The Buridan-Volpin Derivation System

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The Buridan-Volpin Derivation System; Properties and Justification

Proefschrift

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Preface

Being much impressed with prof. David Isles' treatment of logic as a purely syntactic discipline, I tried to follow in his footsteps by expanding on his work in this field.

Much of Chapters 2 and 3 is simply a rehash of his work in [15], albeit that some pains have been taken to make matters formally exact (perhaps to the point of punctiliousness, but it is logic after all) with regard to numerical notation, reduction, reference grammars, and the actual derivation system. Chapter 3 closely follows [15] with one rather important exception, the different interpretations prof. Isles and I have of the notion of inheritance. This is extensively argued in Chapter 6, where one can also find such topics as reference versus reduction, the notion of induction, realism versus nominalism and reference grammars.

Chapter 2 presents the Buridan-Volpin (BV) derivation system[1]. In Chapter 3 we discuss the BV proofs of the totality of Addition, Multiplication and Exponentiation and the failure of the traditional proof of Euclid's lemma in the BV system.

From Chapter 4 onward, the work becomes more original and overall has the goal to show that the Buridan-Volpin (BV) system is an acceptable alternative as a first-order logic for the traditional logic. This is done by proving that the basic properties desired of a first-order logic hold for the Buridan-Volpin system.

In Chapter 4 we prove the Cut-elimination theorem for BV minus Induction and point out why the proof does not go through for BV including Induction. Consequently, we are able to give a non-constructive consistency proof for BV minus Induction.

[1] Buridan for the Medieval logician who claimed that the meaning of terms is dependent on their function and place within an argument and Volpin for Yessenin Volpin, the father of ultraintuitionism.

In Chapter 5 we show that - under certain conditions - a version of the Lőwenheim-Skolem theorem holds for BV including Induction.

Chapter 6 is an attempt to argue that the BV system is not only theoretically acceptable as a first-order logic, but worth actual consideration as a standard first-order logic, at the very least for those of finitistic or nominalistic inclination. The latter is particularly important as so much work yet remains to be done with regard to the properties, proof-theoretical and otherwise, of the BV system. Whether or not this system is truly worth propagating as an alternative to traditional first-order logic remains to be seen, yet I do not doubt that it is worth making the effort to find out. With this thesis I hope to make a contribution to the start of this work, and possibly interest others to take on this project, begun by prof. Isles, as well.

Contents

Chapter 1

Introduction

Logic traditionally claims to be a purely (at least in theory, if not necessarily in practice) syntactic discipline, a trait that underlies most of its distinguishing properties and accounts for its usefulness as a higher-order tool in the study of semantically circumscribed disciplines like mathematics, formal language theory and automata theory. The present writing argues that to make good on this claim, it is required that logic shed itself of the last vestiges -mainly unnoticed in logical practice as performed today- of semantic content remaining within it, to wit: 1) the assumption that the domain of a variable is to be fixed in advance and independent of the derivations in which said variable occurs, and 2) the assumption that to every well-formed numerical notation instance there corresponds a number. These two assumptions are rarely made explicit, as they are considered to have no influence on the properties of derivations and formal systems, and are hence harmless because inert. We will now proceed, working on the foundations laid by prof. David Isles in his articles [12], [13], [14] and [15], to show that avoiding these assumptions yields a formal system of derivation that is substantially different from those that do hold these assumptions, that its proof-theoretical properties differ from the standard ones and that it is a viable alternative as a model of mathematical practice in the sense that it has the properties traditionally associated with a first-order logic. But first we must define the system in question.

1.1 The Problem

Most mathematicians assume - usually unconsciously - that there exists a set $\mathbb{N}$ of natural numbers, containing the elements 0, 1, 2, 3, ... and nothing else. The terms 5 + 7 and 12 are thought to refer to the same natural number in the set $\mathbb{N}$. They assume that the natural numbers 0, 1, 2, ... and the set $\mathbb{N}$ of all natural numbers exist in some Platonic universe, in other words, that these mathematical objects have been created by God and that we, human beings, may discover properties of these objects.

From an intuitionistic point of view the natural numbers are individual constructions in my mind, are created by myself and $\mathbb{N}$ is identified with a construction project: I have a construction project starting with 0 and next, having constructed the natural number n, I know how to construct its successor sn or $n + 1$.

Peano arithmetic PA formalizes the theory of natural numbers by i) introducing a formal logical language with one unary function symbol s for successor, two binary function symbols + and $\cdot$ for addition and multiplication respectively and one binary predicate symbol = for equality, and ii) by postulating a number of axioms in the language which specify the properties s, +, $\cdot$ and = should satisfy. The idea is that all true statements about the successor function, additon and multiplication, like $x \cdot (y + z) = x \cdot y + x \cdot z$ may be logically deduced from Peano's axioms.

An important axiom in PA is the well-known *induction axiom* which says that in order to prove $\forall x[A(x)]$ (where A is a formula in the language of PA) it suffices to show that 0 has property A and that if x has property A then also its successor sx has property A:

$$A(0) \wedge \forall x[A(x) \rightarrow A(sx)] \rightarrow \forall y[A(y)]$$

Notice that the formula $A(x)$ may contain quantifiers which by most mathematicians are also supposed to range over the set $\mathbb{N}$ of all natural numbers. Therefore, one cannot argue that induction only involves the numbers from 0 to n: the property of n being established may be a formula containing quantifiers ranging over the set $\mathbb{N}$ of all natural numbers. In other words, the induction axiom assumes - implicitly - that the set $\mathbb{N}$ exists and the number n in question is supposed to be an element of it. So, the induction axiom involves an *impredicative* concept of number. Consequently, there is a semantic component in Peano's Arithmetic, although it is meant as a purely syntactic enterprise.

Most mathematicians, both classical and intuitionistic, take for granted that the impredicativity inherent in the induction axiom is harmless, that discourse within the domain of numbers is meaningful. However, from a nominalistic and purely syntactic point of view, numbers are symbolic expressions: 0, $s0$, $ss0$, ... A construction does not exist until it is made and when a new construction is made it is not a selection from a pre-existing collection.

To tackle the problem mentioned above, Edward Nelson in [20] simply drops the induction axiom from Peano's Arithmetic. Because the resulting theory is too weak to be of much arithmetical interest, Nelson considers Robinson's theory Q in the language of Peano's Arithmetic with the familiar non-logical (arithmetical) axioms for s, $+$ and $\cdot$ and the following extra axiom R (Robinson):

$$x \neq 0 \rightarrow \exists y[sy = x].$$

Q may be considered as a minimal axiomatization of arithmetic; it does not contain the induction axiom. It turns out to be convenient to reformulate Q as an open theory by adjoining a unary function symbol p (predecessor) and to postulate the following axiom instead of the axiom R:

$$px = y \leftrightarrow sy = x \vee (x = 0 \wedge y = 0)$$

Nelson defines a theory T to be *predicative* iff T is interpretable in Q. In his book [20] Nelson develops *predicative arithmetic* and gives an idea of its limitations. In particular, he shows that the use of induction in the proof of the closure of the natural numbers under exponentiation is impredicative in his sense. This evidence, together with the obvious practical impossibilities of computing the numerical value of even modest exponential expressions (think of 2^{16} or 2^{65536}), leads him to conclude that exponentiation is not total. His proof of the Consistency Theorem for Q is finitary, but the question remains whether it can be established predicatively.

Hilbert's program was to secure the foundations of classical mathematics by giving a finitary consistency proof for it. His program was influenced by his controversy with Brouwer: finitary methods are accepted by the intuitionists. In fact, Gődel in his 1933 paper [10] gave an interpretation of classical arithmetic within intuitionistic arithmetic, hence establishing that classical arithmetic is no less secure than intuitionistic arithmetic. But Hilbert's program was shown to be untenable by Gődel [9] in his famous second incompleteness theorem from 1931, in which he showed that no consistency proof of Peano's Arithmetic (or an extension thereof) can be proved within Peano's Arithmetic (or the extension in question) itself, and hence certainly not by finitary methods. From a nominalist point of view, but also when we take Hilbert's formalism literally, one may criticize Hilbert's program because it accepts impredicative finitary methods and hence it still contains a semantic element.

The approach taken by Davis Isles in [12, 13, 14, 15] is to declare the domain of the variable y in the induction axiom to be N where N is recursively defined by: $0 \in N$ and if $\alpha \in N$, then $s\alpha \in N$. The syntactic objects $0, s0, ss0, \ldots$ are called *numerals* and syntactic objects like $s0 + s0$ may be reduced by rewrite rules to $ss0$. Two syntactic terms may be called equal if they reduce to the same normal form; for instance, $s0+s0 = ss0+0$, because both terms reduce via rewrite rules to the same numeral $ss0$.

Chapter 2

The Buridan-Volpin system

David Isles assumes that numerical notations are fundamentally syntactic objects, and that they do not necessarily correspond with or refer to a semantic content (a number). Rewrite rules reduce numerical notations, like $s0 + ss0$ and $ss0 \cdot sss0$, to numerals ($sss0$, resp. $ssssss0$, in our examples) as normal forms. Note also that the induction rule will, in this syntactic environment, only hold for normal forms as it considers only numerals. This amounts to replacing the semantic reference-relation (a numerical notation refers to an independently existing number) by the syntactic reduction relation (a numerical notation reduces to a normal form). Still informally, also assume that the only requirement placed on a derivation step is preservation of truth. The latter does not require that the domains of the variables be fixed, rather it requires that, for certain kinds of derivation steps, the relations between the domains of the variables used in the derivation be explicited within the derivation. This elicits information to become explicit which is not taken into account in a traditional derivation. The validity of a derivation then becomes dependent on the possibility of meeting all the requirements placed on the domains of the variables. All of this remains a little vague, so for clarity's sake let us turn to a formal definition of the system.

2.1 Numerical Notation

Definition 2.1 (Numeral Numerical Notation)
i) L_N is the language of numeral numerical notation, $L_N := \{0, s, (,)\}$
ii) The numeral numerical notation N is recursively defined by:

a) $0 \in N$;
b) if $\alpha \in N$, then $s(\alpha) \in N$.
Nothing else is in N if not constructed in finitely many steps from a) and b).

Definition 2.2 (Set of function symbols)
$\mathbf{F} := \{f_\alpha \mid \alpha \in N\}$. We call $\mathbf{F}$ the set of function symbols.
Examples of function symbols are $+$ (addition) and $\cdot$ (multiplication).

Definition 2.3 (Functional Numerical Notation)
For all $F \subset \mathbf{F}$
i) $L_{N_F} := L_N \cup F$. L_{N_F} is the language of F-functional numerical notation.
ii) The F-functional numerical notation N_F is recursively defined by:
a) $0 \in N_F$
b) if $\alpha \in N_F$, then $s(\alpha) \in N_F$
c) if $\alpha, \beta \in N_F$, then, for all $\gamma \in N$: $f_\gamma(\alpha, \beta) \in N_F$ iff $f_\gamma \in F$

Informally, N_F contains all numerals [clause a) and b)] and all other numerical notations that can be constructed by means of the function symbols in F [clause c)].

In this book we will only consider N, the addition $(+)$ numerical notation AN, the multiplication $(+, \cdot)$ numerical notation MN and the exponential $(+\ ,\ \cdot\ ,\ \hat{}\ )$ numerical notation EN, all formed according to the above definitions. Note that trivially $N \subset AN \subset MN \subset EN$. All of these are natural numerical notations, but only N is called the numeral numerical notation.

Unless otherwise noted, in this manuscript variables will have ranges in either N, MN or EN.

2.2 Reduction

Definition 2.4 (Immediate Reduction Relation)
Let $\alpha, \beta, \gamma, \delta \in EN$. Then $\alpha \rightharpoonup \beta$ (read: α immediately reduces to β) is recursively defined by:
i) $\alpha \rightharpoonup \alpha$.
ii) $(\alpha \cdot 0) \rightharpoonup 0$.
iii) $\alpha^0 \rightharpoonup s(0)$.

iv) if $\alpha \rightharpoonup \gamma$ and $\beta \rightharpoonup \delta$, then
$$\begin{array}{l} s(\alpha) \rightharpoonup s(\gamma); \\ (\alpha + \beta) \rightharpoonup (\gamma + \delta); \\ (\alpha \cdot \beta) \rightharpoonup (\gamma \cdot \delta); \\ \alpha^{\beta} \rightharpoonup \gamma^{\delta}. \end{array}$$

v) if $\alpha \rightharpoonup \gamma$ and $\beta \rightharpoonup \delta$, then
$$\begin{array}{l} (\alpha + 0) \rightharpoonup \gamma; \\ (\alpha + s(\beta)) \rightharpoonup s(\gamma + \delta); \\ (\alpha \cdot s(\beta)) \rightharpoonup ((\gamma \cdot \delta) + \gamma); \\ \alpha^{s(\beta)} \rightharpoonup ((\gamma^{\delta}) \cdot \gamma). \end{array}$$

Definition 2.5 (Reduction Relation)
Let $\alpha, \beta, \gamma \in EN$. Then $\alpha \rightharpoondown \beta$ (read: α reduces to β) is recursively defined by:
i) $\alpha \rightharpoondown \alpha$
ii) if $\alpha \rightharpoonup \beta$, then $\alpha \rightharpoondown \beta$
iii) if $\alpha \rightharpoonup \gamma$ and $\gamma \rightharpoondown \beta$, then $\alpha \rightharpoondown \beta$

Theorem 2.1 (Normalization of Reduction)
For all $\alpha \in EN$, there exists a unique $\beta \in N$ such that $\alpha \rightharpoondown \beta$.

Proof sketch: Eliminate $\rightharpoondown$ in favour of $\rightharpoonup$ using Definitions 2.4 and 2.5, then apply structural induction following 2.4 iv) and v). This will give a proof of existence. Elaboration is left to the reader.

Definition 2.6 (Equality)
Let $\alpha, \beta \in EN$.
i) $\alpha = \beta$ (α is reductively identical to β) := there exists a $\gamma \in N$ such that:
a) $\alpha \rightharpoondown \gamma$ and
b) $\beta \rightharpoondown \gamma$
For instance: $s(0) + s(0) = ss(0) + 0$ as both reduce (in respectively two and one reduction steps) to $ss(0)$, but not $s(0) + 0 = ss(0) + 0$ as these respectively reduce to $s(0)$ and $ss(0)$
ii) $\alpha =_s \beta$ (α is syntactically identical to β) := $\alpha \rightharpoonup \beta$ and $\beta \rightharpoonup \alpha$.

For instance: $s(0) =_s s(0)$, but not $s(0) + 0 =_s s(0)$ as $s(0)$ does not reduce to $s(0) + 0$. Note that also not $s(0) + 0 =_s 0 + s(0)$ as neither of these two expressions reduces immediately (or otherwise) to the other.

Note that reductive identity corresponds to traditional identity in that it supposes normalization to the same normal form (which is our equivalent to referring to the same number), and that syntactic identity is only met between objects that are symbol for symbol identical.

This defines the structure of numerical notations, the notions of reduction and identity within them and their relation, and shows that reduction, and hence identity, is sound. This means that the reduction relation can be defined precisely and that every notation reduces to a unique normal form.

Next we consider the Buridan-Volpin (from now on: BV) derivation system. For brevity's sake we suppose all standard definitions concerning formal logical language to have been given, as well as the notions of free and bound occurrences of variables. In order to define our derivation system, we will need to define equiformity and rectifiedness of formulas.

Definition 2.7 (equiform, rectified) Two formulas are *equiform* iff they differ (to the point of syntactic identity) only in the naming of their bound variables. A formula is *rectified* iff the names of all its bound variables are distinct from each other and from its free variables.

We also need the notion of the domain of a variable, which we will consider a primitive (and hence undefined) notion. The *domain* of x, $dom(x)$ - following Isles, also called the *range* of x, $r(x)$ - is the set of possible substitution values, taken from the natural numerical notations, for x. Note that N is the smallest of these, so $N \subseteq dom(x)$ always holds. Other potential domains are AN, MN and EN.
The domain of a functional expression, $dom(f(x, y))$ with $f \in \mathbf{F}$, is the set of all expressions $f(a, b)$ with $a \in dom(x)$ and $b \in dom(y)$.

2.3 Reference Relation and Grammar

Because we will not assume the domain of a variable to have been fixed in advance of the derivation, but rather that the derivation supplies the conditions the domains of the variables need to meet, we will need a way of keeping track of the conditions placed on the domains of the variables. This will be done by a reference grammar, consisting of reference relations.

Definition 2.8 (Reference Relations)
i) $x \to y := dom(y) \subseteq dom(x)$ (read: x refers to y)
ii) $[x, y] := x \to y$ and $y \to x$ (i.e. $dom(x) = dom(y)$)
iii) $N \to x := dom(x) \subseteq N$ and $x \to N := N \subseteq dom(x)$.
iv) $[N, x] := N \to x$ and $x \to N$ (i.e. $dom(x) = N$).
(Symmetrically for $[x, N]$)

v) Let $f \in \mathbf{F}$. $f(x,y) \to z := dom(z) \subseteq dom(f(x,y))$. (Symmetrically for $z \to f(x,y)$, analogously for $f(x,y) \to N$ and $N \to f(x,y)$)
vi) Let $f \in \mathbf{F}$. $[f(x,y),z] := f(x,y) \to z$ and $z \to f(x,y)$ (i.e. $dom(f(x,y)) = dom(z)$). (Symmetrically for $[z, f(x,y)]$, analogously for $[f(x,y), N]$.

$\to$ is called a *reference arrow*, $x \to y$ (or any of its possible variations) a reference.

Definition 2.9 (Reference Grammars)
A reference grammar G (or RG) is defined inductively as follows:
i) $\emptyset$ is a reference grammar;
ii) if G is a reference grammar, then so is $G \cup \{x_\alpha \to x_\beta\}$ for all $\alpha, \beta \in N$;
iii) if G is a reference grammar, then so is $G \cup \{f_\gamma(\alpha,\beta) \to x_\delta\}$ for all $\alpha,\beta,\gamma,\delta \in N$;
iv) if G is a reference grammar, then so is $G \cup \{x_\alpha \to f_\delta(\beta,\gamma)\}$ for all $\alpha,\beta,\gamma,\delta \in N$;
v) if G is a reference grammar, then so is $G \cup \{f_\iota(\alpha,\beta) \to f_\kappa(\gamma,\delta)\}$ for all $\alpha,\beta,\gamma,\delta,\iota,\kappa \in N$;
vi) if in any of the above a variable is replaced by N, this also yields a reference grammar;
vii) nothing else is a reference grammar.

Informally, a reference grammar is any set (possibly empty) of reference-arrows between variables and/or functional expressions and/or N. The function of a reference grammar is to keep track of the conditions that the derivation places on the domains of the variables (in order to preserve truth in every step of the derivation). This means that changes to the reference grammar are now a fundamental part of the definition of the derivation system, as can be seen below in the formal definition of the derivation system by the fact that every derivation rule now also explicitly states what, if any, changes in the reference grammar it entails.

2.4 Buridan-Volpin Calculus

Now we can formally present the Rules of the Buridan-Volpin (BV) calculus as given by Isles [15] (presupposing the standard Gentzen [7] notions):

Definition 2.10 (derivation) A *derivation* D with end sequent $\Gamma \Rightarrow \Delta$ and reference grammar G, represented as $D \vdash [\Gamma \Rightarrow \Delta; G]$, in the sequent

calculus BV (including Induction) is a tree of sequents using rules of inference of four kinds: logical, structural, cut and induction. Because different variables in a derivation may have different ranges or domains, more care than is usual must be taken to distinguish different occurrences of variables. If $A[x]$ denotes a formula containing a free variable x at zero or more specified locations and t is a term, then $A[x/t]$ denotes the result of replacing x by t at all the specified locations.
Initial sequents or axioms are of the form $F \Rightarrow F$ where F is a quantifier-free formula. If G is the empty reference grammar, then $[F \Rightarrow F; G]$ is a derivation D.

