A_n(s, t) \\ &=: LS_3 \vee LS_4. \end{aligned}$$

First, by (2.14) we have a.s.

$$\begin{aligned} LS_3 &\leq \limsup_{n \rightarrow \infty} \sup_{(s,t) \in J_n} \frac{(1 - t) |W_n(h(t)) - W_n(h(s))|}{(k_n \log n)^{1/2}} \\ &\leq \limsup_{n \rightarrow \infty} \sup_{\substack{|u-v| \leq k_n \\ 0 \leq u \leq h(\theta) \\ v \geq 0}} \frac{|W_n(u) - W_n(v)|}{(2k_n \log(1/k_n))^{1/2}} \end{aligned}$$

as $\log(1/k_n)/\log n \rightarrow 1/2$ as $n \rightarrow \infty$. Hence $LS_3 \leq 1$ a.s., because of (2.11).

Next, since G is continuous $\|(q_n - I)/(1 - G)\|_0^\theta \sim \|(h \circ q_n - h)(1 - I)^2\|_0^\theta$ a.s. as $n \rightarrow \infty$, so that

$$\begin{aligned}
 LS_4 &= \limsup_{n \rightarrow \infty} \sup_{(s,t) \in K_n} \frac{(1-t) |W_n(h(t)) - W_n(h(s))|}{(\log n \| (h \circ q_n - h)(1-I)^2 \|_0^\theta)^{1/2}} \\
 &\leq \limsup_{n \rightarrow \infty} \sup_{(s,t) \in K_n} \frac{|W_n(h(t)) - W_n(h(s))|}{|h(t) - h(s)|^{1/2} (\log n)^{1/2}} \\
 &\leq \limsup_{n \rightarrow \infty} \sup_{\substack{|u-v| > k_n \\ 0 \leq u, v \leq 2h(\theta)}} \frac{|W_n(u) - W_n(v)|}{|u-v|^{1/2} (2 \log(1/k_n))^{1/2}} \quad \text{a.s.}
 \end{aligned}$$

Applying (2.12) yields $LS_4 \leq 1$ a.s. Hence the proof of (2.15) is completed. Now it remains to show that if $0 < \theta < T_G$,

$$LI := \liminf_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \|r_n\|_0^\theta / (\|b_n/(1-G)\|_0^\theta)^{1/2} \geq 1 \quad \text{a.s.} \quad (2.19)$$

Using similar steps as in the upper bound part of this proof we find that it suffices to show that

$$\liminf_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \frac{\|(1-I)(W_n \circ h - W_n \circ h \circ q_n)\|_0^\theta}{(\|b_n/(1-G)\|_0^\theta)^{1/2}} \geq 1 \quad \text{a.s.}$$

Let

$$h(q_n(t)) = h(t) + n^{-1/2} b_n(t) h'(a(n, t)), \quad (2.20)$$

where $|a(n, t) - t| \leq n^{-1/2} |b_n(t)|$. Then, with h^i denoting the inverse of h ,

$$\begin{aligned}
 LI &= \liminf_{n \rightarrow \infty} \frac{n^{1/4}}{(\log n)^{1/2}} \\
 &\quad \times \frac{\|(1-h^i)(W_n - W_n \circ (I + n^{-1/2}(b_n \circ h^i) h'(a(n, h^i))))\|_0^{h(\theta)}}{(\|(b_n \circ h^i) h'(a(n, h^i))(1-h^i)^2\|_0^{h(\theta)})^{1/2}} \quad \text{a.s.}
 \end{aligned}$$

Write for $v \in [0, n]$,

$$\begin{aligned}
 \psi_{\theta,n}(v) &= h^i(vh(\theta)/n), \\
 \pi_n(v) &= 1 - \psi_{\theta,n}(v), \\
 f_n(v) &= (n^{1/2}/h(\theta)) b_n(\psi_{\theta,n}(v)) h'(a(n, \psi_{\theta,n}(v))),
 \end{aligned} \quad (2.21)$$

and observe that for any standard Wiener process W_n , the process \bar{W}_n defined by

$$\bar{W}_n(v) = (n/h(\theta))^{1/2} W_n(vh(\theta)/n), \quad v \geq 0,$$

is again a standard Wiener process. So by changing variables ($v = (n/h(\theta))t$), we obtain

$$LI = \liminf_{n \rightarrow \infty} \frac{\|\pi_n(\bar{W}_n - \bar{W}_n \circ (I + f_n))\|_0^n}{(\log n)^{1/2} (\|\pi_n^2 f_n\|_0^n)^{1/2}} \quad \text{a.s.} \quad (2.22)$$

