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A UNIFYING APPROACH TO FUNCTIONAL LAWS OF THE ITERATED LOGARITHM AND GLIVENKO-CANTELLI THEOREMS FOR WEIGHTED EMPIRICAL PROCESSES[1)]

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1. INTRODUCTION AND MAIN RESULTS

Let $X_1, X_2, \ldots$ be a sequence of independent uniform $[0,1]^d$, $d \in \mathbb{N}$, random vectors and for each $n \in \mathbb{N}$ let

$$F_n(t) = n^{-1} \sum_{i=1}^{n} \prod_{j=1}^{d} 1_{[0,t_j]}(X_{ij}), \quad t \in [0,1]^d,$$

denote the empirical distribution function (df) based on the first n of these random vectors. (Here X_{ij} is the j-th component of X_i and t_j the j-th component of t). A considerable amount of research has been concerned with the almost sure behaviour of the weighted process V_n defined by

$$(1.1) \qquad V_n(t) = V_n(\alpha,q,d)(t) = \left(\frac{n}{2 \log\log n}\right)^{\alpha} \frac{F_n(t)-|t|}{q(|t|)}, \quad t \in [0,1]^d,$$

for the special cases $\alpha=0$ and $\alpha=1/2$, where $|t| = \pi_{j=1}^{d} t_j$. (The weight function q will be specified later on; throughout we adopt the convention $0/0=0$.) For $\alpha=0$ this behaviour is described by the well known weighted Glivenko-Cantelli theorem in Lai [12] and Wellner [14] for d=1 and Mason [13] for $d \in \mathbb{N}$, whereas the behaviour for the case $\alpha=1/2$ is known as the functional law of the iterated logarithm for the weighted empirical process, established in James ([10], d=1) and Alexander ([1], $d \in \mathbb{N}$).

It is one of the purposes of this paper to establish a strong law for $V_n(\alpha,q,d)$ with $0 \le \alpha \le 1/2$, thus unifying all the results mentioned above.

Key words and phrases: Glivenko-Cantelli theorem, law of the iterated logarithm, order statistic, weighted empirical process.

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In order to present this theorem define the class of weight functions

$$Q = \{q:[0,1]\to[0,\infty): q \text{ bounded away from 0 on } [\delta,1] \text{ for every } \delta > 0\}$$

and for $\tau \in \mathbf{R}$ the subclasses

$$Q_\tau = \{q\in Q: q/I^\tau \text{ is non-increasing on } (0,\delta] \text{ for some } \delta > 0\},$$

where I denotes the identity function on $(0,1]$; furthermore for $d \in \mathbf{N}$, $\alpha \in [0,1/2]$ and $q \in Q_{1-\alpha}$ write

$$A_{d,\alpha}(q) = \int_0^{e^{-e}} \frac{(\log(1/\sigma))^{d-1}}{(q(\sigma))^{1/(1-\alpha)}(\log\log(1/\sigma))^{\alpha/(1-\alpha)}}\, d\sigma.$$

Finally, let $B([0,1]^d)$ denote the space of bounded real-valued functions defined on $[0,1]^d$ with the supremum norm and $F([0,1]^d)$ denote the set of functions $f \in B([0,1]^d)$ such that there exists a function $\mathring{f}:[0,1]^d \to \mathbf{R}$ with

$$\int_{\pi_{j=1}^d [0,t_j]} \mathring{f}(s)d|s| = f(t), \quad t \in [0,1]^d,$$

$$\int_{[0,1]^d} (\mathring{f}(t))^2 d|t| \le 1 \quad \text{and } f(1,\dots,1) = 0.$$

For any $q \in Q$, set $F_q([0,1]^d) = \{f(\cdot)/q(|\cdot|) : f \in F([0,1]^d)\}$.