The **logical rules** deal with the quantifiers and with the connectives:

$\forall L$ **universal quantification left** **universal quantification right** $\forall R$

$$(\forall L)\ \frac{A[x/t], \Gamma \Rightarrow \Delta\ \ [G]}{\forall x A, \Gamma \Rightarrow \Delta\ \ [G']} \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]\ \ [G]}{\Gamma \Rightarrow \Delta, \forall x A\ \ [G']}\ (\forall R)$$

$$G' = G \cup \{x \to t\} \qquad G' = G \cup \{a \to x\}$$

$\exists L$ **existential quantification left** **existential quantification right** $\exists R$

$$(\exists L)\ \frac{A[x/a], \Gamma \Rightarrow \Delta\ \ [G]}{\exists x A, \Gamma \Rightarrow \Delta\ \ [G']} \qquad \frac{\Gamma \Rightarrow \Delta, A[x/t]\ \ [G]}{\Gamma \Rightarrow \Delta, \exists x A\ \ [G']}\ (\exists R)$$

$$G' = G \cup \{a \to x\} \qquad G' = G \cup \{x \to t\}$$

In $(\forall R)$ and $(\exists L)$ a is a parameter (free variable) called the *eigenparameter* of the inference. It is required that a does not occur in the sequent which concludes the inference and that x is a new varaiable.

$\to L$ **implication left** **Implication right** $\to R$

$$(\to L)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1\ \ [G_1] \qquad B, \Gamma_2 \Rightarrow \Delta_2\ \ [G_2]}{A \to B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\ \ [G]} \qquad \frac{A, \Gamma \Rightarrow \Delta, B\ \ [G]}{\Gamma \Rightarrow \Delta, A \to B\ \ [G]}\ (\to R)$$

$$G = G_1 \cup G_2$$

$\wedge L$ **conjunction left** **conjunction right** $\wedge R$

$$(\wedge L)\ \frac{A, \Gamma \Rightarrow \Delta\ \ [G]}{A \wedge B, \Gamma \Rightarrow \Delta\ \ [G]} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A\ \ [G_1] \qquad \Gamma_2 \Rightarrow \Delta_2, B\ \ [G_2]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \wedge B\ \ [G]}\ (\wedge R)$$

$$G = G_1 \cup G_2$$

$\vee L$ **disjunction left** **disjunction right** $\vee R$

$$(\vee L)\ \frac{A, \Gamma_1 \Rightarrow \Delta_1\ [G_1] \qquad B, \Gamma_2 \Rightarrow \Delta_2\ [G_2]}{A \vee B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\ [G]} \qquad \frac{\Gamma \Rightarrow \Delta, A\ [G]}{\Gamma \Rightarrow \Delta, A \vee B\ [G]}\ (\vee R)$$

$$G = G_1 \cup G_2$$

In $(\to L)$, $(\wedge R)$ and $(\vee L)$ all bound variables in the right subtree are distinct from all those in the left subtree and no free variable of the left subtree is an eigenparameter of the right subtree, and conversely.

In $(\wedge L)$ and $(\vee R)$ the formula B is rectified and all its bound variables are distinct from all other bound variables in the derivation to that point.

$\neg L$ **negation left** **negation right** $\neg R$

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, A\ [G]}{\neg A, \Gamma \Rightarrow \Delta\ [G]} \qquad \frac{A, \Gamma \Rightarrow \Delta\ [G]}{\Gamma \Rightarrow \Delta, \neg A\ [G]}\ (\neg R)$$

The **structural rules** deal with permutation, weakening and contraction.

PL **permutation left** **permutation right** PR

$$(PL)\ \frac{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta\ [G]}{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta\ [G]} \qquad \frac{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2\ [G]}{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2\ [G]}\ (PR)$$

WL **weakening left** **weakening right** WR

$$(WL)\ \frac{\Gamma \Rightarrow \Delta\ [G]}{A, \Gamma \Rightarrow \Delta\ [G]} \qquad \frac{\Gamma \Rightarrow \Delta\ [G]}{\Gamma \Rightarrow \Delta, A\ [G]}\ (WR)$$

In (WL) and (WR) no free variable in A is an eigenparameter of the derivation to that point. A is rectified and all of its bound variables are distinct from all other bound variables in the derivation to that point.

CL **contraction left** **contraction right** CR

$$(CL)\ \frac{A_1, A_2, \Gamma \Rightarrow \Delta\ [G']}{A, \Gamma \Rightarrow \Delta\ [G]} \qquad \frac{\Gamma \Rightarrow \Delta, A_1, A_2\ [G']}{\Gamma \Rightarrow \Delta, A\ [G]}\ (CR)$$

$$G = G' \cup \{[x_i, x_{i_1}, x_{i_2}]\} \qquad G = G' \cup \{[x_i, x_{i_1}, x_{i_2}]\}$$

x_i, x_{i_1} and x_{i_2} are the corresponding bound variables in A, A_1 and A_2 respectively. A_1, A_2 and A are equiform and all bound variables in A are distinct from all bound variables in the derivation to that point.

cut rule (C)

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A_1 \ [G_1] \qquad A_2, \Gamma_2 \Rightarrow \Delta_2 \ [G_2]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \ [G]} (C)$$

$$G = G_1 \cup G_2 \cup \{[x_{i_1}, x_{i_2}]\}$$

A_1 and A_2 are equiform; x_{i_1} and x_{i_2} are the corresponding bound variables in A_1 and A_2. All bound variables in one subderivation are distinct from all of those in the other and no free variable in one subderivation is an eigenparameter in the other.

induction rule (I)

$$\frac{F_1[a], \Gamma \Rightarrow \Delta, F_2[a/s(a)] \ [G']}{F_1[a/0], \Gamma \Rightarrow \Delta, \forall y F_2[a/y] \ [G]} (I)$$

$$G = G' \cup \{[x_{i_1}, x_{i_2}]\} \cup \{dom(y) = N\} \cup \{a \rightarrow y\}$$

F_1 and F_2 are equiform with x_{i_1} and x_{i_2} being their corresponding bound variables. The new bound variable y is called an *induction variable*. For clarity in the examples to follow, Greek letters will be used to denote induction variables.

The conditions on the variables in applying the different rules serve to keep the reference grammar conditions on variables properly distinct. If one were to reuse the names of a (bound) variable, this variable (in its second occurrence) would be lumbered with the restrictions their first occurrence had given rise to. Conversely, the first occurrence of the variable would also be subject to the restrictions placed on its second occurrence. This would endanger the idea behind the reference grammar, as unnecessary interactions between reference grammar conditions could arise. It also runs contrary to the notion that a variable is only restricted by derivation rules it occurs in. The same consideration holds for free variables and eigenvariables, as well as variables in distinct branches of the derivation that join at a derivation rule.

Concerning notation, an expression of the form x_{i_j} refers to the $i-th$ variable in the formula with subscript j. For instance, x_{1_2} as used with regards to the definition of the Cut rule would refer to the first variable in the formula A_2. When no confusion can result, we will often replace j with l and r, standing for respectively the left-hand side and right-hand side formula.

At the risk of belabouring the obvious, it bears pointing out that the condition $dom(y) = N$ in the reference grammar after application of the induction rule is crucial. It states that the domain of the induction variable is limited to N, and this is what creates the possibility of contradiction in the reference grammars. No other derivation rule explicitly limits the domain of a variable, at most they state that the domain of one variable must be a subset (but not necessarily a proper subset) of the domain of another variable. But these kinds of conditions would always be met by simply assigning the maximally large domain to all variables. This would even be so in the case where functional expressions enter the reference grammar (see e.g. Section 3.4), as these would automatically be contained in the maximally large domain that every variable would be assigned. Only the explicit limitation of the domain of the induction variable to N by the induction rule causes the possibility of inconsistency in reference grammars, and this is crucial in distinguishing BV from traditional logic. If there were no possibility of inconsistency in the reference grammars, then BV would be proof-theoretically identical to traditional logic and hence constitute a distinction without a significant difference.

Axioms are sequents of the form $A \Rightarrow A$ $[\emptyset]$ (A quantifier-free). A BV-derivation is then defined as a traditional Gentzen-derivation (starting either with an axiom or $\bot\Rightarrow\ \emptyset$ $[\emptyset]$), with the added complexity of the reference grammar and the conditions this places on the use of the derivation rules. We will write $RG(D)$ for the reference grammar of a derivation D.

From a proof-theoretical point of view, a BV-derivation is significantly different from a traditional Gentzen-derivation. Traditionally, if a formal derivation can be given, the conclusion is considered to be valid (possibly conditionally, depending on whether or not there are premisses). However, this obfuscates matters and fails to take into account the relations between the domains of the variables in the derivation that are required by the derivation, by fixing the domain of the variables in advance of the derivation (usually one domain fits all, many-sorted logic allows for distinct domains for distinct variables but still assumes them to be fixed in advance of the actual derivation). This means information is lost in the derivation concerning relations between domains of variables in the derivation. BV-derivations elicit this information by not assuming domains to be fixed but keeping track, in the reference grammar (above: G), of the relations between domains of variables required by the derivation rules (notably: quantification, contraction, cut and induction, with induction crucially limiting the domain of the induction variable to N). This extra information is not redundant, it constitutes an extra set of requirements that needs to be satisfied before one can ascribe validity to a BV-derivation. Not only must the derivation be formally correct, as is the case traditionally, it must also be possible to satisfy the conditions on domains set forth in the reference grammar. If these conditions are unsatisfiable, no model for the derivation can exist as any assignation of domains to variables in the derivation would result in contradiction, and hence such a derivation would be void and invalid.

2.5 Examples of BV-derivations

It is important to take the restrictions mentioned for the applications of the inference rules carefully into account, in particular those for the quantifier rules $(\forall R)$ and $(\exists L)$, where the eigenparameter a may not occur in the sequent which is the conclusion of the inference. These conditions on the eigenparameter prevent a derivation of $\forall u \exists v[P(u,v)] \Rightarrow \exists y \forall x[P(x,y)]$ (every person has a mother does not imply that there is one mother of all persons). Let us try to construct a derivation of $\forall u \exists v[P(u,v)] \Rightarrow \exists y \forall x[P(x,y)]$, an attempt doomed to failure:

$$\begin{array}{lll} & P(a,b) \Rightarrow P(a,b) & \\ (\forall R) & P(a,b) \Rightarrow \forall x[P(x,b)] & a \to x \end{array}$$

However, the eigenparameter a may not occur in the conclusion of the inference rule $(\forall R)$!

Similarly, if we try to apply rule $(\exists L)$:

$$\begin{array}{lll} & P(a,b) \Rightarrow P(a,b) & \\ (\exists L) & \exists v[P(a,v)] \Rightarrow P(a,b) & b \to v \end{array}$$

Again, the eigenparameter b may not occur in the conclusion of the inference rule $(\exists L)$.

On the other hand, a BV-derivation of the converse sequent, $\exists y \forall x[P(x,y)] \Rightarrow \forall u \exists v[P(u,v)]$ is possible:

$$\begin{array}{lll} & P(a,b) \Rightarrow P(a,b) & \\ (\exists R) & P(a,b) \Rightarrow \exists v[P(a,v)] & v \to b \\ (\forall L) & \forall x[P(x,b)] \Rightarrow \exists v[P(a,v)] & x \to a \\ (\forall R) & \forall x[P(x,b)] \Rightarrow \forall u \exists v[P(u,v)] & a \to u \\ (\exists L) & \exists y \forall x[P(x,y)] \Rightarrow \forall u \exists v[P(u,v)] & b \to y \end{array}$$

In the derivation above, at the right side, we have only mentioned the new references induced by the rules indicated at the left.

Notice that now the eigenparameter a does not occur in the conclusion $\forall x[P(x,b)] \Rightarrow \forall u \exists v[P(u,v)]$ of inference rule $(\forall R)$ and that the eigenparameter b does not occur in the conclusion $\exists y \forall x[P(x,y)] \Rightarrow \forall u \exists v[P(u,v)]$ of inference rule $(\exists L)$.

2.6 Reference Grammar and Models

Given a reference grammar $RG(D)$ of a derivation D, it is possible to determine the models of a derivation D of $\Gamma \Rightarrow \Delta$. Start with assigning to every variable the maximally large domain (in our case, EN). Now consider $RG(D)$ and apply all restrictions to the domains of the variables that occur in it. A possible way to do this is to follow the derivation and apply restrictions as they occur chronologically, taking care to backtrack (every time a restriction is added it is possible that the restriction on the domain of a variable also impacts the domains of other variables due to dependencies previously established in the derivation, and every time a dependency is added it is possible that a previous restriction comes to apply to a new variable). This way, at the end of the derivation one will either have found inconsistency (in which case no assignment of domains will constitute a model for the derivation) or have an assignment of domains that constitutes a model for the derivation. This model will also have the property of being maximally large (i.e., expanding any domain will cause it to no longer be a model of the derivation, as it would violate the conditions of $RG(D)$). Smaller models can be created by restricting domains of variables, but one should always make sure that the conditions given in $RG(D)$ are not violated (more backtracking for every restriction one wishes to apply).

Alternatively, one could take the complete $RG(D)$ and apply the restrictions directly from it. This yields the same result (pace prof. Isles). Examples will be given in Chapter 3.

Chapter 3

Distinction without a Difference?

One could argue that all of the above is well and good, but that it remains to be seen whether there actually exist such derivations as to yield validity traditionally and invalidity under BV standards. If fixing domains in advance yields no formal inconsistencies, why should we consider this extra information of the reference grammar to be relevant. Are we not then in the land of distinction without a difference? To this end we will proceed to give an example of a derivation where this is in fact the case (it is valid traditionally but invalid under BV standards). In previous articles by prof. D. Isles, [13] [15], a similar, though less formally expanded, point has been brought forward. However, it seems that formal explicitation has its benefits as some of the points made by prof. Isles in [15] do not hold up under closer scrutiny. However, a derivation can be given for which they do hold, albeit for slightly different reasons and concerning a theorem of significantly higher complexity. First we will consider prof. Isles' points with regard to the totality of addition, multiplication and exponentiation, made formally exact in the BV-system given above. These points will ultimately be rejected, but in doing so we will clarify the status of the conditions set forth in the reference grammar, which in turn will yield a truly problematic derivation, a distinction with a distinct difference. First we must give axioms for equality, addition (A), multiplication (M) and exponentiation (P(ower)).

Definition 3.1 (ent - elementary number theory) The system **ent** (elementary number theory) consists of the following axioms:

E_1: $\forall x(x = x)$
E_2: $\forall x, y(x = y \rightarrow y = x)$
E_3: $\forall x, y, z((x = y \wedge y = z) \rightarrow x = z)$
E_4: $\forall x, y(x = y \rightarrow s(x) = s(y))$
E_5: $\forall x \neg(s(x) = 0)$
A_1: $\forall x(A(x, 0, x))$
A_2: $\forall x, y, z(A(x, y, z) \rightarrow A(x, s(y), s(z)))$
M_1: $\forall x(M(x, 0, 0))$
M_2: $\forall x, y, z, w(M(x, y, z) \wedge A(z, x, w) \rightarrow M(x, s(y), w))$
P_1: $\forall x(P(x, 0, s(0)))$
P_2: $\forall x, y, z, w(P(x, y, z) \wedge M(z, x, w) \rightarrow P(x, s(y), w))$
E_6: Let Q be either A, M or P: $\forall x, y, z, w(x = y \rightarrow ((Q(x, z, w) \rightarrow Q(y, z, w)) \wedge (Q(z, x, w) \rightarrow Q(z, y, w)) \wedge (Q(z, w, x) \rightarrow Q(z, w, y))))$
E_7: $\forall x, y(s(x) = s(y) \rightarrow x = y)$
E_8: Let Q be either A, M or P: $\forall x, y, z, w(Q(z, w, x) \wedge Q(z, w, y) \rightarrow x = y)$

Notice that all these axioms hold in any term model whose domain is EN and where the equality relation is either reductive identity $=$ or syntactic identity $=_s$ (see definition 2.6).

Now we shall consider the following BV-derivations for totality of A, M and P, and their reference grammars. For practical reasons, most notably length of formulae, the reference grammar has been omitted from the derivations themselves and is presented afterwards. For ease of presentation and easier reading, reference arrows are grouped together rather than given individually.

3.1 Addition

Let D_1 be the following BV-derivation of $\forall x_2 \forall \alpha \exists y[A(x_2, \alpha, y)]$. In this derivation A_1 and A_2 abbreviate respectively the first and second definitional formula for addition.

$$\dfrac{\dfrac{\dfrac{A(a,c,b) \Rightarrow A(a,c,b) \qquad A(a,s(c),s(b)) \Rightarrow A(a,s(c),s(b))}{A(a,c,b), A(a,c,b) \rightarrow A(a,s(c),s(b)) \Rightarrow A(a,s(c),s(b))} \rightarrow L}{A(a,c,b), A(a,c,b) \rightarrow A(a,s(c),s(b)) \Rightarrow \exists y[A(a,s(c),y)]} \exists R}{A(a,c,b), \forall x_1, y_1, z_1[A(x_1,y_1,z_1) \rightarrow A(x_1,s(y_1),s(z_1))] \Rightarrow \exists y[A(a,s(c),y)]} \forall L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists y_2[A(a,c,y_2)], A_2 \Rightarrow \exists y[A(a,s(c),y)]}\ \exists L}{\exists y[A(a,0,y)], A_2 \Rightarrow \forall\alpha\exists y[A(a,\alpha,y)]}\ I}{\forall x_4\exists y[A(x_4,0,y)], A_2 \Rightarrow \forall\alpha\exists y[A(a,\alpha,y)]}\ \forall L}{\forall x_4\exists y[A(x_4,0,y)], A_2 \Rightarrow \forall x_2\forall\alpha\exists y[A(x_2,\alpha,y)]}\ \forall R \qquad \aleph$$

$$\dfrac{\dfrac{\dfrac{A(a_1,0,a_1) \Rightarrow A(a_1,0,a_1)}{\forall x[A(x,0,x)] \Rightarrow A(a_1,0,a_1)}\ \forall L}{A_1 \Rightarrow \exists y_3[A(a_1,0,y_3)]}\ \exists R}{A_1 \Rightarrow \forall x_3\exists y_3[A(x_3,0,y_3)]}\ \forall R \qquad \beth$$

$$\dfrac{\beth \quad \aleph}{A_1, A_2 \Rightarrow \forall x_2\forall\alpha\exists y[A(x_2,\alpha,y)]}\ C$$

For the sake of clarity, we will give a detailed rendering of the reference arrows, if any, generated by each step in the derivation. This in order that it may be clear how the reference grammar is constructed. Note that $\rightarrow$ is transitively closed and the reference grammar should be read with this in mind. Note that $N \rightarrow \alpha$ is just another way of saying $dom(\alpha) = N$, as the condition $\alpha \rightarrow N$ is met ex hypothesi.