Now to show that the right side of (2.22) is not smaller than one a.s., we can make use of the following proposition, which constitutes a generalization of Proposition 1 in Deheuvels and Mason [8]. In our proposition we will abuse notation by again using sequences of functions $\{\pi_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ and a sequence $\{\bar{W}_n\}_{n=1}^\infty$ of Wiener processes. These sequences are defined below and are not related to the above sequences with the same names. However, we will apply the Proposition with $\pi_n, f_n,$ and \bar{W}_n as above. Let $\{\pi_n\}_{n=1}^\infty$ be a sequence of decreasing functions satisfying:

($\pi 1$) there exists some $c > 0$, such that

$$c \leq \|\pi_n\|_0^n \leq 1 \quad \text{for all } n \geq 1,$$

($\pi 2$) $\limsup n \|\pi'_n\|_0^n < \infty$.

For any $\gamma > 1, a > 1, \eta > 0, v \geq 1$ we denote by $\mathfrak{F}_\pi(\gamma, a, \eta, v)$ the subclass of all sequences $\{f_n\}_{n=1}^\infty$ of real-valued functions defined on $[0, \infty)$ such that

($F_\pi 1$) for all $n \geq 3$,

$$\gamma^{-1} n^{1/2} / \log^2 n \leq \|\pi_n^2 f_n\|_0^n \leq \gamma n^{1/2} \log^2 n,$$

($F_\pi 2$) for all $n \geq v$,

$$\begin{aligned} M_n(\pi_n^2 f_n) &:= \max \left\{ \inf_{s \in I_n} \pi_n^2(s) f_n(s), \inf_{s \in I_n} (-\pi_n^2(s) f_n(s)) \right\} \\ &\geq a^{-1} \|\pi_n^2 f_n\|_0^n, \end{aligned}$$

for some closed interval $I_n \subset [0, n]$ of length $\eta n e^{-(\log \log n)^2}$,

(F_3) for all $n \geq 1, 0 \leq s + f_n(s)$ for $s \in [0, n]$.

Let $\mathfrak{F}_\pi = \bigcap_{a > 1} (\bigcup_{\gamma > 0} \bigcup_{\eta > 0} \bigcup_{v \geq 1} \mathfrak{F}_\pi(\gamma, a, \eta, v))$. (Here γ, a, η are assumed to be rational and v an integer.)

PROPOSITION. *With the above notation and $\{\bar{W}_n\}_{n=1}^\infty$ being any sequence of standard Wiener process on $[0, \infty)$ sitting on a joint probability space, we have with probability one for all sequences $\{f_n\}_{n=1}^\infty \in \mathfrak{F}_\pi$,*

$$\liminf_{n \rightarrow \infty} R_n(\pi_n, f_n) \geq 1,$$

where $R_n(\pi_n, f_n) = \{\|\pi_n^2 f_n\|_0^n \log n\}^{-1/2} \|\pi_n(\bar{W}_n \circ (I + f_n) - \bar{W}_n)\|_0^n$.

Proof. Choose any $\{f_n\}_{n=1}^\infty \in \mathfrak{F}_\pi(\gamma, a, \eta, \nu)$, where $\gamma > 1, a > 1, \eta > 0$ are rationals and ν is a positive integer. Define

$$h_n(k) = \gamma^{-1} a^k n^{1/2} / \log^2 n$$

$$\text{for } k = -3, -2, -1, 0, \dots, k(n) := [\log_a (c^{-2} \gamma^2 \log^4 n)] + 1$$

and

$$I_n(m) = [m \delta n \exp(-(\log \log n)^2), (m + 1) \delta n \exp(-(\log \log n)^2)]$$

$$\text{for } m = 0, 1, \dots, m(n) := [\delta^{-1} \exp((\log \log n)^2)] + 1,$$

with $\delta = \eta/6$. Let $I_n = [\lambda_n, \rho_n]$. By $(F_\pi 1)$ and $(\pi 1)$, for all $n \geq 3$ we can find an $0 \leq l_n \leq k(n)$ such that

$$h_n(l_n - 1) \leq \|\pi_n^2 f_n\|_0^n / \pi_n^2(\rho_n) \leq h_n(l_n). \tag{2.23}$$