THEOREM 1. Let $d \in \mathbf{N}$, $\alpha \in [0,1/2]$ and $q \in Q_{1-\alpha}$. If $0 \le \alpha < 1/2$ and $A_{d,\alpha}(q) < \infty$, then

$$\text{(1.2)} \qquad \lim_{n\to\infty} \sup_{t\in[0,1]^d} |V_n(t)| = 0 \qquad \text{a.s.}$$

If $\alpha=1/2$ and $A_{d,\alpha}(q) < \infty$, then almost surely the sequence $\{V_n\}_{n=1}^\infty$ is relatively compact in $B([0,1]^d)$ with set of limit points equal to $F_q([0,1]^d)$. Conversely, if $0 \le \alpha \le 1/2$ and $A_{d,\alpha}(q)=\infty$, then

$$\text{(1.3)} \qquad \limsup_{n\to\infty} \sup_{t\in[0,1]^d} |V_n(t)| = \infty \qquad \text{a.s.}$$

Our next result is a refinement of Theorem 1 for the one dimensional case. In this refinement we describe the behavior of a version of the process V_n which is truncated from below at $X_{k:n}$(k fixed), the k-th order statistic of $X_1,\dots,X_n$, thus elucidating the influence of the extreme order statistics on the original process V_n. Of course, specializing this result to $\alpha=0$ and $\alpha=1/2$ respectively, yields the Glivenko-Cantelli theorem and the functional law of the iterated logarithm for this truncated process. For $d=1$, $k \in \mathbf{N}$, $\alpha \in [0,1/2]$ and $q \in Q$ define W_n by

$$(1.4) \qquad W_n(t) = W_n(\alpha,q,k)(t) = \begin{cases} \left(\dfrac{n}{2\ \mathrm{loglog} n}\right)^{\alpha} \dfrac{(F_n(t)-t)}{q(t)} & \text{for } X_{k:n} \le t \le 1 \\ 0 & \text{for } 0 \le t < X_{k:n}, \end{cases}$$

and

$$B_{k,\alpha}(q) = \int_0^{e^{-e}} \frac{\sigma^{k-1}}{(q(\sigma))^{k/(1-\alpha)}(\mathrm{loglog}(1/\sigma))^{k\alpha/(1-\alpha)}}\, d\sigma.$$

THEOREM 2. Let $d=1$, $k \in \mathbf{N}$ fixed, $\alpha \in [0,1/2]$ and $q \in Q_{1-\alpha}$. If $0 \le \alpha < 1/2$ and $B_{k,\alpha}(q) < \infty$, then

$$(1.5) \qquad \lim_{n\to\infty} \sup_{t\in[0,1]} |W_n(t)| = 0 \quad \text{a.s.}$$

If $\alpha=1/2$ and $B_{k,\alpha}(q) < \infty$, then almost surely the sequence $\{W_n\}_{n=1}^{\infty}$ is relatively compact in $B([0,1])$ with set of limit points equal to $F_q([0,1])$. Conversely, if $0 \le \alpha \le 1/2$ and $B_{k,\alpha}(q) = \infty$, then

$$(1.6) \qquad \limsup_{n\to\infty} \sup_{t\in[0,1]} |W_n(t)| = \infty \quad \text{a.s.}$$

Finally, we present two results which are (almost) corollaries to Theorems 1 and 2 respectively. These corollaries deal with the almost sure behaviour of weighted Cramér-von Mises-type statistics. It is shown that simple applications of Theorems 1 and 2 give sufficient conditions for convergence to zero or relative compactness of these statistics. These conditions are then shown to be very close to being necessary.