$\forall L : x_1 \rightarrow a, y_1 \rightarrow c, z_1 \rightarrow b$
$\exists R : y \rightarrow s(b)$
$\exists L : b \rightarrow y_2$
$I : Dom(\alpha) = N, [y, y_2], c \rightarrow \alpha$
$\forall L : x_4 \rightarrow a$
$\forall R : a \rightarrow x_2$
$\forall L : x \rightarrow a_1$
$\exists R : y_3 \rightarrow a_1$
$\forall R : a_1 \rightarrow x_3$
$C : [x_3, x_4], [y_3, y]$

This in itself constitutes the reference grammar for the derivation, though it is a bit of a muddle and therefore it seems advisable to add some structure to the representation of the reference grammar. In the examples below, the representation of the reference grammar gives strings of reference arrows of maximal length. Note that there may be points of intersection between these strings, so it may not suffice to scan every string independently. Putting all reference arrows together we get the following *reference graph* for derivation D_1 which indicates what relations should hold between the ranges of the variables in the derivation if the inference rules are to preserve truth.

The reference grammar $RG(D_1)$ of derivation $D_!$ looks as follows:

$$\begin{array}{lllllllll}
 & & & & & & & & x_1 \\
 & & & & & & & & \downarrow \\
 & & x & \rightarrow & a_1 & \rightarrow & [x_3, x_4] & \rightarrow & a \rightarrow x_2, \\
 & & & & \uparrow & & & & \\
z_1 \rightarrow b \rightarrow [y, y_2, y_3] \rightarrow s(b), & & & & & & & & \\
y_1 \rightarrow c \rightarrow N \rightarrow \alpha & & & & & & & &
\end{array}$$

To check consistency of the reference grammar, one initially assigns the largest domain (in our case EN) to all variables. Then one checks all the restrictions on domains that occur in the reference grammar by backtracking to see to which variables they apply. Next one checks for each variable to which restrictions apply whether inconsistency occurs (for an example of inconsistency, see Section 3.3). If no inconsistency occurs, all assignments of domains to variables that are allowed by the restrictions are acceptable as a model of the derivation (in essence one constructs the most indulgent assignment possible and all smaller ones are also acceptable).

Definition 3.2 (soul of a derivation) A derivation becomes a *proof* if it is possible to assign ranges in EN to its bound variables in a manner consistent with its reference graph G. Yessenin Volpin [27] calls this assignment a *soul* to the derivation.

In the final sequent $\Gamma \Rightarrow F$ of a proof, F is called a theorem from assumptions Γ.

For example, in the reference grammar of derivation D_1 all variables are assigned EN in the term model EN except for α which is assigned N. $\forall x_2 \forall \alpha \exists y[A(x_2, \alpha, y)]$ is an **ent**-theorem based on the **ent**-proof D_1 where x_2 and y inherit EN as range from D_1 and α inherits N.

3.2 Multiplication

Let S stand for $A(d_1, c_1, e_1), M(c_1, b_1, d_1)$. Let D_2 be the following BV-derivation of $\forall z_2 \forall \beta \exists w_1[M(z_2, \beta, w_1)]$. In this derivation M_1 and M_2 abbreviate respectively the first and second definitional formula for multiplication, and *add* abbreviates the formula expressing totality of addition.

$$\dfrac{\dfrac{S, (M(c_1, b_1, d_1) \wedge A(d_1, c_1, e_1) \rightarrow M(c_1, s(b_1), e_1)) \Rightarrow M(c, s(b_1), e_1)}{S, \forall \bar{x}, \bar{y}, \bar{z}, \bar{w}[M(\bar{x}, \bar{y}, \bar{z}) \wedge A(\bar{z}, \bar{x}, \bar{w}) \rightarrow M(\bar{x}, s(\bar{y}), \bar{w})] \Rightarrow M(c_1, s(b_1), e_1)}\ \forall L}{S, \forall \bar{x}, \bar{y}, \bar{z}, \bar{w}[M(\bar{x}, \bar{y}, \bar{z}) \wedge A(\bar{z}, \bar{x}, \bar{w}) \rightarrow M(\bar{x}, s(\bar{y}), \bar{w})] \Rightarrow \exists w_1[M(c_1, s(b_1), w_1)]}\ \exists R$$

$$\dfrac{}{\exists y[A(d_1, c_1, y)], M(c_1, b_1, d_1), M_2 \Rightarrow \exists w_1[M(c_1, s(b_1), w_1)]}\ \exists L$$

$$\dfrac{}{\forall x_2 \forall \alpha \exists y[A(x_2, \alpha, y)], M(c_1, b_1, d_1), M_2 \Rightarrow \exists w_1[M(c_1, s(b_1), w_1)]}\ \forall L$$

$$\dfrac{}{add, \exists w_2[M(c_1, b_1, w_2)], M_2 \Rightarrow \exists w_1[M(c_1, s(b_1), w_1)]}\ \exists L$$

$$\dfrac{}{add, \exists w_1[M(c_1, 0, w_1)], M_2 \Rightarrow \forall \beta \exists w_1[M(c_1, \beta, w_1)]}\ I$$

$$\dfrac{}{add, \forall u_1 \exists w_1[M(u_1, 0, w_1)], M_2 \Rightarrow \forall z_2 \forall \beta \exists w_1[M(z_2, \beta, w_1)]}\ \forall L \forall R$$

א

$$\dfrac{\dfrac{M(c, 0, 0) \Rightarrow M(c, 0, 0)}{\forall \hat{x}[M(\hat{x}, 0, 0)] \Rightarrow \exists w[M(c, 0, w)]}\ \exists R \forall L}{M_1 \Rightarrow \forall u \exists w[M(u, 0, w)]}\ \forall R$$

ב

$$\dfrac{ב \qquad א}{add, M_1, M_2 \Rightarrow \forall z_2 \forall \beta \exists w_1[M(z_2, \beta, w_1)]}\ C$$

First we list the reference arrows going from top to bottom in derivation D_2:
$\forall L$: $\bar{x} \rightarrow c_1$, $\bar{y} \rightarrow b_1$, $\bar{z} \rightarrow d_1$, $\bar{w} \rightarrow e_1$
$\exists R$: $w_1 \rightarrow e_1$
$\exists L$: $e_1 \rightarrow y$
$\forall L$: $x_2 \rightarrow d_1$, $\alpha \rightarrow c_1$
$\exists L$: $d_1 \rightarrow w_2$
I: $b_1 \rightarrow \beta$, $dom(\beta) = N$, $[w_1, w_2]$
$\forall L$: $u_1 \rightarrow c_1$
$\forall R$: $c_1 \rightarrow z_2$

$\exists R$: $\hat{x} \to c$
$\forall L$: $w \to 0$
$\forall R$: $c \to u$
C $[u, u_1]$, $[w, w_1]$

Derivation D_2 uses some axioms of **ent**, Induction and the **ent**-theorem *add*, i.e., $\forall x_2 \forall \alpha \exists y[A(x_2, \alpha, y)]$. This assumption *add* is based on and inherits its variable ranges from derivation D_1. To emphasize this ancestry, the notation $\langle D_1, D_2\rangle$ is used. So $\langle D_1, D_2\rangle$ denotes a derivation D_2 which has an occurrence of an assumption formula F, in our case *add*, which is a theorem via a derivation D_1 and where the ranges of the variables in F are inherited from D_1.

The reference graph of derivation $\langle D_1, D_2\rangle$ is $RG(D_1) \cup RG(D_2)$, where $RG(D_2)$ consists of the followinge references:

$$\begin{array}{ccccccccc} & & & & \alpha & & \\ & & & & \downarrow & & \\ \hat{x} & \to & c & \to [u,u_1] \to & c_1 & \to & z_2 \\ & & & & \uparrow & & \\ & & & & \bar{x} & & \end{array} \qquad \begin{array}{ccccccccc} & & y & & 0 & & x_2 & & \\ & & \uparrow & & \uparrow & & \downarrow & & \\ \bar{w} & \to & e_1 & \leftarrow & [w,w_1,w_2] & \leftarrow & d_1 & \leftarrow & \bar{z} \end{array}$$

$\bar{y} \to b_1 \to N \to \beta$

Since in derivation D_1 all variables can be assigned the same range EN, except the induction variable α which should be assigned range N, it is easy to give a soul to $\langle D_1, D_2\rangle$: c_1 and z_2 inherit the range N from α and to β should be assigned N. All other variables can be assigned the same range EN.

3.3 Exponentiation

Let S stand for $M(b_2, a_2, c_2), P(a_2, d_2, b_2)$. Let D_3 be the following BV-derivation of $\forall w_4, \gamma \exists v_1[P(w_4, \gamma, v_1)]$. In this derivation P_1 and P_2 abbreviate respectively the first and second definitional formula for exponentiation, and *mul* abbreviates the formula expressing totality of multiplication.

$$\dfrac{\dfrac{S, (P(a_2, d_2, b_2) \wedge M(b_2, a_2, c_2)) \to P(a_2, s(d_2), c_2) \Rightarrow P(a_2, s(d_2), c_2)}{S, \forall \bar{x}, \bar{y}, \bar{z}, \bar{w}, [P(\bar{x}, \bar{y}, \bar{z}) \wedge M(\bar{z}, \bar{x}, \bar{w}) \to P(\bar{x}, s(\bar{y}), \bar{w})] \Rightarrow P(a_2, s(d_2), c_2)]}\ \forall L}{S, \forall \bar{x}, \bar{y}, \bar{z}, \bar{w}, [P(\bar{x}, \bar{y}, \bar{z}) \wedge M(\bar{z}, \bar{x}, \bar{w}) \to P(\bar{x}, s(\bar{y}), \bar{w})] \Rightarrow \exists v_1[P(a_2, s(d_2), v_1)]}\ \exists R$$

$$\overline{\exists w_1[M(b_2,a_2,w_1)], P(a_2,d_2,b_2), P_2 \Rightarrow \exists v_1[P(a_2,s(d_2),v_1)]}\ \exists L$$
$$\overline{\forall z_2,\beta\exists w_1[M(z_2,\beta,w_1)], P(a_2,d_2,b_2), P_2 \Rightarrow \exists v_1[P(a_2,s(d_2),v_1)]}\ \forall L$$
$$\overline{mul, \exists v_2[P(a_2,d_2,v_2)], P_2 \Rightarrow \exists v_1[P(a_2,s(d_2),v_1)]}\ \exists L$$
$$\overline{mul, \exists v_1[P(a_2,0,v_1)], P_2 \Rightarrow \forall\gamma\exists v_1[P(a_2,\gamma,v_1)]}\ I$$
$$\overline{mul, \forall u_2\exists v_1[P(u_2,0,v_1)], P_2 \Rightarrow \forall w_4,\gamma\exists v_1[P(w_4,\gamma,v_1)]}\ \forall L\forall R$$

א

$$\frac{P(a_3,0,s(0)) \Rightarrow P(a_3,0,s(0))}{\forall\hat{u}[P(\hat{u},0,s(0))] \Rightarrow P(a_3,0,s(0))}\ \forall L$$
$$\overline{\forall\hat{u}[P(\hat{u},0,s(0))] \Rightarrow \exists v[P(a_3,0,v)]}\ \exists R$$
$$\overline{\forall\hat{u}[P(\hat{u},0,s(0))] \Rightarrow \forall u_3\exists v[P(u_3,0,v)]}\ \forall R$$

ב

$$\frac{\text{ב} \qquad \text{א}}{mul, P_1, P_2 \Rightarrow \forall w_4,\gamma\exists v_1[P(w_4,\gamma,v_1)]}\ C$$

Going through derivation D_3 from top till bottom one encounters the following references:
$\forall L$: $\bar{x} \to a_2$, $\bar{y} \to d_2$, $\bar{z} \to b_2$, $\bar{w} \to c_2$
$\exists R$: $v_1 \to c_2$
$\exists L$: $c_2 \to w_1$
$\forall L$: $z_2 \to b_2$, $\beta \to a_2$
$\exists L$: $b_2 \to v_2$
I: $N \to \gamma$, $[v_1, v_2]$, $d_2 \to \gamma$
$\forall L$: $u_2 \to a_2$
$\forall R$: $a_2 \to w_4$
$\forall L$: $\hat{u} \to a_3$
$\exists R$: $v \to s(0)$
$\forall R$: $a_3 \to u_3$
C: $[u_2, u_3]$, $[v, v_1]$

So, the reference grammar of $\langle\langle D_1, D_2\rangle, D_3\rangle$ is the reference grammar of $\langle D_1, D_2\rangle$ united with the reference grammar $RG(D_3)$ of D_3 which consists of the following references:

$$\begin{array}{ccccccc} & & & \beta & & \\ & & & \downarrow & & \\ \hat{u} \rightarrow a_3 \rightarrow & [u_2, u_3] & \rightarrow & a_2 & \leftarrow \bar{x} \\ & & & \downarrow & & \\ & & & w_4 & & \end{array} \qquad \begin{array}{ccccccc} & \bar{z} & & & & \bar{w} & \\ & \downarrow & & & & \downarrow & \\ z_2 \rightarrow & b_2 & \rightarrow & [v, v_1, v_2] & \rightarrow & c_2 & \rightarrow w_1 \\ & & & \downarrow & & & \\ & & & s(0) & & & \end{array}$$

$$\bar{y} \rightarrow d_2 \rightarrow N \rightarrow \gamma$$

There is no problem with the totality of addition and multiplication, but prof. Isles offers two arguments which purport to show that the derivation of the totality of exponentiation is no longer acceptable in a BV context. In [12] he claims that the argument is circular, as the reference grammar (from now on: RG) of the derivation $\langle\langle D_1, D_2\rangle, D_3\rangle$ contains the condition $N \rightarrow v_1$ (see next paragraph for explicitation), i.e. the domain of v_1 is N, with v_1 the variable that occupies the third argument-place of the P-relation in the formula stating the totality of exponentiation. In other words, the very thing that the derivation sets out to prove is contained in the RG as a condition on the domains of the variables. This, so prof. Isles claims in his article, is a circular argument and hence no proof at all.

$RG(D_1)$ contains $N \rightarrow \alpha$, $RG(D_2)$ contains $\alpha \rightarrow c_1$, hence $N \rightarrow c_1$.
$RG(D_2)$ contains $c_1 \rightarrow z_2$, hence $N \rightarrow z_2$.
$RG(D_3)$ contains $z_2 \rightarrow v_1$ (using transitive closure of $\rightarrow$), hence $N \rightarrow v_1$.
Notice that $RG(D_2)$ contains $w_1 \rightarrow c_1 \rightarrow y$ and $RG(D_3)$ contains $z_2 \rightarrow w_1$. Hence, the reference grammar of $\langle\langle D_1, D_2\rangle, D_3\rangle$ contains $N \rightarrow z_2 \rightarrow w_1 \rightarrow c_1 \rightarrow y$.

I think there is no reason to discard the derivation as a proof on the basis of this argument. The point is that to draw the conclusion that the derivation is circular, one must consider the conditions in the reference grammar to have the same status as logical assumptions in the derivation. Only then can the argument be considered circular. However, there is no good reason to do so. The conditions set forth in the RG are to be considered not as extra assumptions of the logical derivation, as prof. Isles does when he considers the proof to be circular, but as what they in fact just are by their very nature: a set of conditions on the domains of the variables in the derivation that must be satisfiable in order to yield a valid derivation. The satisfiability of the RG is however not in question in prof. Isles' argument. The fact that $N \rightarrow v_1 \in RG$ does not yield contradiction within the RG, nor is it contradictory to the result proved. In fact, the totality of addition and multiplication, as proved above, implies that the values of respectively y and w_1 are also numerals, meaning in terms of RG that $N \to y$ and $N \to w_1$, but these arrows do not occur in the RG's of their respective derivations and hence y in $\forall x_2 \forall \alpha \exists y[A(x_2, \alpha, y)]$ and w_1 in $\forall z_2 \forall \beta \exists w_1[M(z_2, \beta, w_1)]$ are assigned EN as domain. If, as prof. Isles seems to claim in his argument, RG-conditions are liable to create inconsistency or circularity with the logical assumptions or conclusions of the derivation, then the totality of addition and multiplication would also not be proved by these derivations as they would yield inconsistency. But prof. Isles makes no such claim in these cases. This is a first reductio ad absurdum of the argument. A second reductio ad absurdum of this argument is that, if RG-conditions were to be of the same category as logical assumptions, then $N \to v_1 \in RG$ would reduce the proof of totality to a one-step argument, which it ostensibly isn't.

Up to this point we have been very informal about the way variables inherit RG-conditions from previous derivations (we have simply added reference grammars to each other by taking their union), but the following argument hinges exactly on this point so it is now time to be more specific. Assume derivations D_1 and D_2, where the latter uses the former as a lemma (i.e. the succedent of the concluding sequent of D_1 occurs in the antecedent of the concluding sequent of D_2). Without loss of generality, assume there is only one variable in common between the two derivations, say x.

Prof. Isles now holds (see [15]) that in RG_2, x should be given exactly the conditions that pertain to it in RG_1, x 'inherits' these conditions from RG_1, and any conditions placed on x in RG_2 are ignored.

I also hold that the conditions pertaining to x in RG_1 should be added to RG_2, but I do not hold that any further conditions placed on x in RG_2 should be ignored. Rather I argue that the conditions x has thus 'inherited' from RG_1 should be subject to further strengthening of the conditions if RG_2 gives occasion to do so.

In essence, it boils down to this. Both prof. Isles and I agree that the RG of a lemma must be joined to the RG of the main derivation (with respect to shared variables). The difference is that prof. Isles argues that all conditions in the new RG must then stand as they are, whereas I argue that the conditions in the new RG must be brought in accordance with each other by applying whatever restrictions on the conditions that have been added. The following discussion of prof. Isles' second argument for rejecting the validity of the BV-derivation of the totality of exponentiation should serve as an example of the notion of inheritance as well as show that the difference between prof. Isles' view and mine is not merely cosmetic. This point will be expanded upon in the philosophical chapter of the thesis.

Prof. Isles' second argument can be found in [14] and [15], where he claims that RG(D_3) is inconsistent: z_2 inherits N from D_2, w_1 inherits EN from D_2 and $z_2 \to w_1 \in RG(D_3)$, from which follows that $N \to EN \in RG(D_3)$, which would make $RG(D_3)$ inconsistent as $N \neq EN$ and both $N \subset EN$ and $EN \subset N$ should hold. The crucial point in this argument is the notion of inheritance. Specifically, prof. Isles defines it in such a manner that if a variable inherits a domain due to a lemma, then that variable must be assigned this domain in all derivations that use this lemma regardless of any other conditions that may have been imposed on this variable in the main derivation. This may sound reasonable at first glance, but it means that any and all extra conditions placed on the variable in question in the main derivation are ignored. In the case of totality of exponentiation this means that $z_2 \to ... \to w_1$ is ignored, but since $N \to z_2$ (from D_2), this means that $N \to w_1$ is ignored. But $RG(D_3)$ is no longer to be considered inconsistent if this extra information (leading to $N \to w_1$) is considered relevant, since then we can no longer conclude that $N \to EN \in RG(D_3)$. And there seems to be no reason at all to disregard this information. Rather the contrary, there are excellent reasons for not disregarding it. I offer the following considerations in support:

1. It conforms to the basic idea behind BV-derivations, i.e. to elicit information on the relations between the domains of variables in a derivation. Prof. Isles' notion explicitly suppresses said information.

2. The only reason to assign EN to w_1 in $RG(D_2)$ is the tacit assumption that no restrictions should be applied unless required. In D_2 there are no restrictions on w_1 that constrain its domain, but in D_3 there are and so these should be applied.

3. Prof. Isles' definition of inheritance yields a distinction between a proof given from the axioms and the same proof given by use of lemmata. In this particular case, the former would give a valid BV-proof of the totality of exponentiation while the latter wouldn't. This is so contrary to intuition, mathematical practice and the very notion of a lemma, that I regard it as a counter-argument of prof. Isles' position. After

all, use of lemmata comes only from a need for brevity and efficiency, a lemma (introduced in a derivation by means of the Cut rule) is intended to be shorthand for its entire proof. Any definition that yields a difference between writing out all the steps of a proof or using lemmata as shorthand seems counterintuitive.