Hence by $(F_\pi 2)$, for all $n \geq \max(\nu, 3)$,

$$\pi_n^2(\rho_n) h_n(l_n - 2) \leq a^{-1} \|\pi_n^2 f_n\|_0^n \leq M_n(\pi_n^2 f_n) \leq \|\pi_n^2 f_n\|_0^n \leq \pi_n^2(\rho_n) h_n(l_n). \tag{2.24}$$

Now

$$R_n(\pi_n, f_n) \geq \left\{ \sup_{s \in I_n} |\bar{W}_n(s + f_n(s)) - W_n(s)| \right\} / ((\log n) h_n(l_n))^{1/2} =: A_n. \tag{2.25}$$

Furthermore by $(\pi 1)$ and $(\pi 2)$ there exist $K > 0, \nu_1 > 1$ such that for $n \geq \nu_1$,

$$\begin{aligned} \pi_n(\lambda_n) / \pi_n(\rho_n) &= 1 + \{ \pi_n(\lambda_n) - \pi_n(\rho_n) \} / \pi_n(\rho_n) \\ &\leq 1 + |c^{-1}(\rho_n - \lambda_n) \pi'_n(\theta_n)| \\ &\leq 1 + K \eta n e^{-(\log \log n)^2} / n \leq a^{1/2}, \end{aligned} \tag{2.26}$$

with $\lambda_n \leq \theta_n \leq \rho_n$. Also we may choose an $1 \leq m \leq m(n)$ such that

$$I_n(m - 1) \cup I_n(m) \subset I_n. \tag{2.27}$$

Suppose first that $M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (\pi_n^2(s) f_n(s))$. Then by (2.23), (2.24), and (2.26) we have for all $n \geq \nu_1$ and $s \in I_n$,

$$\begin{aligned} h_n(l_n - 3) &\leq h_n(l_n - 2) \pi_n^2(\rho_n) / \pi_n^2(\lambda_n) \\ &\leq M_n(\pi_n^2 f_n) / \pi_n^2(\lambda_n) \leq f_n(s) \\ &\leq \|\pi_n^2 f_n\|_0^n / \pi_n^2(\rho_n) \leq h_n(l_n). \end{aligned} \tag{2.28}$$

So for $s \in I_n$ and n large enough,

$$|f_n(s) - h_n(l_n)| \leq (1 - a^{-3}) h_n(l_n)$$

and, thus,

$$\begin{aligned} \sup_{s \in I_n} |\bar{W}_n(s + f_n(s)) - \bar{W}_n(s)| &\geq \sup_{s \in I_n(m)} |\bar{W}_n(s + h_n(l_n)) - \bar{W}_n(s)| \\ &\quad - \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)|, \end{aligned}$$

where $\tau = 1 - a^{-3}$. Hence in this case,

$$A_n \geq \Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau),$$

where

$$\begin{aligned} \Delta_n(a, \gamma, \delta) &= \min_{-3 \leq k \leq k(n)} \min_{0 \leq m \leq m(n)} \sup_{s \in I_n(m)} |\bar{W}_n(s + h_n(k)) - \bar{W}_n(s)| / (h_n(k) \log n)^{1/2} \end{aligned}$$

and

$$\begin{aligned} D_n(a, \gamma, \tau) &= \max_{-3 \leq k \leq k(n)} \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \tau h_n(k)} |\bar{W}_n(s + t) - \bar{W}_n(s)| / (h_n(k) \log n)^{1/2}. \end{aligned}$$

Next, suppose $M_n(\pi_n^2 f_n) = \inf_{s \in I_n} (-\pi_n^2(s) f_n(s))$. Then similarly as in the preceding case one shows that for $s \in I_n$ and n large enough,

$$|f_n(s) + h_n(l_n - 3)| \leq \tau h_n(l_n)$$

and $0 \leq s + f_n(s) \leq s - h_n(l_n - 3)$. Thus,

$$\begin{aligned} \sup_{u \in I_n} |\bar{W}_n(u + f_n(u)) - \bar{W}_n(u)| &\geq \sup_{u \in I_n} |\bar{W}_n(u - h_n(l_n - 3)) - \bar{W}_n(u)| \\ &\quad - \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)|. \end{aligned}$$