COROLLARY 1. Let $d \in \mathbf{N}$, $\alpha \in [0,1/2]$, $r > 0$, $q \in Q$ and assume $q^{r(1-\alpha)/(r(1-\alpha)+1)}/(I^{1-\alpha}(\mathrm{loglog}(1/I))^{\alpha/(r(1-\alpha)+1)})$ is non-increasing on $(0,\delta]$ for some $\delta > 0$. If $0 \le \alpha < 1/2$ and

$$\tilde{A}_{d,\alpha,r}(q) := \int_0^{e^{-e}} \frac{(\log(1/\sigma))^{d-1}}{(q(\sigma))^{\frac{r}{r(1-\alpha)+1}}(\mathrm{loglog}(1/\sigma))^{\frac{\alpha r}{r(1-\alpha)+1}}}\, d\sigma < \infty,$$

then

(1.7) $$\lim_{n\to\infty} \int_{[0,1]^d} |V_n(t)|^r d|t| = 0 \qquad \text{a.s.}$$

If $\alpha=1/2$ and $\tilde{A}_{d,\alpha,r}(q) < \infty$, then almost surely the sequence

$$\{ \int_{[0,1]^d} |V_n(t)|^r d|t| \}_{n=1}^{\infty}$$

is relatively compact in R (with the absolute value metric) with set of limit points equal to

$$\{ \int_{[0,1]^d} (|f(t)|/q(|t|))^r d|t| \; : \; f \in F([0,1]^d) \} .$$

However, for any $0 \leq \alpha \leq 1/2$ the choice $q = I^{1-\alpha+1/r}$ gives

(1.8) $$\limsup_{n\to\infty} \int_{[0,1]^d} |V_n(t)|^r d|t| = \infty \qquad \text{a.s.}$$

COROLLARY 2. Let $d=1$, $k \in N$ fixed, $\alpha \in [0,1/2]$, $r > 0$, $q \in Q$ and assume $q^{r(1-\alpha)/(r(1-\alpha)+k)}/(I^{((1-\alpha)+r(1-\alpha)^2)/(r(1-\alpha)+k)}(\log\log(1/I))^{k\alpha/(r(1-\alpha)+k)})$ is non-increasing on $(0,\delta]$ for some $\delta > 0$. If $0 \leq \alpha < 1/2$ and

$$\tilde{B}_{k,\alpha,r}(q) := \int_0^{e^{-e}} \frac{\sigma^{\frac{(k-1)(1-\alpha)r}{r(1-\alpha)+k}}}{q(\sigma))^{\frac{kr}{r(1-\alpha)+k}}(\log\log(1/\sigma))^{\frac{k\alpha r}{r(1-\alpha)+k}}}\, d\sigma < \infty,$$

then

(1.9) $$\lim_{n\to\infty} \int_0^1 |W_n(t)|^r dt = 0 \qquad \text{a.s.}$$

If $\alpha=1/2$ and $\tilde{B}_{k,\alpha,r}(q) < \infty$, then almost surely the sequence

$$\{ \int_0^1 |W_n(t)|^r dt \}_{n=1}^{\infty}$$

is relatively compact in R with set of limit points equal to

$$\{ \int_0^1 (|f(t)|/q(t))^r dt \; : \; f \in F([0,1]) \}.$$

However, for any $0 \leq \alpha \leq 1/2$ the choice $q = I^{1-\alpha+1/r}$ gives

(1.10) $$\limsup_{n\to\infty} \int_0^1 |W_n(t)|^r dt = \infty \qquad \text{a.s.}$$

For results on the closely related problem of the almost sure behavior of the weighted quantile process see Einmahl and Mason [7]. This section will be concluded by a number of remarks. All the proofs of the results are deferred to section 2.

Remark 1. Restricting the X_i's to the uniform distribution in this paper is too severe. For d=1 all the results remain true, mutatis mutandis, when the X_i's have a (common) df F on R; the same holds true for the multivariate results, provided the X_i's have a df F on $[0,1]^d$ with a continuous density (w.r.t. Lebesgue measure) that is bounded away from 0 and ∞. For the sake of brevity, we do not give the routine details here.