4. Most damning is the fact that prof. Isles' definition of inheritance leads to a notion of deduction in which conservation of truth is no longer guaranteed in every step of the deduction. This, in essence, renders the notions of deduction and proof moot, as well as formal logic itself. The conditions on the variables of a derivation follow directly from the definitions of the derivation rules. The definitions of the derivation rules are, in the spirit of Ockham and common sense, conservative, i.e., no conditions or restrictions are applied save those necessary for preservation of truth in every step/application of a deduction rule. If a condition is not met, preservation of truth cannot be guaranteed and hence there is no proof. Prof. Isles' notion of inheritance fails to enforce the conditions placed on variables introduced in lemmata after introduction of the lemmata by means of the Cut rule. Therefore this notion fails to meet conditions on variables (necessary for the guarantee of truth preservation), thereby removing the guarantee of truth preservation in every step. Such a derivation is no proof, as it fails to guarantee the truth of the conclusion given truth of the premisses. Be you as liberal-minded as you like, the buck has to stop somewhere, and quite clearly that needs to happen (well) before this point. Conditions set forth in derivation rules may not be ignored, on pain of losing the guarantee of truth preservation, so let's not go there, especially since there is no need.

In reply to a previous version of this thesis, in defence against the critique above, prof. Isles has shared an insight of his that he has recently written down in [16]. Prof. Isles states that the BV calculus provides a definition of derivations, but a derivation only becomes a proof when it can be given a 'soul' (terminology due to Y. Volpin), i.e., an assignment of domains to the variables (said assignment being consistent with the RG). But this assignment is not given by taking the RG of the whole derivation, but rather by moving down the derivation, building up the reference grammar as you go and checking at each deduction step that it remains valid. This means that one must be careful at Induction, Contraction and Cut, as these require for validity that the domains of certain bound variables in the premisses are identical (bound variables in F_1 and F_2 for Induction, A_1 and A_2 for Contraction and Cut). Looking at the derivation of totality of exponentiation, when one reaches the Induction rule there is a problem. At that time, v_1 has domain EN (it is as yet unrestricted), whereas v_2 has range N ($z_2 \to b_2 \to v_2$ and z_2 inherits domain N from proof D_2). Therefore the validity of the Induction rule is not guaranteed in this case.

There are two (related) reasons I disagree with it. Firstly, this seems to me to rest on an interpretation of the notion of a deduction step (and reference grammar) that is overly 'chronological'. There is only a problem when one sees the deduction step as first getting premisses and only then (later in time) leading to a conclusion, with a corresponding change in reference grammar. I would hold that there is no sooner and later in this matter, the deduction step is an inseparable whole (though different parts of it may be distinguished). Given that the reference grammar for all of the potentially offending deduction steps (Induction, Contraction, Cut) takes the necessity of identical domains for the bound variables in the premisses into account (easily seen by checking the definitions of the deduction steps), I see no reason to doubt the validity of a deduction step on these grounds. (With regard to the example given concerning totality of exponentiation, it should be noted that the use of the Induction rule leads to identification of the domains of the bound variables v_1 and v_2.)

Secondly (assuming for a moment that there is a 'sooner' and 'later' with respect to deduction steps), the assignment of domain EN to v_1 is entirely a matter of convention. It is because we have arbitrarily decided to start by giving all the variables the maximally large domain (see Section 2.6) that one can say that v_1 has the domain EN at this point. We could as well have agreed to assign no domains at all at the start, assign only such domains as are required by the reference grammar during the derivation, and then assign the maximally large domain (or any other domain that does not violate the reference grammar) to any unrestricted variables left at the conclusion of the derivation. Given this procedure, v_1 would not yet have been assigned a domain at the relevant point in the derivation (although obviously the application of Induction would then necessitate the assignment of N to v_1).

From my point of view, the deduction step is sound. The reference grammar takes the need to identify the domains of the variables in the premisses into account, and the fact that we write this down next to the conclusion instead of the premisses is merely a matter of convenience and convention.

So, I would still claim that $\langle\langle D_1, D_2\rangle, D_3\rangle$ is a valid proof of the totality of exponentiation in BV, pace prof. Isles. However, as shown in [15], the proof of Euclid's lemma as given in [17] will offer an example of a proof that is traditionally valid but not so under BV.

3.4 Euclid's Lemma

Lemma 157 of Kleene [17] expresses the existence of a common multiple of the natural numbers $1, 2, \ldots, n$. It is a crucial step to prove Kleene's Theorem 162 [17] which is Euclid's theorem stating that there are infinitely many prime numbers.

The function of Lemma 157 is to guarantee that every multiple of the least common multiple of an initial segment of the natural numbers, say up till n, may be represented in Peano Arithmetic, and hence, in particular $n!$. Lemma 157 is also needed for the construction of Gődel's beta-function (Kleene [17], pp. 239-241), which is used to permit quantification over finite sequences of natural numbers in Peano Arithmetic. In particular, the beta-function is used in showing that the class of arithmetically definable functions is closed under primitive recursion, and therefore includes all primitive recursive functions.

In this section we shall show that the derivation of Kleene's Lemma 157 is not reliable in the sense that the reference arrows in the BV derivation cannot be satisfied. The derivations in question are based on the following axioms **ENT** for elementary number theory; see Kleene [17], p. 82. Notice that all these axioms hold in any term model whose domain is MN and where the equality relation is either reductive identity $=$ or syntactic identity $=_s$ (see Definition 2.6).

$Eq1 : \forall x(x = x)$
$Eq2 : \forall x, y(x = y \to y = x)$
$Eq3 : \forall x, y, z(x = y \wedge y = z \to x = z)$
$Eq4 : \forall x, y(x = y \to s(x) = s(y))$
$Eq5 : \forall x, y, z(x = y \to (x + z = y + z))$
$Eq6 : \forall x, y, z(x = y \to (z + x = z + y))$
$Eq7 : \forall x, y, z(x = y \to (x \cdot z = y \cdot z))$
$Eq8 : \forall x, y, z(x = y \to (z \cdot x = z \cdot y))$
$Eq9 : \forall x, y, z(x = y \to (x = z \to y = z))$
$Eq10 : \forall x, y, z(x = y \to (z = x \to z = y))$

$S1 : \forall x \neg (s(x) = 0)$
$S2 : \forall x, y(s(x) = s(y) \to x = y)$
$Ad1 : \forall x(x + 0 = x)$
$Ad2 : \forall x, y(x + s(y) = s(x + y))$
$Mu1 : \forall x(x \cdot 0 = 0)$
$Mu2 : \forall x, y(x \cdot s(y) = x \cdot y + x)$

Some preliminaries and abbreviative definitions are required:
A: $\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
B: $\forall u \forall v[u \cdot v = v \cdot u]$
D: $\forall x \forall y[y < s(x) \to y \leq x]$
Let $F(a, b)$ be the property defined by:

$$F(a, b)\text{: } 0 < b \wedge \forall x[0 < x \leq a \to \exists y[x \cdot y = b]]$$

First of all notice that assumption B is proved by induction and consequently the reference arrows $N \to u$ and $N \to v$ apply.

$F(a, b)$ states that if b is not zero, then for all the numbers, save zero, running up to and including a, there can be found a corresponding number such that their product is b. This means that every number, save zero, up to and including a must be a divisor of b, but this can only be true if b is a common multiple of a and all its predecessors, or a multiple thereof. So, $F(a, b)$ is true iff a common multiple of a and all its predecessors exists and it divides b without remainder. Kleene's Lemma 157 is the statement that $\forall \delta \exists x[F(\delta, x)]$ can be formally deduced from **ENT**.

Let D_4 be the following BV-derivation of $\forall \delta \exists x(F(\delta, x))$ (which essentially states that for any number a common multiple of that number and all its predecessors exists, see above).

$$\dfrac{\dfrac{\begin{array}{c}[D_0] \\ \vdots \\ A, b \cdot c = d \Rightarrow (b \cdot (c \cdot s(a)) = d \cdot s(a)\end{array}}{A, b \cdot c = d \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\ \exists R}{A, \exists w^\star(b \cdot w^\star = d) \Rightarrow \exists w_1(b = w_1 = d \cdot s(a))}\ \exists L \qquad \alpha$$

$$\begin{array}{c}[D_1] \\ \vdots \\ D, 0 < b, b < s(a), F(a, d) \Rightarrow \exists w(b \cdot w = d)\end{array} \qquad \beta$$

$$\frac{\beta \qquad \alpha}{A, D, 0 < b, b < s(a), F(a,d) \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\,C \qquad \gamma$$

$$\begin{array}{c} [D_2] \\ \vdots \\ B, b = s(a) \Rightarrow \exists w_2(b \cdot w_2 = d \cdot s(a)) \end{array} \qquad \epsilon$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\gamma \qquad \epsilon}{F(a,d), A, B, D, 0 < b, b \leq s(a) \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a)), \exists w_2(b \cdot w_2 = d \cdot s(a))}\,\vee L}{F(a,d), A, B, D, 0 < b, b \leq s(a) \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\,CR}{F(a,d), A, B, D, 0 < d \cdot s(a), 0 < b, b \leq s(a) \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\,WL}{F(a,d), A, B, D, 0 < d \cdot s(a) \wedge 0 < b \wedge b \leq s(a) \Rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\,\wedge L}{F(a,d), A, B, D \Rightarrow 0 < d \cdot s(a) \wedge 0 < b \wedge b \leq s(a) \rightarrow \exists w_1(b \cdot w_1 = d \cdot s(a))}\,\rightarrow R}{F(a,d), A, B, D \Rightarrow \forall z_1(0 < d \cdot s(a) \wedge 0 < z_1 \wedge z_1 \leq s(a) \rightarrow \exists w_1(z_1 \cdot w_1 = d \cdot s(a)))}\,\forall R}{A, B, D, F(a,d) \Rightarrow \exists x_1(F(s(a), x_1))}\,\exists R}{A, B, D, \exists x(F(a,x)) \Rightarrow \exists x_1(F(s(a), x_1))}\,\exists L}{A, B, D, \exists x(F(0,x)) \Rightarrow \forall \delta \exists x(F(\delta, x))}\,I \qquad \eta$$

$$\begin{array}{c} [D_3] \\ \vdots \\ ENTax \Rightarrow \exists x_2(F(0, x_2)) \end{array} \qquad \theta$$

$$\frac{\theta \qquad \eta}{ENTax, A, B, D \Rightarrow \forall \delta \exists x(F(\delta, x))}\,C$$

Technically, all the assumptions A, B, D and the formula F should be given distinct variables, but we are mainly concerned with B here. The proof of B can be found in [17] p.186 and - because it uses induction - gives rise to, amongst others, the reference arrow $N \rightarrow v$. From ϵ we get $v \rightarrow d$ (left as exercise for the reader). In η we have $\exists L$ which gives $d \rightarrow x$, I

which gives $[x, x_1]$ and $\exists R$ which gives $x_1 \to d \cdot s(a)$. Together these yield $N \to v \to d \to x \to x_1 \to d \cdot s(a)$. This requires that $dom(d \cdot s(a)) \subseteq N$, which is impossible as N by definition contains no multiplicative terms. So no assignment of domains of variables is possible for D_4 and hence it is not a BV-proof. Note that this does not mean that Euclid's Lemma is no longer a theorem under BV (an appropriate BV proof might still exist), merely that the traditional proof does not establish it as a theorem in BV.

Two points need to be made now. Firstly, D_4 is valid traditionally but not in the BV structure. This shows that the BV structure is proof-theoretically distinct from traditional structures in that the notion of proof differs, which was the goal of this chapter. Secondly, this does not yet constitute a proof of the fact that there is also a difference between BV and traditional structures at the level of theorems. It is now still possible that the same theorems admit of proof in both structures, albeit that not all traditional proofs can be accepted in the BV structure (and hence alternative proofs would need to be given). In this particular case, Euclid's Lemma could still be a BV-theorem (albeit with a proof different from the traditional one).

3.5 Feasibility and Bounded Arithmetic

For Brouwer it was natural to idealize that a mathematician will live forever. So, he may make a construction project, like constructing the natural numbers, that will last forever and never finish. And the set $\mathbb{N}$ of all natural numbers is identified with this construction project. Brouwer's view implies that the I-ness cannot think of its own mortality: 'After all, the concept of time, as well as space, belong to my ideas, but my I-ness is completely separate from it.' (from Brouwer's confession in 1898, see van Stigt [22] Appendix 1).

However, in the past, several mathematicians, logicians and philosophers, like van Dantzig [5], van Bendegem [1] and Yessenin-Volpin [27], have raised doubts about the concrete existence of large numbers such as $10^{10^{10}}$ formed with exponential terms. Exponentiation seems to give rise to numbers which are intuitively infeasible or non-constructible.

From this point of view it is interesting that David Isles argues in [12, 14, 15] that the totality of exponentiation, $\forall x \forall y \exists z[P(x, y, z)]$, cannot be properly shown in the Buridan-Volpin system, while we have argued that it can; see the discussion in Section 3.3.

As shown by Parikh in [21] there is a gap between exponentiation and the feasible operations of addition and multiplication in models of arithmetic in the following sense: there is a non-standard model M of Peano Arithmetic and there are two distinct functions f_1 and f_2 on M which both satisfy the axioms characterizing exponentiation.

In the same paper Parikh defines Δ_0 to be the set of predicates definable by bounded formulas, i.e., formulas with all quantifiers bounded by a term t. Next he introduces the theory $I\Delta_0$ of *bounded arithmetic* as the first-order theory with non-logical symbols $0, s, +, \ \cdot$ and $\leq$, containing the familiar axioms for these symbols, but with the induction axiom restricted to bounded formulas. More precisely, let $L = \{0, s, +, \ \cdot \ \leq\}$ be the usual first-order language for arithmetic. An L-formula is called Δ_0 or *bounded*, if all its quantifiers occur in the context $\forall x[x \leq t(y) \to \dots]$ or $\exists x[x \leq t(y) \wedge \dots]$, where t is a term of L not containing x. The system $I\Delta_0$ of *bounded arithmetic* contains the following axioms:

$0 \leq 0 \wedge \neg s0 \leq 0$

$\forall x \forall y[sx = sy \to x = y]$

$\forall x \forall y[x \leq sy \leftrightarrow (x \leq y \vee x = sy)]$

$\forall x[x + 0 = x] \wedge \forall x \forall y[x + sy = s(x + y)]$

$\forall x[x \cdot 0 = 0 \wedge x \cdot s0 = x] \wedge \forall x \forall y[x \cdot sy = (x \cdot y) + x]$

$\forall x \forall z[A(x, 0) \wedge \forall y \leq z[A(x, y) \to A(x, sy)] \to \forall y \leq z[A(x, y)]]$ for any formula A in Δ_0 (Bounded Induction)

Parikh [21] proved that there is no formula, bounded or not, which defines exponentiation as a provably total function of $I\Delta_0$. Also, if $A(x, y) \in \Delta_0$ and $I\Delta_0 \vdash \forall x \exists y[A(x, y)]$, then there is a term t in L such that $I\Delta_0 \vdash \forall x \exists y \leq t(x)[A(x, y)]$ (see [21]). Gaifman and Dimitracopoulos [6] showed that there is a Δ_0 formula $exp(x, y, z)$ - to be read as $x^y = z$ - which can be proved in $I\Delta_0$ to have all the usual basic properties of the exponential function with the exception - because of the result just mentioned - of totality $\forall x \forall y \exists z[exp(x, y, z)]$.

It is also known that the Euclid proof of the infinity of primes can easily be formalized in $I\Delta_0 + exp$, but clearly requires *exp*; see [26]. To the best of our knowledge it is not yet known whether this theorem can be proved in $I\Delta_0$ alone. Wilkie and Paris prove in [26] that the consistency of $I\Delta_0$ cannot be derived from $I\Delta_0$ and even not from $I\Delta_0 + exp$. Even the consistency of Robinson's arithmetic Q cannot be formally derived from $I\Delta_0 + exp$.

Lipton [19] proved that the set of Δ_0 predicates is equal to the set of predicates recognized by constant alternation, *linear time* Turing machines. And Myhill proved that every Δ_0 formula defines a predicate which can be computed in *linear space* by a deterministic Turing machine. To the best of our knowledge, the converse question is still open.

More generally, one may introduce an *arithmetical hierarchy* of formulas: $\Sigma_0 = \Pi_0$ is by definition the class of all bounded formulas; Σ_{n+1} is the class of all formulas of the form $\exists x[\phi]$ with $\phi \in \Pi_n$ and Π_{n+1} is the class of all formulas of the form $\forall x[\phi]$ with $\phi \in \Sigma_n$. A set $X \subseteq \mathbb{N}$ is Σ_n, resp. Π_n, if it is defined by a Σ_n, resp. Π_n, formula. And X is Δ_n if it is both Σ_n and Π_n.

Hájek and Pudlák [11] construct in Definition 3.12 a Σ_0 ($= \Delta_0$) formula $Exp(x, y, z)$ and prove in Theorem 3.15 that the formulas (c1) and (c2) are provable in $I\Sigma_0$: (c1) $Exp(x, \overline{0}, z) \leftrightarrow z = \overline{1}$ and (c2) $Exp(x, y + \overline{1}, z) \leftrightarrow \exists v[Exp(x, y, v) \wedge z = v \cdot x]$. In the definition of Exp they use recursion on binary notation instead of the ordinary recursion in order to guarantee that Exp is a Σ_0 formula. In their book [11] Hájek and Pudlák pay much attention to subsystems of Peano Arithmetic, including bounded arithmetic and related theories.

As is well-known, exponentiation plays an important role in the arithmetization of metamathematics and hence in the proof of Gődel's incompleteness theorem. To the best of our knowledge it is still an open question whether $I\Delta_0$ can formalize the metamathematics needed for Gődel's incompleteness theorems (Gődel, 1931). However, it is known that $I\Sigma_1$ suffices to prove all necessary things for the arithmetization. It is also known that the bounded arithmetic $I\Delta_0 + \Omega_1$ already suffices for arithmetization and to show Gődel's first and second incompleteness theorem. See, for example, p. 77 of Alessandro Berarducci and Rineke Verbrugge [2]. And so does the bounded arithmetic S^1_2, see Buss [3], Chapter 7. Both these systems are comparable in strength: they are weaker than $I\Delta_0$ + exponentiation, but stronger than $I\Delta_0$ in that they prove totality of a function that grows about as fast as $2^{|x|^2}$.

For more results on bounded arithmetic see Buss [3, 4], Wilkie-Paris [26], Hájek-Pudlák [11] and Krajíček [18]. In this context it should also be mentioned that, for instance, Buss [3] uses quite a different arithmetization of metamathematics than Kleene and consequently does not seem to need Euclid's Lemma, that is Kleene's [17] Lemma 157.

The question naturally arises what we mean by a *feasible* number. A similar problem occurs in theoretical computer science where it is not enough to know that an algorithm will sooner or later stop and deliver an output. No, we want to know that the algorithm stops within a feasible time. For instance, the satisfiability problem SAT for classical propositional logic is the question whether a given formula is satisfiable, i.e., has (at least) a 1 in its truth-table. The computation of one line in the truth table of a given formula A can be done realistically: the time required to do so is a polynomial of the complexity of A. So, SAT is non-deterministically decidable in polynomial time: non-deterministically choose a line in the truth table and compute the truth value of the given formula. However, when the formula has been built from n atomic formulas, its truth-table will consist of 2^n rows and one just has to check whether one of these rows has 1 as truth-value for the formula in question. Of course, there is a trivial algorithm which eventually will produce the answer to this question: simply check every row for a 1. But, if $n = 64$, that is, if the formula has been built from 64 atomic formulas, the algorithm in question may not stop within a feasible amount of steps; it may take many human lifetimes in order to check all lines in the truth table for a 1 and produce an answer; see de Swart [23], Section 2.3.1 and 2.5.3, plus all the literature on NP-completeness of SAT.