Note that there exists $v_2 \geq \max(v, 3)$ such that for all $n \geq v_2$,

$$h_n(l_n) \leq h_n(k(n)) \leq \frac{1}{2} \delta n \exp(-(\log \log n)^2). \tag{2.29}$$

Hence we have $\{u = s + h_n(l_n - 3) : s \in I_n(m - 1)\} \subset I_n(m - 1) \cup I_n(m) \subset I_n$, so that in the present case,

$$\begin{aligned} A_n &\geq (h_n(l_n) \log n)^{-1/2} \left\{ \sup_{s \in I_n(m-1)} |\bar{W}_n(s + h_n(l_n - 3)) - \bar{W}_n(s)| \right. \\ &\quad \left. - \sup_{0 \leq s \leq 2n} \sup_{0 \leq t \leq \tau h_n(l_n)} |\bar{W}_n(s + t) - \bar{W}_n(s)| \right\} \\ &\geq (h_n(l_n - 3)/h_n(l_n))^{1/2} (\Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau)). \end{aligned}$$

Hence, in both cases possible we have

$$A_n \geq a^{-3/2} \Delta_n(a, \gamma, \delta) - D_n(a, \gamma, \tau).$$

Now from (slight modifications of) Lemmas 1 and 2 in Deheuvels and Mason [8] we obtain with probability one uniformly over all sequences $\{f_n\}_{n=1}^\infty \in \mathfrak{F}_\pi(\gamma, a, \eta, \nu)$,

$$\liminf_{n \rightarrow \infty} R_n(\pi_n, f_n) \geq a^{-3/2} - 2(1 - a^{-3})^{1/2}. \quad (2.30)$$

Observing that the right side of (2.30) can be chosen arbitrary close to one for a suitable choice of $a > 1$ completes the proof. ■

Let us now finish the proof of Theorem 3. Observe that $(\pi 1)$ and $(\pi 2)$ are easily checked for π_n as defined in (2.21). So it suffices to verify the conditions $(F_\pi 1)$, $(F_\pi 2)$, and $(F 3)$ in case

$$\begin{aligned} f_n(s) &= (n^{1/2}/h(\theta)) b_n(\psi_{\theta,n}(s)) h'(a(n, \psi_{\theta,n}(s))) \\ &= -s + (n/h(\theta)) h(q_n(\psi_{\theta,n}(s))), \end{aligned}$$

where the last equality follows from (2.20). So $(F 3)$ is immediate. To check $(F_\pi 1)$ remark that for any $0 < \bar{\theta} < T_G$,

$$1 \leq h'(u) \leq (1 - \bar{\theta})^{-2} (1 - G(\bar{\theta}))^{-1}, \quad 0 \leq u \leq \bar{\theta}.$$

Using condition $(\pi 1)$ and Fact 3 we see that $\{f_n\}_{n=1}^n$ satisfies $(F_\pi 1)$ almost surely for γ large enough.

Finally, we show that $\{f_n\}_{n=1}^\infty$ satisfies $(F_\pi 2)$ almost surely. Let $\kappa_n = \eta n \exp(-(\log \log n)^2)$. For any $s, t \in [0, n]$,

$$\begin{aligned} |\pi_n^2(t) f_n(t) - \pi_n^2(s) f_n(s)| &\leq |(\pi_n^2(t) - \pi_n^2(s)) f_n(t)| + |\pi_n^2(s)(f_n(t) - f_n(s))| \\ &=: d_1(s, t) + d_2(s, t). \end{aligned}$$

First,

$$\begin{aligned} d_1(s, t) &\leq 2 \|\pi_n\|_0^n \|f_n\|_0^n |\pi_n(t) - \pi_n(s)| \\ &\leq 2c^{-2} \|\pi_n^2 f_n\|_0^n \|\pi_n'\|_0^n |t - s|, \end{aligned}$$

where for the last inequality $(\pi 1)$ is used twice. Hence, uniformly over all intervals I_n of length κ_n , we have a.s. as $n \rightarrow \infty$ that

$$\sup_{s, t \in I_n} d_1(s, t) = O((\kappa_n/n) \|\pi_n^2 f_n\|_0^n) = o(\|\pi_n^2 f_n\|_0^n).$$