Remark 2. Of course, it is also possible to weight our processes in the right tail, i.e. that we also divide them by $\tilde{q}(1-|t|)$ for some weight function $\tilde{q}$. From symmetry considerations it is immediate that the one dimensional processes show the same behaviour in the right tail as in the left one. In the multivariate case the situation is a little more complicated. Looking at the proofs, however, it is easily seen how the present results have to be modified to obtain the right tail versions. (See e.g. Einmahl and Mason ([6], formula (3.36)) and Einmahl, Mason and Ruymgaart ([8], Théorème 3.4).)

Remark 3. Results closely related to our Theorem 1 for d=1 and $0 \le \alpha < 1/2$ can be found in Andersen, Giné and Zinn ([2], pp. 53-55). It is also interesting to compare Corollary 1 for d=1, $\alpha=1/2$, r=2 and $q=I^{1/2}$ with Theorem 3.1 in Csáki [3]. Using Remark 2, Csáki's result is in fact contained in Corollary 1, apart from showing that

$$\sup_{f \in \mathcal{F}([0,1])} 2 \int_0^1 ((f(t))^2/(t(1-t)))dt = 1.$$

Remark 4. The proofs of Theorem 1 and 2 rely heavily on results in Einmahl and Mason [6] and Einmahl, Haeusler and Mason [5], respectively. Taking these results as a starting point the present proofs are short and simple. It is also easily seen from the papers just mentioned that the case $\alpha=1/2$ is mathematically the hardest one; the results for $0 \le \alpha < 1/2$ are more or less corollaries to those for $\alpha=1/2$.

Remark 5. The conditions on the weight function in Theorems 1 and 2 are very mild. Note that neither continuity nor monotonicity of q (itself) is assumed. Needless to say the monotonicity conditions on q appear quite technical. We mention, however, that every smooth weight function for which the integral $\tilde{A}$(or $\tilde{B}$) is finite satisfies the corresponding condition.

Finally note that it is tacitly assumed that the weight functions are such that no measurability problems occur.

2. PROOFS

For convenient reference later on, we begin by recording a number of facts. Facts 1 and 2 are the main tools for proving Theorems 1 and 2 respectively. In the first three facts $\{a_n\}_{n=1}^{\infty}$ denotes a sequence of positive real numbers.

FACT 1. (Einmahl and Mason [6].) Let $d \in \mathbf{N}$ and $\alpha \in [0,1/2]$. If $\sum_n a_n(\log(1/a_n))^{d-1} < \infty$ and $na_n\downarrow$, then

$$\lim_{n\to\infty} a_n^{1-\alpha} \sup_{t\in[0,1]^d} \frac{n|F_n(t)-|t||}{|t|^{1-\alpha}} = 0 \qquad \text{a.s.}$$

FACT 2. (Einmahl, Haeusler and Mason [5].) Let $d=1$, $k \in \mathbf{N}$ fixed and $\alpha \in [0,1/2]$. If $\sum_n n^{k-1}a_n^k < \infty$ and $na_n\downarrow$, then

$$\lim_{n\to\infty} a_n^{1-\alpha} \sup_{X_{k:n}\le t\le 1} \frac{n|F_n(t)-t|}{t^{1-\alpha}} = 0 \qquad \text{a.s.}$$

FACT 3. (Kiefer [11].) Let $d=1$, $k \in \mathbf{N}$ fixed and $na_n\downarrow$. Then $P(X_{k:n}\le a_n \text{ i.o.}) = 1$ (or 0) according as $\sum_n n^{k-1}a_n^k = \infty$ (or $< \infty$).

FACT 4. (James [10].) If, for some $\delta > 0$, h is positive and non-increasing on $(0,\delta]$ and $\int_0^\delta (\sigma h(\sigma))^{-1}d\sigma < \infty$, then

$$\lim_{\sigma\downarrow 0} h(\sigma)/\log(1/\sigma) = \infty.$$

PROOF OF THEOREM 1. If we take a quick look at the published proofs for the cases $\alpha=0$ and $\alpha=1/2$ (refer to the introduction for the proper references) we see that the proof of Theorem 1 is an easy consequence of the following proposition combined with the functional law of the iterated logarithm for the unweighted empirical process (cf. Finkelstein [9] for d=1 and Wichura [15] for $d\ge 2$).