In $I\Sigma_0$ one has induction for bounded formulas. It is well-known that bounded formulas define easily-computable sets; in particular, they are computable in linear space. In Chapter V of [11] Hájek and Pudlák give some deeper connections.

Chapter 4

BV minus I: Cut-Elimination, Consistency

In Chapter 2 we have defined the BV derivation system. In Chapter 3 we have shown that BV is significantly different from traditional logical derivation systems, in that not all traditionally valid proofs remain valid under BV. This seems enough to support the, rather modest, claim that BV is not prima facie superfluous. In order to claim that BV is truly interesting however, it will be necessary to prove that it is a genuine logic rather than a mere derivation system. There may be contention as to what constitutes a genuine logic, but consistency is a property universally accepted as at least necessary to deserve the predicate of genuineness. In this chapter we will prove the consistency of BV without induction (I) by means of Cut-elimination for BV without induction (I), mainly because Cut-elimination will also prove useful in proving other properties (but for this see the next section). For the sake of brevity we will use the term BV in this chapter to refer to the BV system (defined in Section 2.4) without induction (I). Versions of the Cut-elimination proof for traditional logic in [24] and [25] have been used as templates, and extended to deal with the complications arising from the presence of reference grammars. Note that the presence of $\perp$ does away with the need for $\neg L$ and $\neg R$, so these rules are not considered, neither are the structural rules. Also note that in this proof we speak of identical formulae instead of equiform formulae. This is done for the sake of simplicity, and does not affect the proof itself (equiform formulae would give rise to the need for several distinct variables, which are then given identical domains in the reference grammar via Cut and Contraction, none of which materially affects the proof but would increase the complexity of the reduction steps, hence the decision to ignore the distinction between identity and equiformity in this case).

4.1 Cut-Elimination

Theorem 4.1 (Cut-elimination for BV)
Any proof in BV (minus Induction) that contains the Cut rule can be replaced by a BV proof of the same theorem that does not contain the Cut rule.

Note that the requirement that the cutfree proof is BV valid entails only that the reference grammar of the cutfree proof does not yield inconsistency, it does not entail that the reference grammars of the original proof and the cutfree proof be identical. Some preliminary remarks on terminology.

Definition 4.1 (level of cut) The *level* of a cut is the number of derivation steps in the deductions of the premisses of the cut.

Definition 4.2 (rank of cut) The *rank* of a cut on a formula F is the complexity of F (notation: c(F)), with complexity 1 for atomic formulas, 1 added for every quantifier or negation applied to a formula and the complexity of a formula with a connective is the sum of the complexities of the constituent formulas plus 1.

For example: let A and B be atomic, then $c(A) = c(B) = 1$; $c(\neg A) = c(A)+1$, $c(\neg A \wedge B) = c(\neg A) + c(B) + 1$ and $c(\forall x[A]) = c(\exists x[A]) = c(A) + 1$.

Definition 4.3 (principal formula) The *principal* formula in an application of a BV rule is the one that is constructed in that application. For instance, the principal formula in rule $\forall L$ and in rule $\forall R$ is the formula $\forall xA$. All other formulae are called *secondary*.

Proof of the Cut-elimination theorem The proof is by induction on rank, with a subinduction on level. The induction basis (i.e., the case where both rank and level are reduced to 1 and hence the simplest possible form of a derivation in which Cut occurs) is trivial:

$$\frac{A \Rightarrow A\ \ [\emptyset] \qquad A \Rightarrow A\ \ [\emptyset]}{A \Rightarrow A\ \ [\emptyset]}\, C$$

can simply be replaced by

$$A \Rightarrow A \;\; [\emptyset]$$

removing Cut from the proof (and retaining an identical reference grammar, the empty set).

It now suffices to prove that any occurrence of the Cut rule in the derivation can be replaced by a derivation that:

- ends in the same formula and,
- either Cut is removed or the rank or level (possibly both) of the cut formula are reduced,
- and whose reference grammar does not yield inconsistency (for the derivation as a whole).

Let D_l be the derivation of the left-hand premise of the Cut occurrence and D_r the derivation of the right-hand premise of the Cut occurrence. By induction hypothesis, we may assume D_l and D_r to be cutfree. This leaves three possibilities:

1. At least one of D_l, D_r is an axiom.
2. D_l and D_r are not axioms, and the cut formula is not principal in at least one of the premisses of the Cut occurrence.
3. D_l and D_r are not axioms, and the cut formula is principal in both premisses of the Cut occurrence.

Case 1. One of the premisses is an axiom, say D_l. (We leave it to the reader to check that the right-hand cases are symmetrical.) If the cut formula differs from the principal formula of the axiom, we get

$$\frac{P, \Gamma \Rightarrow \Delta, P, A_1 \;\; [G_1] \qquad A_2, \Gamma' \Rightarrow \Delta' \;\; [G_2]}{P, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', P \;\; [G_1 \cup G_2 \cup [x_{i_1}, x_{i_2}]]} C$$

which can be replaced by

$$P, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', P \;\; [G_1 \cup G_2]$$

Cut is removed and the reference grammar differs only in that the condition $[x_{i_1}, x_{i_2}]$ is left out. Since we know that the variables in A_1 and A_2 are new and cannot occur in the remainder of the derivation, the consistency conditions of both reference grammars are still identical.

If the cut formula is identical to the principal formula of the axiom, we get

$$\dfrac{P, \Gamma \Rightarrow \Delta, P \ [G_1] \qquad P, \Gamma' \Rightarrow \Delta' \ [G_2]}{P, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ [G_1 \cup G_2]} \, C$$

which can be replaced by

$$\dfrac{\dfrac{P, \Gamma' \Rightarrow \Delta' \ [G_2]}{P, \Gamma', \Gamma \Rightarrow \Delta' \ [G_2]} \, WL}{P, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ [G_2]} \, PL, WR$$

Cut is removed and the reference grammar is reduced (and can hence not cause inconsistency).

This leaves the case where $\perp$ is the cut formula. Assuming (without loss of generality) D_l derives $\Gamma \Rightarrow \Delta, \perp \ [G_1]$ and D_r derives $\perp, \Gamma' \Rightarrow \Delta' \ [G_2]$ and using R as a symbol for any derivation rule we get

$$\dfrac{\Gamma'' \Rightarrow \Delta'', \perp \ [G_0]}{\Gamma \Rightarrow \Delta, \perp \ [G_1]} \, R \quad (\alpha)$$

$$\dfrac{(\alpha) \qquad \perp, \Gamma' \Rightarrow \Delta' \ [G_2]}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ [G_1 \cup G_2]} \, C$$

which can be replaced by

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta'', \perp \ [G_0] \qquad \perp, \Gamma' \Rightarrow \Delta' \ [G_2]}{\Gamma'', \Gamma' \Rightarrow \Delta'', \Delta' \ [G_0 \cup G_2]} \, C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ [G_1 \cup G_2]} \, R$$

The level of Cut is reduced and the reference grammar is identical.

This concludes case 1.

Case 2. The deductions of the premisses are not axioms and the cut formula (D) is not on both sides principal, say it is not principal on the left. (We leave it to the reader to check that the right-hand cases are symmetrical.) This leaves us with two cases: the left premise is the conclusion of a 1-premise rule or of a 2-premise rule. In case of a 1-premise rule we get

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A\ \ [G_0]}{\Gamma \Rightarrow \Delta, A\ \ [G_1]}\, R \quad (\alpha)$$

$$\frac{(\alpha) \qquad A, \Gamma' \Rightarrow \Delta'\ \ [G_2]}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_1 \cup G_2]}\, C$$

which can be replaced by

$$\cfrac{\cfrac{\Gamma_0 \Rightarrow \Delta_0, A\ \ [G_0] \qquad A, \Gamma' \Rightarrow \Delta'\ \ [G_2]}{\Gamma_0, \Gamma' \Rightarrow \Delta_0, \Delta'\ \ [G_0 \cup G_2]}\, C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_1 \cup G_2]}\, R$$

The level of Cut is reduced and the reference grammar is identical.

In case of a 2-premise rule we get

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A\ \ [G_0] \qquad \Gamma_1 \Rightarrow \Delta_1, A\ \ [G_1]}{\Gamma \Rightarrow \Delta, A\ \ [G_2]}\, R \quad (\alpha)$$

$$\frac{(\alpha) \qquad A, \Gamma' \Rightarrow \Delta'\ \ [G_3]}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_2 \cup G_3]}\, C$$

which can be replaced by

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A\ \ [G_0] \qquad A, \Gamma' \Rightarrow \Delta'\ \ [G_3]}{\Gamma_0, \Gamma' \Rightarrow \Delta_0, \Delta\ \ [G_0 \cup G_3]}\, C \quad (\alpha)$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A\ \ [G_1] \qquad A, \Gamma' \Rightarrow \Delta'\ \ [G_3]}{\Gamma_1, \Gamma' \Rightarrow \Delta_1, \Delta'\ \ [G_1 \cup G_3]}\, C \quad (\beta)$$

$$\frac{(\alpha) \qquad (\beta)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_2 \cup G_3]}\, R$$

In both instances the level of Cut is reduced, and the reference grammar is identical.

This concludes case 2.

Case 3. The deductions of the premisses are not axioms and the cut formula is principal in both premisses. Reduction steps must now be given for all connectives and quantifiers.

For $\wedge$ we get

$$\dfrac{\Gamma \Rightarrow \Delta, A\ [G_0] \qquad \Gamma \Rightarrow \Delta, B\ [G_1]}{\Gamma \Rightarrow \Delta, A \wedge B\ [G_0 \cup G_1]} \wedge R \quad (\alpha)$$

$$\dfrac{A, B, \Gamma' \Rightarrow \Delta'\ [G_2]}{A \wedge B, \Gamma' \Rightarrow \Delta'\ [G_2]} \wedge L \quad (\beta)$$

$$\dfrac{(\alpha) \qquad (\beta)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A \wedge B_l}}, x_{i_{A \wedge B_r}}]\}]}$$

which can be replaced by

$$\dfrac{\Gamma \Rightarrow \Delta, B\ [G_1] \qquad A, B, \Gamma' \Rightarrow \Delta'\ [G_2]}{A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ [G_1 \cup G_2 \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\}]} C \quad (\alpha)$$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A\ [G_0] \qquad (\alpha)}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'\ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\} \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]} C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\} \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]} CL, CR$$

While the number of Cuts has doubled, both are of lower rank than the original Cut thereby satisfying the induction condition. The reference grammar is identical, given that identity of the domains of the variables in a compound formula (i.c. $\wedge$) reduces to identity of the domains of the variables of the constituent formulas of the compound formula. This is to say that $[x_{i_{A \wedge B_l}}, x_{i_{A \wedge B_r}}]$ constitutes exactly the same condition as the combination of $[x_{i_{A_l}}, x_{i_{A_r}}]$ and $[x_{i_{B_l}}, x_{i_{B_r}}]$.

For $\rightarrow$ we get

$$\dfrac{\Gamma' \Rightarrow A, \Delta'\ [G_1] \qquad \Gamma', B \Rightarrow \Delta'\ [G_2]}{\Gamma', A \rightarrow B \Rightarrow \Delta'\ [G_1 \cup G_2]} \rightarrow L \quad (\alpha)$$

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B, \Delta \ \ [G_0]}{\Gamma \Rightarrow A \to B, \Delta \ \ [G_0]} \to R \qquad (\alpha)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A\to B_l}}, x_{i_{A\to B_r}}]\}]} \, C$$

which can be replaced by

$$\dfrac{\dfrac{\dfrac{\Gamma, A \Rightarrow B, \Delta \ \ [G_0] \qquad \Gamma', A \Rightarrow \Delta' \ \ [G_1]}{\Gamma, \Gamma' \Rightarrow B, \Delta, \Delta' \ \ [G_0 \cup G_1 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]} \, C \qquad \Gamma', B \Rightarrow \Delta' \ \ [G_2]}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta'\Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\} \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\}]} \, C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\} \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\}]} \, CL, CR$$

The new Cuts are of lower rank than the original Cut and the reference grammar is identical (for the reason given under the $\wedge$ case).

For $\vee$ we get

$$\dfrac{\Gamma \Rightarrow \Delta, A, B \ \ [G_0]}{\Gamma \Rightarrow \Delta, A \vee B \ \ [G_0]} \vee R \quad (\alpha)$$

$$\dfrac{A, \Gamma' \Rightarrow \Delta' \ \ [G_1] \qquad B, \Gamma' \Rightarrow \Delta' \ \ [G_2]}{A \vee B, \Gamma' \Rightarrow \Delta' \ \ [G_1 \cup G_2]} \vee L \quad (\beta)$$

$$\dfrac{(\alpha) \qquad (\beta)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A\vee B_l}}, x_{i_{A\vee B_r}}]\}]} \, C$$

which can be replaced by

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A, B \ \ [G_0] \qquad A, \Gamma' \Rightarrow \Delta' \ \ [G_1]}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', B \ \ [G_0 \cup G_1 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]} \, C \qquad B, \Gamma' \Rightarrow \Delta' \ \ [G_2]}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\} \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\}]} \, C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \ \ [G_0 \cup G_1 \cup G_2 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\} \cup \{[x_{i_{B_l}}, x_{i_{B_r}}]\}]} \, CL, CR$$

The new Cuts are of lower rank than the original Cut and the reference grammar is identical for the reasons given above.

For $\forall$ we get

$$\dfrac{\Gamma \Rightarrow \Delta, A[x/y]\ \ [G_0]}{\Gamma \Rightarrow \Delta, \forall x A\ \ [G_0 \cup \{y \rightarrow x\}]}\ \forall R \quad (\alpha)$$

$$\dfrac{\dfrac{\forall x A, A[x/t], \Gamma' \Rightarrow \Delta'\ \ [G_1]}{\forall x A, \forall x A, \Gamma' \Rightarrow \Delta'\ \ [G_1 \cup \{x \rightarrow t\}]}\ \forall L}{\forall x A, \Gamma' \Rightarrow \Delta'\ \ [G_1 \cup \{x \rightarrow t\}]}\ CL \quad (\beta)$$

$$\dfrac{(\alpha) \qquad (\beta)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup \{y \rightarrow x\} \cup G_1 \cup \{x \rightarrow t\} \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]}\ C$$

which can be replaced by

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A[x/y]\ \ [G_0]}{\Gamma \Rightarrow \Delta, \forall x A\ \ [G_0 \cup \{y \rightarrow x\}]}\ \forall R \qquad \forall x A, A[x/t], \Gamma' \Rightarrow \Delta'\ \ [G_1]}{A[x/t], \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup \{y \rightarrow x\} \cup G_1 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]}\ C \quad (\alpha)$$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A[x/t]\ \ [G_0 \cup \{y \rightarrow t\}] \qquad (\alpha)}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'\ \ [G_0 \cup \{y \rightarrow t\} \cup \{y \rightarrow x\} \cup G_1 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]}\ C}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup \{y \rightarrow t\} \cup \{y \rightarrow x\} \cup G_1 \cup \{[x_{i_{A_l}}, x_{i_{A_r}}]\}]}\ CL, CR$$

Some elucidation may be helpful here, as this case is slightly more complex than the previous ones.

Firstly, one might wonder how we got to $\Gamma \Rightarrow \Delta, A[x/t]\ \ [G_0 \cup \{y \rightarrow t\}]$ as the left-hand premise in the final derivation. This results from taking the derivation of $\Gamma \Rightarrow \Delta, A[x/y]\ \ [G_0]$ and substituting t for y, which yields the desired formula and also necessitates the appropriate addition to the reference grammar.

Secondly, the reference grammars of the derivations seem to be different. More specifically, the reference grammar of the latter derivation contains a condition that does not seem to occur in the reference grammar of the original derivation, i.e. $y \rightarrow t$. If so, this would be problematic as this could potentially yield cases where the latter derivation becomes invalid while the original derivation remains valid, thus contradicting the contention that the latter derivation is an appropriate replacement for the former. However, note that the reference grammar is transitively closed with respect to $\rightarrow$ and that the reference grammar of the original derivation contains both $y \rightarrow x$ and $x \rightarrow t$, meaning that $y \rightarrow t$ is also, albeit implicitly, present in the reference

grammar of the original derivation.
This leaves only one difference between the two reference grammers, to wit, the reference grammar of the original derivation contains a condition, $x \to t$ that is not present in the reference grammar of the replacing derivation. We have met with such a situation before in this proof, namely under case 1 with the cut formula being identical to the principal formula of the axiom, and kind of glossed it there, but it deserves a little more attention than I have given it there. The point is that a reduction of the reference grammar is not a problem as it does not give rise to the possibility of the original derivation being valid while the replacement derivation is invalid (the reference grammar of the replacement derivation is smaller than that of the original derivation, so if the conditions for the original derivation are met then so are a fortiori the conditions for the latter derivation). The possibility does arise that the replacement derivation could be valid while the original derivation proves to be invalid. This would turn an invalid proof into a valid one by replacing the original derivation with the latter one. This may seem counterintuitive, but it is in fact a property of BV derivations that is to be expected. On the other hand, as we've seen in Chapter 3, a previously valid proof can become invalid when extra information is added to the RG. It will then be necessary to find alternative proofs, assuming that these exist, for them that are BV valid. Hence, a situation where a reference grammar is reduced, as was the case here, is not only not problematic, it is in fact a situation much to be desired as it will give the opportunity to find such alternative proofs.

Thirdly and lastly, there are two Cuts instead of the original one. The first Cut, right-hand side on $\forall x A$, is reduced in level with regard to the original Cut (it now appears before $A[x/t]$ has been eliminated). The second Cut, on $A[x/t]$, is reduced in rank with regard to the original Cut. Thus the induction condition is satisfied. However, it does become clear from this reduction step that the removal of Cut from a proof is not a straightforward continuing reduction of complexity. One Cut is replaced by two, of which one is of the same rank and only one single level lower, while the other is of lower rank but of higher level (one derivation step is added).

The $\exists$ case is analogous to the $\forall$ case and left as an exercise for the reader.
$\square$

4.1.1 The problem with the Induction Rule

One might hope that the proof of the Cut-elimination theorem may be extended to the Buridan-Volpin system including the Induction rule. We thought we had done so, but evidently our desire to do so was so great that we made a serious mistake, as carefully pointed out to us by professor David Isles. We are grateful to him for this important contribution.

Notice that the Induction rule I introduces a universal quantifier at the right, just as the logical rule $(\forall R)$ does:

$$\frac{F_1(a), \Gamma \Rightarrow \Delta, F_2(a/s(a))}{F_1(a/0), \Gamma \Rightarrow \Delta, \forall y[F_2(a/y)]}$$

Hence, for logic we have the rule $(\forall R)$, but for arithmetic we have the extra $\forall$-quantifier rule in the form of Induction. It seems that Cut-elimination does hold for predicate logic, but no longer for arithmetic anymore.

In this subsection we hope to clarify the problems caused by the Induction rule.