Next by $(\pi 1)$ and standard manipulations

$$\begin{aligned} d_2(s, t) &\leq |f_n(t) - f_n(s)| \\ &\leq (n^{1/2}/h(\theta)) \|b_n\|_0^\theta |h'(a(n, \psi_{\theta,n}(t))) - h'(a(n, \psi_{\theta,n}(s)))| \\ &\quad + (n^{1/2}/h(\theta)) \|h'\|_0^{\theta \vee q_n(\theta)} |b_n(\psi_{\theta,n}(t)) - b_n(\psi_{\theta,n}(s))| \\ &=: d_3(s, t) + d_4(s, t). \end{aligned}$$

As $h' \geq 1$ on $[0, \theta]$, it follows that $\psi_{\theta,n}$ is a Lipschitz function: for all $n \geq 1$ and all $s, t \in [0, n]$,

$$|\psi_{\theta,n}(t) - \psi_{\theta,n}(s)| \leq (h(\theta)/n) |t - s|. \tag{2.31}$$

Moreover, h' is uniformly continuous on $[0, \bar{\theta}]$ for $0 < \bar{\theta} < T_G$, since G is assumed to be continuous. Hence, also using (2.13), we have uniformly over all intervals I_n of length κ_n that a.s. as $n \rightarrow \infty$,

$$\sup_{s, t \in I_n} d_3(s, t) = o(n^{1/2} \|b_n\|_0^\theta) = o(\|\pi_n^2 f_n\|_0^n),$$

where the last “equality” follows from the fact that $\|\pi_n^2 f_n\|_0^n \geq c^2 n^{1/2} \|b_n\|_0^\theta / h(\theta)$.

Observe that for any $s, t \in [0, n]$,

$$\begin{aligned} d_4(s, t) &\leq (n^{1/2}/h(\theta)) \|h'\|_0^{\theta \vee q_n(\theta)} \|r_n\|_0^\theta \\ &\quad + n^{1/2} \|h'\|_0^{\theta \vee q_n(\theta)} |a_n(\psi_{\theta,n}(t)) - a_n(\psi_{\theta,n}(s))| \\ &=: d_5 + d_6(s, t). \end{aligned}$$

From (2.13), the upper bound part of this proof and the fact that h' is bounded from above on $[0, \bar{\theta}]$ for $0 < \bar{\theta} < T_G$, we see that a.s. as $n \rightarrow \infty$,

$$d_5 = O(n^{1/4}(\log n)^{1/2} (\log \log n)^{1/4}) = o(\|\pi_n^2 f_n\|_0^n),$$

where for the last “equality” $(F_{\pi 1})$ is applied. From (2.31) we get with the help of Schäfer [14, Corollary 3.2] or Aly *et al.* [1, Theorem 2.1] that a.s. as $n \rightarrow \infty$,

$$\begin{aligned} &\sup_{\substack{I_n: |I_n| \leq \kappa_n \\ I_n \subset [0, n]}} \sup_{s, t \in I_n} d_6(s, t) \\ &\leq n^{1/2} \|h'\|_0^{\theta \vee q_n(\theta)} \sup_{\substack{J_n: |J_n| \leq h(\theta)\kappa_n/n \\ J_n \subset [0, \theta]}} \sup_{u, v \in J_n} |a_n(u) - a_n(v)| \\ &= O(n^{1/2}(\log n)^{1/2} (\kappa_n/n)^{1/2}). \end{aligned}$$

As $(\kappa_n \log n)^{1/2} = o(\|\pi_n^2 f_n\|_0^n)$, we can conclude that uniformly over all intervals $I_n \subset [0, n]$ of length κ_n we have a.s. as $n \rightarrow \infty$,

$$\sup_{s, t \in I_n} |\pi_n^2(s) f_n(s) - \pi_n^2(t) f_n(t)| = o(\|\pi_n^2 f_n\|_0^n).$$

Hence (F_{π^2}) holds almost surely ($a > 1$ arbitrary), finishing the proof. ■

Proof of Theorem 1. The derivation of the limit finite dimensional distributions of r_n follows the lines of the proof of Theorem 3 in Beirlant *et al.* [3]. We only sketch the proof.