PROPOSITION 1. Let $d \in \mathbf{N}$, $\alpha \in [0,1/2]$ and $q \in Q_{1-\alpha}$. If $A_{d,\alpha}(q) < \infty$, then for every $\varepsilon > 0$ there exists an $0 < \eta < 1$ such that

$$\text{(2.1)} \qquad \limsup_{n\to\infty} \sup_{0\le|t|\le\eta} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)} < \varepsilon \qquad \text{a.s.}$$

If $A_{d,\alpha}(q) = \infty$, then for every $0 < \eta < 1$ the limsup in (2.1) is almost surely infinite.

PROOF. Assume $A_{d,\alpha}(q) < \infty$. We first prove

$$\text{(2.2)} \qquad \lim_{n\to\infty} \sup_{0\le|t|\le n^{-1/2}} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)} = 0 \qquad \text{a.s.}$$

Write $\phi = I/q^{1/(1-\alpha)}$ and observe that $\phi\uparrow$ on $(0,\delta]$ for some $\delta > 0$. We now have

$$\int_0^{e^{-e}} \frac{\phi(\sigma)\ (\log(1/\sigma))^{d-1}}{\sigma\ (\text{loglog } (1/\sigma))^{\alpha/(1-\alpha)}}\, d\sigma < \infty.$$

From the change of variables $\sigma = \tau^{-1/2}$ it is immediate that

$$\int_{e^{e/2}}^{\infty} \frac{\phi(\tau^{-1/2})(\log\tau)^{d-1}}{\tau\ (\text{loglog } \tau)^{\alpha/(1-\alpha)}}\, d\tau < \infty$$

and, since $\phi\uparrow$ on $(0,\delta]$,

$$\text{(2.3)} \qquad \sum_n \frac{\phi(n^{-1/2})(\log n)^{d-1}}{n(\text{loglog } n)^{\alpha/(1-\alpha)}} < \infty.$$

Writing $a_n = \phi(n^{-1/2})/(n(\text{loglog} n)^{\alpha/(1-\alpha)})$ and using some elementary analysis it follows from (2.3) that

$$\text{(2.4)} \qquad \sum_n a_n\ (\log(1/a_n))^{d-1} < \infty.$$

We also have $na_n\downarrow$ (ultimately) since $\phi\uparrow$ on $(0,\delta]$.

Observe that

$$\text{(2.5)} \qquad \sup_{0\le|t|\le n^{-1/2}} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)}$$

$$= \sup_{0\le|t|\le n^{-1/2}} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-|t||}{|t|^{1-\alpha}} (\phi(|t|))^{1-\alpha}$$

$$\le a_n^{1-\alpha} \sup_{t\in[0,1]^d} \frac{n|F_n(t)-|t||}{|t|^{1-\alpha}} .$$

Combining (2.5) with (2.4) and Fact 1 yields (2.2).

Next we show that for every $\varepsilon > 0$ there exists an $0 < \eta < 1$ such that

$$\text{(2.6)} \qquad \limsup_{n\to\infty} \sup_{n^{-1/2}\le|t|\le\eta} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)} < \varepsilon \qquad \text{a.s.}$$

Of course $A_{d,\alpha}(q) < \infty$ implies $A_{1,\alpha}(q) < \infty$. Using Fact 4, this implies

$$\text{(2.7)} \qquad \lim_{\sigma\downarrow 0} q(\sigma)/\sigma^{1-\alpha} = \infty.$$