For induction on the right we get

$$\frac{\Gamma_0 \Rightarrow \Delta_0 \;\; [G']}{\Gamma' \Rightarrow \Delta', F(0) \;\; [G_1]} R \qquad (\beta)$$

$$\frac{F(a), \Gamma \Rightarrow \Delta, F(s(a)) \;\; [G_0]}{F(0), \Gamma \Rightarrow \Delta, \forall x F(x) \;\; [G_0 \cup \{Dom x = N\} \cup \{a \to x\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\}]} I \qquad (\alpha)$$

$$\frac{(\beta) \qquad (\alpha)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \forall x F(x) \;\; [G_0 \cup G_1 \cup \{Dom x = N\} \cup \{a \to x\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\}]} C$$

If the Δ_0 in the premiss of rule R would also contain $F(0)$, say $\Delta_0 = \Delta_0', F(0)$, then we might apply the cut rule to $\Gamma_0 \Rightarrow \Delta_0', F(0)$ and the conclusion of (α) and in this way reduce the level of the Cut:

$$\frac{\Gamma_0 \Rightarrow \Delta_0', F(0) \;\; [G']}{\Gamma' \Rightarrow \Delta', F(0) \;\; [G_1]} R$$

$$\frac{\dfrac{\Gamma_0 \Rightarrow \Delta_0', F(0) \;\; [G'] \qquad (\alpha)}{\Gamma_0, \Gamma \Rightarrow \Delta, \Delta_0', \forall x F(x) \;\; [G' \cup G_0 \cup \{Dom x = N\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\} \cup \{a \to x\}]} C}{\Gamma', \Gamma \Rightarrow \Delta, \Delta', \forall x F(x) \;\; [G_1 \cup G_0 \cup \{Dom x = N\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\} \cup \{a \to x\}]} R$$

However, if the $F(0)$ in the conclusion of rule R is the result of weakening right, in other words if $R = (WR)$, there is a serious problem!

For induction on the left we get

$$\frac{F(a), \Gamma \Rightarrow \Delta, F(s(a))\ \ [G_0]}{F(0), \Gamma \Rightarrow \Delta, \forall x F(x)\ \ [G_0 \cup \{Domx = N\} \cup \{a \to x\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\}]}\ I \qquad (\alpha)$$

$$\frac{\Gamma', F(a) \Rightarrow \Delta'\ \ [G_1]}{\Gamma', \forall x F(x) \Rightarrow \Delta'\ \ [G_1 \cup \{x \to a\}]}\ \forall L \qquad (\beta)$$

$$\frac{(\alpha) \qquad (\beta)}{F(0), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup G_1 \cup \{x \to a\} \cup \{Domx = N\} \cup \{a \to x\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\}]}\ C$$

which - as it seems - can be replaced by

$$\frac{\dfrac{F(a), \Gamma \Rightarrow \Delta, F(s(a))\ \ [G_0] \qquad \Gamma', F(s(a)) \Rightarrow \Delta'\ \ [G_1 \cup \{a \to s(a)\}]}{F(a), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup G_1 \cup \{a \to s(a)\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\}]}\ C}{F(0).\Gamma, \Gamma' \Rightarrow \Delta, \Delta'\ \ [G_0 \cup G_1 \cup \{a \to s(a)\} \cup \{[x_{i_{F_l}}, x_{i_{F_r}}]\} \cup \{a \to 0\}]}\ inst.a/0$$

The rank of the new Cut is lower than that of the original. The reference grammars differ but not in a problematical way. As seen above, just before Subsection 4.1.1, the conditions in the original reference grammar that do not occur in the replacing reference grammar are not a problem. This leaves the conditions $a \to s(a)$ and $a \to 0$, which occur in the replacing reference grammar but not in the original. $a \to 0$ is trivially true as 0 is member of any domain, $a \to s(a)$ is also true, as the condition $Dom(s(a)) \subseteq Dom(a)$ is true since all possible domains considered are closed under application of the successor function. Hence the replacement derivation is an acceptable alternative to the original derivation.

One might wonder how we arrived at $\Gamma', F(s(a)) \Rightarrow \Delta'\ [G_1 \cup \{a \to s(a)\}]$ as the right-hand premise in the replacing derivation. This results from taking the derivation of $\Gamma', F(a) \Rightarrow \Delta'\ [G_1]$ and substituting $s(a)$ for a, which also necessitates the appropriate condition to the reference grammar.

Should this not have been clear, the expression $[x_{i_{F_l}}, x_{i_{F_r}}]$ states, analogously to the conditions in the reference grammars for Cut and Induction as given in the definition of these rules, that all bound variables occurring in the F on the left have domains identical to their counterparts occurring in the F on the right.

The expression $inst.x/t$ is used to indicate the instantiation of x with t. This necessitates addition of the condition $x \to t$ to the reference grammar.

4.2 Consistency

Now we can put the Cut-elimination theorem we have just proved to good use. First we will prove the subformula property for BV and then we will be able to give a (non-constructive) proof of the consistency of BV. The consistency proof will not be very informative, being non-constructive, but our main concern here is the result itself as this is sufficient for the project of this thesis. A constructive consistency proof and an evaluation of its implications would be a good subject for follow-up research.

For the purposes of this section, the reference grammars are not relevant. Hence we will not consider them and limit ourselves to the derivation steps themselves.

First now the subformula property for BV. We informally define the notion of 'subformula' in the usual way, to wit:

- any formula is a subformula of itself
- any formula is a subformula of its negation
- for any formula constituted by means of a logical connective, both constituent formulae are subformulae of the complete formula
- for any formula constituted by means of quantification, the formula quantified over is a subformula of the complete formula

Note that for the purposes of this definition we treat equiform formulae as identical. This means that if a formula A is a subformula of formula B, then all formulae equiform to A are subformulae of all formulae equiform to B.

Theorem 4.2 (Subformula property for BV)
All formulae occurring in a Cut-free derivation D are subformulae of the formulae occurring in the concluding sequent of D.

Proof: By induction on the number of derivation steps (i.e., the level of the derivation).

The induction base, a zero step derivation, is trivial as the concluding sequent is the only sequent and hence all occurring formulae are subformulae (note that by the definition of subformula, a formula is a subformula of itself).

As induction hypothesis we assume all Cut-free derivations containing at most n derivation steps to have the subformula property. It remains to prove that all Cut-free derivations containing $n+1$ derivation steps have the subformula property.

Take a random Cut-free derivation D_{n+1} containing $n+1$ derivation steps. Let D_n be the derivation created by removing the last derivation step from D_{n+1}. By the induction hypothesis we know that D_n has the subformula property. Then D_{n+1} can be construed by adding the appropriate derivation step to D_n. Note that this added derivation step cannot be the Cut rule as D_{n+1} is ex hypothesi Cut-free. It is left to the reader to check that for every derivation step (save Cut, which is not under consideration here) the subformula property of D_n also holds for D_{n+1}. Intuitively, all derivation steps (save Cut) retain all formulae from their premisses in their conclusion, either by copying them, or by creating a more complex formula (i.e., a formula that the premiss formula is a subformula of) by means of logical connective or quantifier. □

This concludes the proof of the subformula property. Note that this property holds for BV with induction.

Theorem 4.3 (Consistency of BV)
BV is consistent

Proof: By reductio ad absurdum.

Suppose BV is inconsistent. Then there exists a derivation ending in the sequent:

$$\emptyset \Rightarrow \bot.$$

To this derivation can then be added the following derivation step:

$$\frac{\emptyset \Rightarrow \bot \qquad \bot \Rightarrow \emptyset}{\emptyset \Rightarrow \emptyset}\, C$$

(Note that reference grammar is not a problem here, as the right-hand side premiss has the empty set as reference grammar and therefore no new restrictions obtain)

So there is a proof of the sequent: $\emptyset \Rightarrow \emptyset$.

Hence, by the Cut-elimination theorem, there is a Cut-free proof of this sequent. But this yields inconsistency with the subformula property for BV derivations. The sequent is not an axiom, hence the derivation must have a level higher than 0. But any derivation of level 1 or higher will by definition contain formulae. As the sequent proved contains no formulae, the subformula property is violated. But by the subformula property theorem we know that this derivation (which meets all the requirements of the theorem) should have the subformula property. Hence, contradiction and the original assumption that BV is inconsistent is false. Therefore, BV is consistent. $\square$

This concludes the proof of consistency of BV. Note that, since the Cut-elimination theorem is used, this proof only holds for BV without induction.

Chapter 5

Löwenheim-Skolem (L-S)

In this chapter we consider the Löwenheim-Skolem (L-S) theorem in the context of the BV system. The Lőwenheim-Skolem theorem in classical logic says that if a set Γ of sentences has a model, then it has a denumerable model; see, for instance, [23]. *Skolemization* is the process of replacing $\forall x \exists y[P(x, y)]$ by $\forall x[P(x, f(x))]$, where f is a new function symbol. The underlying idea is that if M is a model with domain D of $\forall x \exists y[P(x, y)]$, then there must be a function $f^* : D \to D$ such that for all elements d in D, $M \models P(a_1, a_2)[d, f^*(d)]$. f^* is called a *Skolem function*.

In Section 5.1 we discuss the paper *A Finite Analog to the Lőwenheim-Skolem Theorem* by David Isles [13]. In this paper Isles considers a classical sequent calculus LK which is very similar to the Buridan-Volpin (BV) derivation system from Chapter 2, except that it does not have the Induction Rule (I). In Section 5.2 we shall attempt to generalize Isles' results for LK in Section 5.1 to the Buridan-Volpin derivation system BV from Chapter 2. On the one hand we take the liberty to adapt some of the notations of David Isles, on the other hand following David Isles we shall speak of the *range* of a variable instead of its *domain*.

5.1 A Finite Analog to the L-S Theorem

5.1.1 Introduction

As before, we consider the possibility that the domains of variables in a derivation need not be fixed in advance. Consequently, the meaning of a formula might differ from derivation to derivation. One would only need to keep track of the structural restrictions placed on variables in the derivation. By way of illustration, consider the following derivation in Gentzen's sequent calculus LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{\forall x[P(x)] \Rightarrow P(a)\ \ [x \to a]}\ \forall L \quad (\alpha) \qquad (\alpha) \quad Q(a) \Rightarrow Q(a)}{\forall x[P(x)], P(a) \to Q(a) \Rightarrow Q(a)} \to L}{\forall x[P(x)], \forall y[P(y) \to Q(y)] \Rightarrow Q(a)\ \ [y \to a]}\ \forall L}{\forall x[P(x)], \forall y[P(y) \to Q(y)] \Rightarrow \forall z[Q(z)]\ \ [a \to z]}\ \forall R$$

$x \to a$, $y \to a$ and $a \to z$ are 'reference arrows' in the reference grammar of this derivation and indicate, respectively, that a has been substituted for x, a has been substituted for y and z has been substituted for a. Any arrow $n \to m$ can be read as saying that any value of m is also a value of n. This means that, in this particular derivation, the soundness of the inference rules requires only that the range of z is a subset of the range of a, and the range of a is a subset of both the range of x and the range of y. To interpret the formulas in this derivation, one would need a domain D, an interpretation of P and Q (which would be a subset of the domain) and a map $r : \{x, y, z, a\} \to \mathcal{P}(D)$ (power set of D) such that $r(z) \subseteq r(a)$, $r(a) \subseteq r(x)$ and $r(a) \subseteq r(y)$. We shall call such an interpretation a *BV interpretation* which is a traditional Tarski model M together with a range function r which assigns to every variable v in the derivation its range $r(v)$ which is a subset of the domain of M. Note that the usual notion of model obtains if the domains of all the variables are considered to be identical to the domain of the derivation D.

The example shows that a derivation gives rise to a many-sorted interpretation in which the domains of the variables are not fixed in advance, but are determined by their use within the derivation. This also means that these interpretations may vary as a derivation grows, since new restrictions might come to obtain.

5.1.2 Buridan-Volpin interpretations and reference grammars

Gentzen's sequent calculus LK is used as it is presented in Girard [8], Kleene [17] or Takeuti [24]. The first-order language L contains all the usual symbols, with F, C and R finite sets of, respectively, function, constant and relation letters. Bound variables are denoted x, y or z (possibly subscripted), free variables are denoted a or b (possibly subscripted). We also use the term *parameter* for a free variable. (Pseudo-)terms and formulas are defined in the usual way, terms being pseudo-terms that contain no bound variables (only parameters or constants) and formulas being pseudo-formulas in which all variables (save parameters) occur only within the scope of a corresponding quantifier. E.g., $x+0$ is a pseudo-term, while $a+0$ is a term; and $\forall x[x+0 = x] \wedge x \cdot 0 = 0$ is a pseudo-formula, while $\forall x[x+0 = x \wedge x \cdot 0 = 0]$ is a formula and even a sentence. $F[s/t]$ denotes the substitution of pseudo-term t for pseudo-term s in pseudo-formula F.

Definition 5.1 (sequent, rectified, equiform)
1. Let Γ, Δ be sequences of finitely many (possibly zero) formulas. Then $\Gamma \Rightarrow \Delta$ is called a sequent, with Γ the antecedent and Δ the consequent.
2. A formula F or sequent $\Gamma \Rightarrow \Delta$ is called *rectified* if all of its quantified variables are distinct.
3. Two formulas F_1 and F_2 are called *equiform* if they differ only in the choice of bound variables.

Definition 5.2 (BV interpretation)
A Buridan-Volpin (BV) interpretation M_r for L (the first-order language) is a regular Tarski interpretation M - see, for instance, [23], Section 4.2 - to which is added a function r that assigns to each variable (free or bound) v a non-empty subset $r(v) \subseteq |M|$ (with $|M|$ the domain of the interpretation M). We call $r(v)$ the *range* of v. The BV interpretation M_r is called a BV substructure of the Tarski interpretation M. The BV interpretation M_r is called 'finite' if each $r(v)$ is finite.

Definition 5.3 (F is true in a BV interpretation)
1. Let F be a formula and M_r a BV interpretation with domain $|M|$. Let α be a function from the set of all free variables (parameters) to D with $\alpha(v) \in r(v)$ for each variable v. The standard Tarski definition $[M_r, \alpha] \models F$ now holds, with the extra requirement that for any variable v holds: $\alpha(v) \in r(v)$.

2. If $\Gamma = [A_1, ..., A_n]$ and $\Delta = [B_1, ..., B_m]$, then $[M_r, \alpha] \models \Gamma \Rightarrow \Delta$ iff $[M_r, \alpha] \models [A_1 \wedge ... \wedge A_n] \rightarrow [B_1 \vee ... \vee B_m]$.

Example 5.1
Let $M = \langle\ \mathbb{N}, E^*, f^*, c^*\ \rangle$ with $E^* = \{\text{even numbers}\}$, $f^*(n) = 2n + 1$ and $c^* = 1$. And let $r(x) = \{2, 4\}$. Then, for any α, $[M_r, \alpha] \models \forall x[E(x)]$, since all elements in the range $r(x)$ have the property E^*, and $\forall x[E(x)] \vdash E(f(c))$, but $[M_r, \alpha] \not\models E(f(c))$, since $f^*(c^*) = 3$ does not have the property E^*.

The example shows that $[M_r, \alpha] \models S_2$ no longer necessarily follows from $S_1 \vdash S_2$ and $[M_r, \alpha] \models S_1$. This shows the need for a reference grammar, in which one keeps track of the limitations on the domains of the variables. As seen in the example, substituting a function term for a variable under a quantification can expand the range of that variable. The reference grammar must hence keep track of these substitutions.

In what follows, if U is a set, then U^* is the set of all finite strings of elements of U.

Definition 5.4 (reference grammar) Let F be a set of function symbols and C be a set of constants.
1. A *reference grammar* G based on $F \cup C$ is a triple (V, T, R) with:
V = the set of all variables (bound or free)
$T = F \cup C \cup \{(,)\}$
R (Rewrite or production rules) is a finite subset of $V \times (V \cup T)^*$ meeting the requirement that if $(v, u) \in R$, then u is a pseudo-term.
2. For any $v \in V$, $u \in (V \cup T)^*$: $G : v \rightarrow u$ ($v \rightarrow u$ when there is no ambiguity) whenever $(v, u) \in R$. Given the trivial productions $v \rightarrow v$ for all $v \in V$, R may be said to contain every variable (free or bound).
For any string $u, v \in (V \cup T)^*$: $G : u \rightarrow v$ exactly when there exist strings $r, s, t \in (V \cup T)^*$ and $w \in V$ such that: $u = rwt$, $v = rst$ and $G : w \rightarrow s$.
$G : t_0 \rightarrow\!\!\rightarrow t_n$ refers to a sequence $G : t_0 \rightarrow t_1$, ..., $G : t_{n-1} \rightarrow t_n$. A string u is called 'terminal' if there exists no string v, distinct from u, so that $u \rightarrow\!\!\rightarrow v$.
3. For $v \in V$, $G(v)$ is the set of terminal strings derivable from v.

Definition 5.5 (compatibility)
A BV structure M_r is said to be *compatible* with a reference grammar G (notation: $M_r \in G$) whenever:
1. If $q \rightarrow\!\!\rightarrow t(q_1, ..., q_n) \in G$, then $t^*(d_1, ..., d_n) \in r(q)$ for each $[d_1, ..., d_n] \in$

$r(q_1) \times ... \times r(q_n)$, with t^* the interpretation of t in M. Particularly:
2. If $q_1 \rightarrow\rightarrow q_2 \in G$ then $r(q_2) \subseteq r(q_1)$ and
3. If $q \rightarrow\rightarrow c$ then $c* \in r(q)$.

It should be noted that any Tarski interpretation of the symbols $F \cup C$ of G is also compatible with G. Note also that the BV interpretation in Example 5.1 is not compatible with the reference grammar $G = \{x \rightarrow f(c)\}$ in the derivation of $E(f(c))$ from $\forall x[E(x)]$ (left as exercise for the reader) since $f^*(c^*) = 3 \notin r(x) = \{2, 4\}$.

Definition 5.6
Let G_1, G_2 be reference grammars. When the condition: if $q \rightarrow t \in G_1$, then $q \rightarrow\rightarrow t \in G_2$ is met, we say that G_2 extends G_1 and write $G_1 \subseteq G_2$.

Let $F_1 = \forall x \exists y[D(x, y)]$ and $F_2 = \forall z \exists w[D(z, w)]$. Then F_1 and F_2 are equiform and rectified. Let $M = \langle \mathbb{N}, D^* \rangle$ with $D^*(n, m) := n > m$. And let $r(x) = \mathbb{N}_0 = \mathbb{N} - \{0\}$, $r(y) = r(z) = r(w) = \mathbb{N}$. Then $M_r \models \forall x \exists y[D(x, y)]$, but $M_r \not\models \forall z \exists w[D(z, w)]$ because $r(z) \not\subseteq r(x)$. Similarly, if $r(x) = r(z) = r(w) = \mathbb{N}_0$ and $r(y) = \mathbb{N}$, then again $M_r \models \forall x \exists y[D(x, y)]$, but $M_r \not\models \forall z \exists w[D(z, w)]$ because $r(y) \not\subseteq r(w)$.

In order to guarantee that from $M_r \models \forall x \exists y[D(x, y)]$ it follows that $M_r \models \forall z \exists w[D(z, w)]$, we have to demand that $r(z) \subseteq r(x)$ and that $r(y) \subseteq r(w)$. We shall work this out more generally.