First with the help of approximation results (cf. Fact 1) one shows that it is possible to construct a sequence $\{W_n\}_{n=1}^\infty$ of standard Wiener processes extended to $(-\infty, \infty)$, in such a way that for $0 < s < \theta$, as $n \rightarrow \infty$,

$$n^{1/4} \left| r_n(s) - (1-s) \left\{ W_n(h(s)) - W_n \left(h(s) - \frac{n^{-1/2} W_n(h(s))}{(1-s)(1-G(s))} \right) \right\} \right| = o_p(1).$$

For any choice of $k \geq 1$ and $0 < s_1 < \dots < s_k < \theta$ fixed, let

$$W_n^{(i)}(x_i) = n^{1/4} \{ W_n(h(s_i) + n^{-1/2} x_i) - W_n(h(s_i)) \}, \quad x_i \in \mathbb{R}, \quad i = 1, \dots, k,$$

and let

$$V_n := -(W_n \circ h) / ((1-I)(1-G)).$$

Using Lemma 2.2 in Beirlant *et al.* [3] one shows that as $n \rightarrow \infty$,

$$(W_n^{(1)}, \dots, W_n^{(k)}, V_n) \xrightarrow{d} (W^{(1)}, \dots, W^{(k)}, V),$$

where $W^{(1)}, \dots, W^{(k)}$ are independent two-sided Wiener processes independent of $V = {}^d V_n$. To this end one only needs to check that

$$n^{1/4} \{ \text{Cov}[W_n(h(s) + n^{-1/2} x), -W_n(h(t)) / ((1-t)(1-G(t)))] \\ - \text{Cov}[W_n(h(s)), -W_n(h(t)) / ((1-t)(1-G(t)))] \} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any $s \in (0, \theta)$ and $x \in \mathbb{R}$. From this weak convergence result one deduces that as $n \rightarrow \infty$,

$$\begin{aligned} & (-(1-s_1) W_n^{(1)}(V_n(s_1)), \dots, -(1-s_k) W_n^{(k)}(V_n(s_k))) \\ & \xrightarrow{d} (-(1-s_1) W^{(1)}(V(s_1)), \dots, -(1-s_k) W^{(k)}(V(s_k))). \end{aligned} \quad (2.32)$$

Since the right side of (2.32) is equal in distribution to

$$\begin{aligned} & ((1-s_1) W^{(1)}(\bar{A}_G(s_1)/(1-s_1)^2), \dots, (1-s_k) W^{(k)}(\bar{A}_G(s_k)/(1-s_k)^2)) \\ & \stackrel{d}{=} (Z_1 |\bar{A}_G(s_1)|^{1/2}, \dots, Z_k |\bar{A}_G(s_k)|^{1/2}), \end{aligned}$$

the result follows. ■

Proof of Theorem 2. This proof can be given along similar lines as that of the upper bound part of Theorem 3. However, it is simpler because no supremum ($0 \leq s \leq \theta$) and no denominator ($(\|\beta_{n,G}\|_0^\theta)^{1/2}$) is involved. Here follows a short proof.

Writing $l_n = n^{1/4}/(\log \log n)^{3/4}$ we have for arbitrary $\varepsilon > 0$, almost surely, the following string of (in)equalities:

$$\begin{aligned} & \limsup_{n \rightarrow \infty} l_n |r_n(s)| \\ &= \limsup_{n \rightarrow \infty} l_n |A_n(s) - A_n(q(s))| \\ &\leq \limsup_{n \rightarrow \infty} l_n \sup_{t: |h(s) - h(t)| \leq |h(s) - h(q_n(s))|} |(1-s)W_n(h(s)) - (1-t)W_n(h(t))| \\ &\leq \limsup_{n \rightarrow \infty} l_n(1-s) \sup_{t: |h(s) - h(t)| \leq \frac{((1+\varepsilon)2h(s)\log \log n)^{1/2}}{n^{1/2}(1-s)(1-G(s))}} |W_n(h(s)) - W_n(h(t))| =: L(s). \end{aligned}$$

It is easily shown that

$$L(s) \leq (1 + \varepsilon)^{1/2} 2^{3/4} h^{1/4}(s)(1-s)^{1/2} (1-G(s))^{-1/2} \quad \text{a.s.}$$

Noting that $\varepsilon > 0$ is arbitrary, yields the desired result. ■

Note added in proof. After completion of our paper, Paul Deheuvels informed us that he and Ming Gu did research on this subject too.

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