Moreover, we have (see e.g. Einmahl [4])

$$\text{(2.8)} \qquad \limsup_{n\to\infty} \sup_{n^{-1/2}\le|t|\le 1} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{|t|^{1-\alpha}}$$

$$\le (2(d+1))^{1/2} 1_{\{1/2\}}(\alpha) \qquad \text{a.s.}$$

But

$$\text{(2.9)} \qquad \sup_{n^{-1/2}\le|t|\le\eta} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)}$$

$$\le \sup_{n^{-1/2}\le|t|\le\eta} \frac{|t|^{1-\alpha}}{q(|t|)} \sup_{n^{-1/2}\le|t|\le 1} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{|t|^{1-\alpha}} .$$

Combination of (2.7)-(2.9) yields (2.6) for $\eta=\eta(\varepsilon)$ small enough.

Now assume $A_{d,\alpha}(q)=\infty$ and let $0 < \eta < 1$ be arbitrary. Write $b_n=1/(n(\log n)^{d+1})$ and note that $\Sigma\, b_n(\log(1/b_n))^{d-1} < \infty$. By some elementary analysis we have $\Sigma\, \phi(b_n)(\log n)^{d-1}/(n(\log\log n)^{\alpha/(1-\alpha)}) = \infty$, with ϕ as before. This in combination with $\phi\uparrow$ on $(0,\delta]$ implies $\Sigma\, a_n(\log(1/a_n))^{d-1} = \infty$, where $a_n = \phi(b_n)/(n(\log\log n)^{\alpha/(1-\alpha)})$. From the Borel-Cantelli lemma we now have almost immediately

$$\text{(2.10)} \qquad P(b_n < |X_n| \le \varepsilon a_n \le \eta \text{ i.o.}) = 1, \text{ for any } \varepsilon > 0.$$

But

$$\text{(2.11)} \qquad \sup_{0\le|t|\le h} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)}$$

$$\ge \sup_{b_n\le|t|\le h} \frac{n^{\alpha}|F_n(t)-|t||}{|t|^{1-\alpha}} \frac{|t|^{1-\alpha}}{q(|t|)(\log\log n)^{\alpha}}$$

$$\ge a_n^{1-\alpha} \sup_{b_n\le|t|\le h} \frac{n|F_n(t)-|t||}{|t|^{1-\alpha}} .$$

Now it is clear from (2.10) and (2.11), noticing that F_n makes a jump of at least $1/n$ in X_n, that

$$\limsup_{n\to\infty} \sup_{0\le|t|\le h} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)} \ge \frac{1}{2\varepsilon^{1-\alpha}} \qquad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ completes the proof of this proposition. []

PROOF OF THEOREM 2. Similarly as in the proof of Theorem 1 it here suffices to prove

PROPOSITION 2. Let $d=1$, $k \in \mathbb{N}$ fixed, $\alpha \in [0,1/2]$ and $q \in Q_{1-\alpha}$. If $B_{k,\alpha}(q) < \infty$, then for every $\varepsilon > 0$ there exists an $0 < h < 1$ such that

$$\text{(2.12)} \qquad \limsup_{n\to\infty} \sup_{X_{k:n}\le t\le h} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)} < \varepsilon \qquad \text{a.s.}$$

If $B_{k,\alpha}(q) = \infty$, then for every $0 < h < 1$ the limsup in (2.12) is almost surely infinite.

PROOF. The proof is similar to that of Proposition 1 and hence will be abridged. The emphasis will be laid on those parts which are different.

Assume $B_{k,\alpha}(q) < \infty$. We first show

$$\text{(2.13)} \qquad \lim_{n\to\infty} \sup_{X_{k:n}\le t\le n^{-1/2}} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)} = 0 \qquad \text{a.s.}$$

Writing again $\phi = I/q^{1/(1-\alpha)}$ and $a_n = \phi(n^{-1/2})/(n(\log\log n)^{\alpha/(1-\alpha)})$, we can show that

(2.14) $$\sum_n n^{k-1} a_n^k < \infty \text{ and } na_n \downarrow \text{ (ultimately)}.$$

We also have

(2.15) $$\sup_{X_{k:n} \le t \le n^{-1/2}} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)} \le a_n^{1-\alpha} \sup_{X_{k:n} \le t \le 1} \frac{n|F_n(t)-t|}{t^{1-\alpha}}.$$

Combining (2.15) with (2.14) and Fact 2 yields (2.13).