Definition 5.7 (positive and negative quantifiers)
1. An occurrence of a quantifier (Qx) is positive (or negative) in a formula F under the following conditions:
a) If $F = (Q\alpha)G$ and $x = \alpha$, then (Qx) occurs positively in F. If $x \neq \alpha$, then if (Qx) occurs positively (negatively) in G, it occurs positively (negatively) in F.
b) If $F = F_1 \rightarrow F_2, \neg F_1, F_2 \vee F_3, F_2 \wedge F_3$, then, if the occurrence of (Qx) is in F_1 and is positive (negative) there, it is negative (positive) in F; if the occurrence is positive (negative) in F_2 or F_3, it is positive (negative) in F.
2. If (Qx) is $\forall x$ and occurs positively in F or is $\exists x$ and occurs negatively in F, then it is called a *positive quantifier* of F. In all other cases it is a *negative quantifier* of F.
3. If (Qx) is a positive quantifier of a formula in Δ or a negative quantifier of a formula in Γ, then it is a positive quantifier of the sequent $\Gamma \Rightarrow \Delta$. In all other cases it is a negative quantifier of the sequent.

Example 5.2
$\exists x$ occurs positively in $\exists x[P(x)]$ and hence, it occurs negatively in $\neg\exists x[P(x)]$. $\exists y$ occurs positively in $\exists y[Q(y)]$ and $\forall x$ occurs positively in $\forall x[P(x)]$. Hence, $\forall x$ occurs negatively in $F = \forall x[P(x)] \rightarrow \exists y[Q(y)]$, while $\exists y$ occurs positively in F.

In $F = \forall x \exists y[P(x, y)]$ the universal quantifier and the existential quantifier both occur positively in F. Hence, the universal quantifier is a positive quantifier of F and the existential quantifier is a negative quantifier of F. Let M_r be a BV interpretation, $F_1 = \forall x_1 \exists y_1[P(x_1, y_1)]$ and $F_2 = \forall x_2 \exists y_2[P(x_2, y_2)]$, so F_1 and F_2 are equiform and rectified. Suppose $r(x_2) \subseteq r(x_1)$ and $r(y_1) \subseteq r(y_2)$. Then from $M_r \models F_1$, it follows that $M_r \models F_2$.

Example 5.3
Consider the formula $F : \forall x[A(x) \rightarrow \neg\exists y \forall z[D(x, y, z)]$. $\forall z$ occurs positively in $\forall z[D]$ and by Definition 5.7 1a, $\forall z$ also occurs positively in $\exists y \forall z[D]$. $\exists y$ also occurs positively in $\exists y \forall z[D]$. Hence, by Definition 5.7 1b, the quantifiers $\exists y$ and $\forall z$ occur negatively in $\neg\exists y \forall z[D]$. Again by Definition 5.7 1b, $\exists y$ and $\forall z$ occur negatively in $A(x) \rightarrow \neg\exists y \forall z[D]$. By Definition 5.7 1a, $\exists y$ and $\forall z$ both occur negatively in $F = \forall x[A(x) \rightarrow \neg\exists y \forall z[D(x, y, z)]]$. By Definition 5.7 2, $\exists y$ is a positive quantifier of F, but $\forall z$ is a negative quantifier of F. Finally, $\forall x$ occurs positively in F and hence, by Definition 5.7 2, is a positive quantifier of F.

Now let $F_1 = \forall x_1[A(x_1) \rightarrow \neg\exists y_1 \forall z_1[D(x_1, y_1, z_1)]$ and $F_2 = \forall x_2[A(x_2) \rightarrow \neg\exists y_2 \forall z_2[D(x_2, y_2, z_2)]$. If we demand that $r(x_2) \subseteq r(x_1)$ - corresponding with the fact that $\forall x_1$ and $\forall x_2$ are positive quantifiers of F_1 and F_2 respectively, and $r(y_2) \subseteq r(y_1)$ - corresponding to the fact that $\exists y_1$ and $\exists y_2$ are positive quantifiers of F_1 and F_2 respectively - and in addition that $r(z_1) \subseteq r(z_2)$ - corresponding to the fact that $\forall z_1$ and $\forall z_2$ are negative quantifiers of F_1 and F_2 respectively - , then it follows that: if $M_r \models F_1$, then also $M_r \models F_2$.

More generally, we have:

Lemma 5.1
Let M_r be a BV interpretation and F_1, F_2 two equiform rectified formulas. Let Qx_1 and Qx_2 be corresponding positive quantifiers of F_1 and F_2 respectively, and Qz_1 and Qz_2 corresponding negative quantifiers. Suppose $r(x_2) \subseteq r(x_1)$ and $r(z_1) \subseteq r(z_2)$ in M_r for all such corresponding positive, resp. negative, quantifiers. Then from $[M_r, \alpha] \models F_1$ it follows that $[M_r, \alpha] \models F_2$.

Proof sketch: From definition 5.7 it follows that for Qx to be a positive quantifier of a formula F means that either Qx is $\forall x$ (Def. 5.7, 2 and 1), or Qx is $\neg\exists x$ (Def. 5.7, 2 and 1b, keeping in mind that $P \to Q$ is equivalent to $\neg P \vee Q$), which is equivalent to $\forall x\ \neg$. So if Qx is a positive quantifier of F, then Qx is a universal quantifier (either immediately or after some rewriting).

Analogously, for Qx to be a negative quantifier of a formula F means that either Qx is $\exists x$ (Def. 5.7, 2 and 1), or Qx is $\neg\forall x$ (Def. 5.7, 2 and 1b), which is equivalent to $\exists x\ \neg$. So if Qx is a negative quantifier of F, then Qx is an existential quantifier (either immediately of after some rewriting).

Given these remarks and the equiformity of F_1 and F_2, the condition $r(x_2) \subseteq r(x_1)$ (on the positive quantifiers) can now be seen to mean that the domain of any universally quantified variable in F_2 must be a subset of the domain of the corresponding variable in F_1. Thus any model for F_1 will also still be a model for F_2 (that which holds for all elements of a set, must necessarily also hold for all elements of a subset thereof).

Analogously, the condition $r(z_1) \subseteq r(z_2)$ (on the negative quantifiers) can now be seen to mean that the domain of any existentially quantified variable in F_1 must be a subset of the domain of the corresponding variable in F_2. Thus any model for F_1 will also still be a model for F_2 (that which holds for at least one element of a subset, must necessarily also hold for at least one element of the comprising set). This concludes the proof sketch of Lemma 5.1.

Definition 5.8 (Derivation in LK)
The notion of 'derivation with reference grammar G in Gentzen's system LK' is defined as in David Isles paper [13]. This derivation system is similar to the BV system from Section 2.4 except that LK does not contain the Induction rule I and that the reference grammars at the two contraction rules and at the Cut rule C are slightly different. These latter changes do not, however, materially change the system from the one described in Section 2.4, so consideration of these points is delayed until Section 5.2.

$$(CL)\ \frac{A_1, A_2, \Gamma \Rightarrow \Delta\ [G]}{A, \Gamma \Rightarrow \Delta\ [G_1]} \qquad \frac{\Gamma \Rightarrow \Delta, A_1, A_2\ [G]}{\Gamma \Rightarrow \Delta, A\ [G_2]}\ (CR)$$

Here A_1, A_2 and A are equiform and all quantifiers of A are distinct from all quantifiers in the derivation to that point. If Qx_{i_1}, Qx_{i_2} and Qx_i are the corresponding positive quantifiers in A_1, A_2 and A and Qy_{j_1}, Qy_{j_2} and Qy_j are the corresponding negative quantifiers in those formulas, then

$$G_1 = G \cup \{x_i \to x_{i_1}, x_i \to x_{i_2}\} \cup \{y_{j_1} \to y_j, y_{j_2} \to y_j\}$$
and
$$G_2 = G \cup \{x_{i_1} \to x_i, x_{i_2} \to x_i\} \cup \{y_j \to y_{j_1}, y_j \to y_{j_2}\}$$

$$\dfrac{\begin{matrix} D_1 & & D_2 \\ \Gamma_1 \Rightarrow \Delta_1, A_1\ [G_1] & & A_2, \Gamma_2 \Rightarrow \Delta_2\ [G_2] \end{matrix}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\ [G]}\ (C)$$

Here all quantifiers in D_1 are distinct from those in D_2 and no parameter in D_1 is an eigenparameter in D_2 (and conversely). A_1 and A_2 are equiform with Qx_{i_1} and Qx_{i_2} the corresponding positive quantifiers and Qy_{j_1} and Qy_{j_2} the corresponding negative quantifiers. Then

$$G = G_1 \cup G_2 \cup \{x_{i_1} \to x_{i_2}\} \cup \{y_{j_2} \to y_{j_1}\}.$$

A_1 and A_2 are called the 'cut formulas'.

Examples of derivations can be found in Chapter 3.

Definition 5.9 (main formula)
1. The new formula introduced in the conclusion of each rule (with the exception of cut) is called the *'main formula'* of the rule. The formula(s) with which it is connected in the premisses of all rules (save weakenings) is (are) called the *'side formula(s)'*.
2. The connective introduced in the main formula of all logical rules is called the *'main connective'* of that rule.

It follows directly from the definitions of the derivation rules that each sequent is rectified, and that any derivation has the 'pure variable property', to wit, no eigenparameter, i.e., eigenvariable (for definition of this term, see definition of $\forall R$ or $\exists L$ in Section 2.4), of a rule occurs anywhere other in a derivation than above that rule. This ensures that mere alphabetic variation does not preclude a derivation from going through. Hence the expressive power of the new system is no less than the traditional one. Note that enlargement of the reference grammar only occurs at quantification rules. Other than that, only cut and contraction change the reference grammar (by identification of formulas).

Definition 5.10 (LK with equality)
If we add equality to the relations, a number of initial sequents become available. This system is called LK_e. Let s, t be terms, f a function letter and R a relation letter.
1. $\Rightarrow s = s$
2. $s_1 = t_1, ..., s_n = t_n \Rightarrow f(s_1, ..., s_n) = f(t_1, ..., t_n)$
3. $s_1 = t_1, ..., s_n = t_n \Rightarrow R(s_1, ..., s_n) \rightarrow R(t_1, ..., t_n)$

Definition 5.11 (compatibility of a BV interpretation)
Let M_r be a BV interpretation for LK_e and D a derivation of $[\Gamma \Rightarrow \Delta; G]$. M_r is said to be compatible with D if $M_r \in G$, i.e., M_r is compatible with G. (See also definition 5.5)

Example 5.4
Let P be a one-place relation letter and c_0 and c_1 individual constants. From now on, reference arrows will be shown in the derivation at the step where they originate, but the entire reference grammar will only be given at the end of the derivation.

$$\frac{P(c_0) \Rightarrow P(c_0)}{\forall x[P(x)] \Rightarrow P(c_0)\ \ [x \rightarrow c_0]} \forall L \qquad (\alpha)$$

$$\frac{P(c_1) \Rightarrow P(c_1)}{\forall z[P(z)] \Rightarrow P(c_1)\ \ [z \rightarrow c_1]} \forall L \qquad (\beta)$$

$$\cfrac{\cfrac{\alpha \qquad \beta}{\forall x[P(x)], \forall z[P(z)] \Rightarrow P(c_0) \wedge P(c_1)} \wedge R}{\forall w[P(w)] \Rightarrow P(c_0) \wedge P(c_1)\ \ [w \rightarrow x, w \rightarrow z]} CL$$

$$G = \{x \rightarrow c_0, z \rightarrow c_1, w \rightarrow x, w \rightarrow z\}$$

A BV interpretation of this derivation would need to have $\{c_0*, c_1*\} \subseteq r(w)$. Hence, given $c_0* \neq c_1*$, $2 \leq |r(w)|$.

In the following derivation, traditionally considered to be a variation of the derivation above, we will see that a BV derivation needs only have $c_0* \in r(x), c_1* \in r(z)$, corresponding to $1 \leq |r(x)|, 1 \leq |r(z)|$. So here we have two traditionally equivalent derivations that nonetheless have different BV interpretations.

$$\frac{P(c_0) \Rightarrow P(c_0)}{\forall x[P(x)] \Rightarrow P(c_0)\ \ [x \to c_0]}\ \forall L \qquad (\alpha)$$

$$\frac{P(c_1) \Rightarrow P(c_1)}{\forall z[P(z)] \Rightarrow P(c_1)\ \ [z \to c_1]}\ \forall L \qquad (\beta)$$

$$\cfrac{\cfrac{\alpha \qquad \beta}{\forall x[P(x)], \forall z[P(z)] \Rightarrow P(c_0) \wedge P(c_1)}\ \wedge R}{\emptyset \Rightarrow \forall x[P(x)] \to (\forall z[P(z)] \to (P(c_0) \wedge P(c_1)))}\ \to R, \to R$$

Let $[D_a, G_a]$ be a subderivation of $[D, G]$. Then there exists a series of subderivations $[D_i, G_i]$, with $i = 1, ..., m$, such that:

1. $[D_1, G_1] = [D_a, G_a]$
2. $[D_m, G_m] = [D, G]$
3. For all i, $[D_i, G_i]$ is an immediate subderivation of $[D_{i+1}, G_{i+1}]$, to wit, the latter consists of the former and one extra derivation step.
4. For all i, $G_i \subseteq G_{i+1}$ (and consequently, $G_a \subseteq G$).

Keeping this in mind, we can now prove

Theorem 5.1 (Soundness)
Let D be a derivation of $[\Gamma \Rightarrow \Delta; G]$. Let M_r be a BV interpretation compatible with D, and $\alpha : \{parameters\} \to |M|$. Then, if $[M_r, \alpha] \models \Gamma$, then $[M_r, \alpha] \models \Delta$.

Proof sketch: The usual inductive proof can be followed. As regards the complications arising from the use of reference grammars, the fact that M_r is compatible with G guarantees closure under substitutions occurring at the quantifier steps, and Lemma 5.1 is used for occurrences of cut or contraction.

Note that in Example 5.1, the BV interpretation M_r, defined by $M = \langle \mathbb{N}; even; f^*, c^* \rangle$ with $c^* = 1$ and $f^*(n) = 2n + 1$ and $r(x) = \{2, 4\}$, is not compatible with the reference grammar $\{x \to f(c)\}$ of the derivation D of $\forall x[E(x)] \to E(f(c)$: $M_r \models \forall x[E(x)]$, but $M_r \not\models E(f(c))$.

5.1.3 The reference grammar of quantifier-normal derivations

In this section it will be proved that, if there is a derivation D of $\Gamma \Rightarrow \Delta$, with all quantifiers in Γ positive, and a BV interpretation M_r compatible with (the reference grammar of) D such that $M_r \models \Gamma$, then there exists a finite BV submodel of M_r compatible with D, say M_r', such that $M_r' \models \Gamma$. In the next section it will be proved that the theorem still holds if the condition on the quantifiers in Γ is omitted.

Definition 5.12 (cycle) Let G be a reference grammar over $F \cup C$.
1. A cycle is a G-derivation of the form $x \rightarrow\rightarrow t(x)$ (or $a \rightarrow\rightarrow t(a)$) where t is a term containing x (or a).
2. G is called *self-embedding* if G contains a cycle $x \rightarrow\rightarrow t(x)$, such that t contains at least one function letter in addition to x.
3. A derivation D is called *self-referential* if the grammar G of D contains a cycle.

Note that the presence of a cycle in the reference grammar of a derivation D might well preclude the existence of a finite BV model compatible with D.

Example 5.5 Let D_1 be the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{F(a,f(b)) \Rightarrow F(a,f(b))}{F(a,f(b)) \Rightarrow F(a,f(b)), F(b,c)}\,WR}{F(a,f(b)) \Rightarrow \exists y[F(a,y)], \exists y_1[F(b,y_1)] \;\; [y \rightarrow f(b), y_1 \rightarrow c]}\,\exists R}{\forall w \forall z[F(w,z)] \Rightarrow \exists y[F(a,y)], \exists y_1[F(b,y_1)] \;\; [w \rightarrow a, z \rightarrow f(b)]}\,\forall L}{\forall w \forall z[F(w,z)] \Rightarrow \forall x \exists y[F(x,y)], \forall x_1 \exists y_1[F(x_1,y_1)] \;\; [a \rightarrow x, b \rightarrow x_1]}\,\forall R}{\forall w \forall z[F(w,z)] \Rightarrow \forall u \exists v[F(u,v)] \;\; [v \rightarrow y, v \rightarrow y_1, x \rightarrow u, x_1 \rightarrow u]}\,CR$$

Note that the reference grammar G_1 of D_1 contains $v \rightarrow y \rightarrow f(b)$ and $b \rightarrow x_1 \rightarrow u$, and hence also $v \rightarrow\rightarrow f(u)$.

Next let D_2 be the following derivation:

$$\dfrac{F(a,b) \Rightarrow F(a,b)}{F(a,b), F(f(b),c) \Rightarrow F(a,b)}\,WL$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F(a,b),F(f(b),c) \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [w_1 \to a, z_1 \to b]}\exists R}{F(a,b),\exists v_2[F(f(b),v_2)] \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [c \to v_2]}\exists L}{F(a,b),\forall u_2\exists v_2[F(u_2,v_2) \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [u_2 \to f(b)]}\forall L}{\exists v_3[F(a,v_3)],\forall u_2\exists v_2[F(u_2,v_2) \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [b \to v_3]}\exists L}{\forall u_3\exists v_3[F(u_3,v_3)],\forall u_2\exists v_2[F(u_2,v_2) \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [u_3 \to a]}\forall L}{\forall u_1\exists v_1[F(u_1,v_1)] \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)\ [u_1 \to u_2, u_1 \to u_3, v_2 \to v_1, v_3 \to v_1]}CL$$

Note that the reference grammar G_2 of D_2 contains the references $u_1 \to u_2 \to f(b) \to f(v_3) \to f(v_1)$, and hence also $u_1 \to\to f(v_1)$.
Finally, we apply Cut to the end sequents of D_1 and D_2 to obtain the sequent $\forall w\forall z[F(w,z)] \Rightarrow \exists w_1\exists z_1[F(w_1,z_1)]$ with the additional references $u \to u_1$ and $v_1 \to v$. Notice that u is a target variable in G_1, while u_1 is a source variable in G_2, and that v is a source variable in G_1, while v_1 is a target variable in G_2. The derivation as a whole (consisting of D_1, D_2 and an application of the Cut rule) is self-referential as its reference grammar contains a cycle:

$$v \underset{G_1}{\to\to} f(u) \underset{\text{cut}}{\to} f(u_1) \underset{G_2}{\to\to} f(f(v_1)) \underset{\text{cut}}{\to} f(f(v).$$

Definition 5.13 (target and source variables)
A variable/parameter is called a *target variable/parameter* in a reference grammar G if it occurs in a term on the right-hand side of a non-trivial production rule; it is called a *source variable/parameter* if it occurs on the left-hand side of such a rule.

Notice that a parameter in a derivation can only be a source parameter if it is an eigenparameter.

Lemma 5.2 Let D be a derivation of $\Gamma \Rightarrow \Delta$ with reference grammar G.
1a) If x is a source variable in G and $\forall x$ occurs in the end sequent $\Gamma \Rightarrow \Delta$ of D, then $\forall x$ occurs either positively in a formula of Γ or negatively in a formula of Δ.
1b) If x is a source variable in G and $\exists x$ occurs in the end sequent $\Gamma \Rightarrow \Delta$ of D, then $\exists x$ occurs either negatively in a formula of Γ or positively in a formula of Δ.
2a) If x is a target variable in G and $\forall x$ occurs in the end sequent $\Gamma \Rightarrow \Delta$

of D, then $\forall x$ occurs either negatively in a formula of Γ or positively in a formula of Δ.
2a) If x is a target variable in G and $\exists x$ occurs in the end sequent $\Gamma \Rightarrow \Delta$ of D, then $\exists x$ occurs either positively in a formula of Γ or negatively in a formula of Δ.