Next we show that for every $\varepsilon > 0$ there exists an $0 < \eta < 1$ such that

(2.16) $$\limsup_{n\to\infty} \sup_{n^{-1/2} \le t \le \eta} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)} < \varepsilon \quad \text{a.s.}$$

From Fact 4 we again obtain that

$$\lim_{\sigma \downarrow 0} q(\sigma)/\sigma^{1-\alpha} = \infty.$$

Now we are in the same position as in the proof of Proposition 1 (cf. (2.7) -(2.9)). Hence we have (2.16).

Finally assume $B_{k,\alpha}(q) = \infty$ and let $0 < \eta < 1$ be arbitrary. Write $b_n = 1/(n(\log n)^2)$, then $\Sigma\, b_n < \infty$. By some elementary analysis we have $\Sigma\, n^{k-1} a_n^k = \infty$ and $na_n \downarrow$, where $a_n = \phi(b_n)/(n(\text{loglog } n)^{\alpha/(1-\alpha)})$. Using Fact 3, this yields

(2.17) $$P(b_n \le X_{k:n} \le \varepsilon a_n \wedge k/(2n) \le \eta \text{ i.o.}) = 1, \text{ for any } \varepsilon > 0.$$

Observe that (if $X_{k:n} \le \eta$)

(2.18) $$\sup_{X_{k:n} \le t \le \eta} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)}$$

$$\ge \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{(k/n - X_{k:n})(\phi(X_{k:n}))^{1-\alpha}}{X_{k:n}^{1-\alpha}}.$$

Now from (2.17) and (2.18)

$$\limsup_{n\to\infty} \sup_{X_{k:n} \le t \le \eta} \left(\frac{n}{\text{loglog } n}\right)^{\alpha} \frac{|F_n(t)-t|}{q(t)} \ge \frac{k}{2\varepsilon^{1-\alpha}} \text{ a.s.}$$

Letting $\varepsilon \downarrow 0$ completes the proof of this proposition. []

PROOF OF COROLLARY 1. We begin with the last part of the corollary. Let Y_n be the random vector among $X_1,\dots X_n$ such that its first component Y_{n1} is the smallest order statistic of $X_{11},\dots,X_{n1}$ From Fact 3 it is immediate that

$$(2.19)\qquad P(Y_{n1}\le 1/(n\log n),\ Y_{n2}\le 1/2,\dots,Y_{nd}\le 1/2 \quad \text{i.o.}) = 1.$$

Now assume, for the time being, that $Y_{n1}\le 1/(n\log n)$, $Y_{n2}\le 1/2,\dots,$ $Y_{nd}\le 1/2$. Then

$$(2.20)\qquad \int_{[0,1]^d} |V_n(t)|^r d|t| = \int_{[0,1]^d} \left(\left(\frac{n}{2\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{|t|^{1-\alpha+1/r}}\right)^r d|t|$$

$$\ge \left(\frac{n}{2\log\log n}\right)^{\alpha r} \int_{[0,1]^d} \left(\frac{|F_n(t)-|t||}{t_1^{1-\alpha+1/r}}\right)^r d|t|$$

$$\ge \left(\frac{n}{2\log\log n}\right)^{\alpha r} \int_{1/2}^{1}\cdots\int_{1/2}^{1}\int_{Y_{n1}}^{1/(2n)} \left(\frac{1}{2nt_1^{1-\alpha+1/r}}\right)^r dt_1 dt_2\dots dt_d$$

$$\ge \left(\frac{1}{2}\right)^{d-1+r(1+\alpha)} \left(\frac{1}{n^{1-\alpha}(\log\log n)^{\alpha}}\right)^r \int_{Y_{n1}}^{1/(2n)} \frac{1}{t_1^{r(1-\alpha)+1}}\, dt_1$$

$$= \left(\frac{1}{2}\right)^{d-1+r(1+\alpha)} \frac{1}{r(1-\alpha)} \left(\left(\frac{1}{(nY_{n1})^{1-\alpha}(\log\log n)^{\alpha}}\right)^r - \left(\frac{2^{1-\alpha}}{(\log\log n)^{\alpha}}\right)^r\right)$$

$$= \left(\frac{1}{2}\right)^{d-1+r(1+\alpha)} \frac{1}{r(1-\alpha)} \left(\left(\frac{(\log n)^{1-\alpha}}{(\log\log n)^{\alpha}}\right)^r - \left(\frac{2^{1-\alpha}}{(\log\log n)^{\alpha}}\right)^r\right).$$

Combination of (2.19) and (2.20) yields (1.8).

To complete the proof of Corollary 1 it is sufficient to establish

PROPOSITION 3. Let $d \in \mathbb{N}$, $\alpha \in [0,1/2]$, $r > 0$, $q \in Q$ and assume $q^{r(1-\alpha)/(r(1-\alpha)+1)}/(I^{1-\alpha}(\log\log(1/I))^{\alpha/(r(1-\alpha)+1)})$ is non-increasing on $(0,\delta]$ for some $\delta > 0$. If $\tilde{A}_{d,\alpha,r}(q) < \infty$, then for every $\varepsilon > 0$ there exists an $0 < \eta < 1$ such that

$$\limsup_{n\to\infty} \int_{|t|\le\eta} \left(\left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)}\right)^{r} d|t| < \varepsilon \quad \text{a.s.} \tag{2.21}$$

PROOF. Let $\varepsilon > 0$. Note that for any $0 < \eta < e^{-e}$

$$\int_{|t|\le\eta} \left(\left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)}\right)^{r} d|t| \tag{2.22}$$

$$\le \left\{\sup_{|t|\le\eta} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)^{\frac{r(1-\alpha)}{r(1-\alpha)+1}}(\log\log(1/|t|))^{\frac{-\alpha}{r(1-\alpha)+1}}}\right\}^{r}$$

$$\cdot \int_{|t|\le\eta} q(|t|)^{\frac{-r}{r(1-\alpha)+1}} \log\log(1/|t|))^{\frac{-\alpha r}{r(1-\alpha)+1}} d|t|.$$

From Proposition 1 we see that for η small enough

$$\limsup_{n\to\infty} \sup_{|t|\le\eta} \left(\frac{n}{\log\log n}\right)^{\alpha} \frac{|F_n(t)-|t||}{q(|t|)^{\frac{r(1-\alpha)}{r(1-\alpha)+1}}(\log\log(1/|t|))^{\frac{-\alpha}{r(1-\alpha)+1}}} < \varepsilon \text{ a.s.} \tag{2.23}$$

Moreover by the change of variables $\sigma=|t|$, $s_2=t_2,\dots,s_d=t_d$, it is immediate that the integral on the right side of (2.22) is bounded from above by

$$\int_0^{\eta} \frac{(\log(1/\sigma))^{d-1}}{(q(\sigma))^{\frac{r}{r(1-\alpha)+1}}(\log\log(1/\sigma))^{\frac{\alpha r}{r(1-\alpha)+1}}} d\sigma,$$

which in turn is bounded by 1 for η small enough, since $\tilde{A}_{d,\alpha,r}(q) < \infty$. Combination of (2.22), (2.23) and the remark just above, yields (2.21). This completes the proof of Proposition 3 and hence of Corollary 1. []

The proof of Corollary 2 can be given along the same lines as the previous one, using Proposition 2 instead of Proposition 1. Therefore it will be omitted.

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