Proof: By induction on the length of the derivation.

$\forall L$ **universal quantification left** **universal quantification right** $\forall R$

$$(\forall L)\ \frac{A[x/t], \Gamma \Rightarrow \Delta\ \ [G]}{\forall x A, \Gamma \Rightarrow \Delta\ \ [G']} \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]\ \ [G]}{\Gamma \Rightarrow \Delta, \forall x A\ \ [G']}\ (\forall R)$$

$$G' = G \cup \{x \to t\} \qquad G' = G \cup \{a \to x\}$$

In rule $(\forall L)$ x is a source variable and $\forall x$ occurs positively in Γ. In rule $(\forall R)$ x is a target variable and $\forall x$ occurs positively in Δ.

$\exists L$ **existential quantification left** **existential quantification right** $\exists R$

$$(\exists L)\ \frac{A[x/a], \Gamma \Rightarrow \Delta\ \ [G]}{\exists x A, \Gamma \Rightarrow \Delta\ \ [G']} \qquad \frac{\Gamma \Rightarrow \Delta, A[x/t]\ \ [G]}{\Gamma \Rightarrow \Delta, \exists x A\ \ [G']}\ (\exists R)$$

$$G' = G \cup \{a \to x\} \qquad G' = G \cup \{x \to t\}$$

In rule $(\exists L)$ x is a target variable and $\exists x$ occurs positively in Γ. In rule $(\exists R)$ x is a source variable and $\exists x$ occurs positively in Δ.

$\neg L$ **negation left** **negation right** $\neg R$

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, A\ \ [G]}{\neg A, \Gamma \Rightarrow \Delta\ \ [G]} \qquad \frac{A, \Gamma \Rightarrow \Delta\ \ [G]}{\Gamma \Rightarrow \Delta, \neg A\ \ [G]}\ (\neg R)$$

For rule $(\neg L)$:
a) if $\forall x$ occurs negatively (resp. positively) in A, then it occurs positively (resp. negatively) in $\neg A, \Gamma$.
b) if $\exists x$ occurs positively (resp. negatively) in A, then it occurs negatively (resp. positively) in $\neg A, \Gamma$.

For rule $(\neg R)$:
a) if $\forall x$ occurs negatively (resp. positively) in A, then it occurs positively (resp. negatively) in $\Delta, \neg A$.
b) if $\exists x$ occurs positively (resp. negatively) in A, then it occurs negatively (resp. positively) in $\Delta, \neg A$.

$\wedge L$ **conjunction left** **conjunction right** $\wedge R$

$$(\wedge L)\ \frac{A, \Gamma \Rightarrow \Delta\ \ [G]}{A \wedge B, \Gamma \Rightarrow \Delta\ \ [G]} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A\ \ [G_1] \qquad \Gamma_2 \Rightarrow \Delta_2, B\ \ [G_2]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \wedge B\ \ [G]}\ (\wedge R)$$

$$G = G_1 \cup G_2$$

For rule $(\wedge L)$: if $\forall x$ or $\exists x$ occurs positively (resp. negatively) in A or in B, then it occurs positively (resp. negatively) in $A \wedge B$.
For rule $(\wedge R)$: if $\forall x$ or $\exists x$ occurs positively (resp. negatively) in A or in B, then it occurs positively (resp. negatively) in $A \wedge B$.

The other derivation rules are dealt with in a similar way. □

For illustration, consider derivation D_1 in Example 5.5. In this example w, z and v are source variables and $\forall w$, $\forall z$ and $\exists v$ occur in the end sequent $\forall w \forall z[F(w,z)] \Rightarrow \forall u \exists v[F(u,v)]$. $\forall w$ and $\forall z$ both occur positively in $\Gamma = \forall w \forall z[F(w,z)]$, while $\exists v$ occurs positively in $\Delta = \forall u \exists v[F(u,v)]$. In the same example u is a target variable and $\forall u$ occurs positively in $\Delta = \forall u \exists v[F(u,v)]$.

Definition 5.14 (normal derivation) A derivation is *normal* if it does not contain an application of the cut rule.

Definition 5.15 (Quantifier-normal derivation) A derivation is called *quantifier-normal* (Q-normal) if all cuts in it have quantifier-free cut formulae.

Notice that the derivations in Section 3.1, 3.2, 3.3 and 3.4 all use quantified formulas in their applications of the Cut rule and hence are not quantifier-normal.

Theorem 5.2 (Gentzen's Normal Form Theorem)
Any derivation $D \vdash [\Gamma \Rightarrow \Delta; G]$ in LK or LK_e can be effectively transformed into a normal, and therefore a fortiori Q-normal, derivation $D' \vdash [\Gamma' \Rightarrow \Delta'; G']$, where Γ' and Δ' are equiform with Γ and Δ respectively.

Proof: The proof is identical to the proofs of the same result given in Takeuti [24], Girard [8] or Kleene [17].

Theorem 5.3
A Q-normal derivation is not self-referential.

Proof: By induction over the given derivation D of $\Gamma \Rightarrow \Delta$ we show that the accompanying reference grammar G does not contain a cycle $x \to\to t(x)$. If G is the empty reference grammar, it clearly does not contain a cycle. Induction hypothesis: suppose that G does not contain a cycle.

In rules $(\forall L)$ and $(\exists R)$, $G' = G \cup \{x \to t\}$, where x is a new variable. So, G' does not contain a cycle.

In rules $(\forall R)$ and $(\exists L)$, $G' = G \cup \{a \to x\}$, where x is a new variable. So, G' does not contain a cycle.

In rules $(\wedge R)$, $(\vee L)$ and $(\to L)$, assume that G_1 and G_2 do not contain a cycle (induction hypothesis). By definition, all quantifiers in the left derivation D_1 are different from those in the right derivation D_2 and no parameter of D_1 is an eigenparameter in D_2 and conversely. Therefore, $G = G_1 \cup G_2$ does not contain a cycle.

In rules (CL) and (CR), the new reference grammars are respectively $G_1 = G \cup \{x_i \to x_{i_1}, x_i \to x_{i_2}\} \cup \{y_{j_1} \to y_j, y_{j_2} \to y_j\}$ and $G_2 = G \cup \{x_{i_1} \to x_i, x_{i_2} \to x_i\} \cup \{y_j \to y_{j_1}, y_j \to y_{j_2}\}$, where Qx_{i_i}, Qx_{i_2} and Qx_i are the corresponding positive quantifiers in A_1, A_2 and A and Qy_{j_i}, Qy_{j_2} and Qy_j are the corresponding negative quantifiers in A_1, A_2 and A. Because all quantifiers in A are distinct from all quantifiers in the derivation up to that point, G_1 and G_2 will not contain a cycle, assuming the induction hypothesis that G does not contain a cycle.

Note that Q-normality means no reference arrows for bounded variables are added by use of the cut rule and that the cut rule is the only rule that allows for reference arrows that connect target and source variables. □

Note that the derivation in example 5.5 is not Q-normal and that it is indeed the application of cut (and the attendant addition of reference arrows) that creates the cycle in the reference grammar. From Theorem 5.3 follows:

Corollary 5.1 If G is the reference grammar of a Q-normal derivation and q is a variable or parameter, then $G(q)$, the set of terminal strings derivable from q, is finite.

Definition 5.16 Suppose D is a derivation of $\Gamma \Rightarrow \Delta$ with reference grammar G. D is said to be *coherent* if there is a BV interpretation M_r compatible with (the reference grammar G of) D such that $M_r \models \Gamma$ (and hence $M_r \models \Delta$). Such an M_r is called a *soul* of D.

Corollary 5.2 Let Γ be a set of formulas all of whose quantifiers are positive. If Γ has a Tarski model M, then any Q-normal derivation D of $\Gamma \Rightarrow \Delta$ has a finite soul that is a BV substructure of M.

Proof: Assume D is a Q-normal derivation of $\Gamma \Rightarrow \Delta$ with all quantifiers Qx_i in Γ positive. That is - see Definition 5.7 - $Qx_i = \forall x_i$ and it occurs positively in Γ or $Qx_i = \exists x_i$ and it occurs negatively in Γ. So, from Lemma 5.2 it follows that x_i can at most occur as source variable in the reference grammar G of D.

Let Qz_k and Qw_j respectively be the positive and negative quantifiers in Δ. That is - see Definition 5.7 - $Qz_k = \forall z_k$ and it occurs positively in Δ or $Qz_k = \exists z_k$ and it occurs negatively in Δ. So, again from Lemma 5.2 it follows that z_k can at most occur as target variable in the reference grammar G of D. Similarly, Qw_j being a negative quantifier of Δ, $Qw_j = \forall w_j$ and it occurs negatively in Δ or $Qw_j = \exists w_j$ and it occurs positively in Δ and hence, by Lemma 5.2, w_j can at most occur as source variable in G.

Let q be any parameter or variable in D which is a source in G. By Theorem 5.3 it then follows that $G(q)$ (the set of terminal strings derivable from q in G) consists of a finite set of terms of the form $t(p_1, \ldots, p_n)$, with p_i a non-source parameter or variable of D. Let M be a Tarski model for Γ. Then a finite BV interpretation M'_r, compatible with G, such that $M'_r \models \Gamma$ (that is to say: M'_r is a finite soul of D) can be constructed. First, define for each non-source variable or parameter p_i the range $r(p_i)$ to be an arbitrary finite subset of the domain of M. Next for each derivation $q \rightarrow\rightarrow t(p_1, \ldots, p_n)$ in G, take for $r(q)$ the minimal subset of the domain of M such that $t^*(r(p_1), \ldots, r(p_n)) \subseteq r(q)$, where t^* is the interpretation of t in M (and taking care not to expand $r(q)$ beyond what is necessary to meet these conditions, which will keep it finite). The method of construction also guarantees that M'_r is a substructure of M. $\Box$

Example 5.6 We shall illustrate the proof of Corollary 5.2 by an example. Let $\Gamma = \forall x_1[P(x_1)] \vee \forall x_2[Q(x_2)]$. And let $M = \langle \mathbb{N}; \geq 0, \text{is even}\rangle$ be a Tarski model of Γ. Let us consider the following Q-normal derivation D of $\Gamma \Rightarrow \Delta$ with $\Delta = \forall z[P(z) \vee Q(z)]$.

Derivation D_1

$$\frac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a) \vee Q(a)} \vee R$$

$$\dfrac{}{\forall x_1[P(x_1)] \Rightarrow P(a) \vee Q(a) \;\; [x_1 \to a]} \forall L$$

$$\dfrac{}{\forall x_1[P(x_1)] \Rightarrow \forall x_3[P(x_3) \vee Q(x_3)] \;\; [a \to x_3]} \forall R$$

Derivation D_2

$$\dfrac{Q(b) \Rightarrow Q(b)}{Q(b) \Rightarrow P(b) \vee Q(b)} \vee R$$

$$\dfrac{}{\forall x_2[Q(x_2)] \Rightarrow P(b) \vee Q(b) \;\; [x_2 \to b]} \forall L$$

$$\dfrac{}{\forall x_2[Q(x_2)] \Rightarrow \forall x_4[P(x_4) \vee Q(x_4)] \;\; [b \to x_4]} \forall R$$

Finally, let D be the derivation

$$\dfrac{D_1 \qquad\qquad D_2}{\forall x_1[P(x_1)] \vee \forall x_2[Q(x_2)] \Rightarrow \forall x_3[P(x_3) \vee Q(x_3)], \forall x_4[P(x_4) \vee Q(x_4)]} \vee L$$

$$\dfrac{}{\forall x_1[P(x_1)] \vee \forall x_2[Q(x_2)] \Rightarrow \forall z[P(z) \vee Q(z)] \;\; [x_3 \to z, x_4 \to z]} CR$$

The reference grammar G of D contains the following references:
$x_1 \to a \to x_3 \to z$ and $x_2 \to b \to x_4 \to z$.
$\forall x_1$ and $\forall x_2$ are the positive quantifiers in Γ, while $\forall z$ is the positive quantifier in Δ. According to Lemma 5.2, x_1 and x_2 can at most occur as source variables in G and z can at most occur as target variable in G.

$G(x_1) = \{z\}$ and also $G(x_2) = \{z\}$. We first assign to z a finite subset $r(z)$ of $|M| = \mathbb{N}$, say $r(z) = \{1, 2\}$. Next, taking into account that $x_1 \to a \to x_3 \to z$ and $x_2 \to b \to x_4 \to z$. we define $r(x_1) = r(a) = r(x_3) = r(z) = \{1, 2\}$ and $r(x_2) = r(b) = r(x_4) = r(z) = \{1, 2\}$. Then M'_r is a finite soul of D which is a BV substructure of M.

5.1.4 Derivations with finite souls

Let D be a Q-normal derivation of $\Gamma \Rightarrow \Delta$ with soul M_r. As the following example shows, it is not always the case that D has a finite soul M'_r that is a BV-substructure of M.

Example 5.7

$$\dfrac{R(a,b) \Rightarrow R(a,b) \qquad R(b,d) \Rightarrow R(b,d)}{R(a,b), R(b,d) \Rightarrow R(a,b) \wedge R(b,d)} \wedge R$$

$$\dfrac{}{R(a,b), R(b,d) \Rightarrow \exists z \exists w[R(a,z) \wedge R(z,w)] \;\; [w \to d, z \to b]} \exists R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R(a,b), \exists y_2[R(b,y_2)] \Rightarrow \exists z \exists w[R(a,z) \wedge R(z,w)] \;\; [d \rightarrow y_2]}\exists L}{R(a,b), \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \wedge R(z,w)] \;\; [x_2 \rightarrow b]}\forall L}{\exists y_1[R(a,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \wedge R(z,w)] \;\; [b \rightarrow y_1]}\exists L}{\forall x_1 \exists y_1[R(x_1,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \wedge R(z,w)] \;\; [x_1 \rightarrow a]}\forall L}{\forall x_1 \exists y_1[R(x_1,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \forall v \exists z \exists w[R(v,z) \wedge R(z,w)] \;\; a \rightarrow v]}\forall R}{\forall x \exists y[R(x,y)] \Rightarrow \forall v \exists z \exists w[R(v,z) \wedge R(z,w)] \;\; [x \rightarrow x_1, x \rightarrow x_2, y_1 \rightarrow y, y_2 \rightarrow y]}CL$$

The reference grammar contains $x \rightarrow x_2 \rightarrow b \rightarrow y_1 \rightarrow y$ and hence also $x \rightarrow\rightarrow y$. Now suppose we take Tarski model M with as domain the natural numbers and R interpreted as 'less than'. It is now impossible to construct a finite BV interpretation M'_r which is a submodel of M and a soul for D, because of the presence of $x \rightarrow\rightarrow y$ in the reference grammar. Suppose such an M'_r could be found, then, being a soul for D, it would need to meet the condition $M'_r \models \forall x \exists y[R(x,y)]$. However, from $x \rightarrow\rightarrow y \in G$ it follows that $r(y) \subseteq r(x)$. Now, the only way both conditions can be met at the same time is if $r(y) = r(x) = \mathbb{N}$ ($r(y) \subset r(x)$ would make it impossible to meet the condition $M'_r \models \forall x \exists y[R(x,y)]$, as would $r(y) = r(x)$ with both of them finite). As we shall see, the reference arrows accompanying the Contraction rule Left (CL) force a soul of D to be infinite.

Definition 5.17 (semantic cycle)
Let Γ be a sequence of formulas and G a reference grammar. Let $< Qx_i, Qy_i >$, with $i = 1, \ldots, n$, be pairs of positive and negative quantifiers each from the same formula in Γ such that Qy_i lies within the scope of Qx_i.

G has a *semantic cycle* in Γ iff G includes derivations

$$x_1 \rightarrow\rightarrow f_1(\ldots y_2 \ldots), x_2 \rightarrow\rightarrow f_2(\ldots y_3 \ldots), \ldots, x_n \rightarrow\rightarrow f_n(\ldots y_1 \ldots).$$

For illustration, in Example 5.7, $\Gamma = \{\forall x \exists y[R(x,y)]\}$ and let G be the reference grammar of the derivation. Then $< \forall x, \exists y >$ is a pair of positive and negative quantifiers from the same formula in Γ such that $\exists y$ lies within the scope of $\forall x$. Because G includes the derivation $x \rightarrow\rightarrow y$, G is said to have a semantic cycle in Γ.

To avoid semantic cycles, we must put more restrictions on Q-normal derivations.

Definition 5.18 (linear derivation)
Let LK_e^* be the system obtained from LK_e (see 5.10) by:

1. Adding the following derivation rules:
   Conjunction Left   Conjunction Right

$$\frac{A_1, A_2, \Gamma \Rightarrow \Delta\ [G]}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta\ [G]}\, CJL \qquad \text{and} \qquad \frac{\Gamma \Rightarrow A_1, A_2, \Delta\ [G]}{\Gamma \Rightarrow A_1 \wedge A_2, \Delta\ [G]}\, CJR$$

   under the condition that A_1 and A_2 are equiform.

2. Use of the contraction rules is limited to cases where the main formulas are quantifier-free.

Derivations in LK_e^* are called *linear derivations.*

If in Example 5.7 we replace the CL step (which is no longer allowed in LK^*) by CJL, we obtain a linear derivation of the sequent

$$\forall x_1 \exists y_1[R(x_1, y_1)] \wedge \forall x_2 \exists y_2[R(x_2, y_2)] \Rightarrow \forall v \exists z \exists w[R(v, z) \wedge R(z, w)]\ (*)$$

The reference grammar of this linear derivation equals the reference grammar of the original derivation minus the reference arrows resulting from the use of CL. Since the arrows generated by CL were crucial in obtaining the semantic cycle $x \rightarrow\rightarrow y$ in $\Gamma = \{\forall x \exists y[R(x, y)]\}$, the reference grammar of the linear derivation contains no semantic cycle in the antecedent of (*).

We will now show that no Q-normal linear derivation of $\Gamma \Rightarrow \Delta$ contains a semantic cycle in Γ; see Corollary 5.3. First it may be noted that in such derivations any quantifier may be traced back through the derivation to its originating quantifier rule occurrence (cut and contraction are the only rules that may interrupt the backtracking of a quantifier, and ex hypothesi both are restricted here to quantifier-free formulas).

Lemma 5.3
Let D be a Q-normal linear derivation of $\Gamma \Rightarrow \Delta$, and suppose the quantifiers Qx and Qy occur in $\Gamma \Rightarrow \Delta$. If the reference grammar G_D of D contains a derivation $x \rightarrow\rightarrow f(\ldots y \ldots)$, then the quantifier rule where Qx is introduced occurs above the quantifier rule where Qy is introduced. (We shall say that *Qx occurs above Qy in D*).

Proof: As y is a target variable, Qy is introduced either by $\forall R$ or by $\exists L$, in either case adding an arrow $a \to y$ to G_D. Note that a is an eigenparameter, and hence does not appear in D below the occurrence of this rule.
Since x is a source variable, Qx is introduced either by $\forall L$ or by $\exists R$, in either case adding an arrow $x \to t(b)$ (with $t(b)$ a term containing at least parameter b) to G_D.
Now assume Qx does not occur above Qy. It follows that $b \neq a$ (as a is an eigenparameter). Ex hypothesi $x \to\to f(\ldots y \ldots)$ occurs in G_D, so it is necessary that there be (at least) arrows $b \to z$ and $z \to s$ (with s a term that would give access to y) in G_D. However, Qz is introduced in exactly one place in D, so that it is impossible for both $b \to z$ and $z \to s$ to occur within G_D. Contradiction. Hence Qx must occur above Qy. □

Corollary 5.3
Let D be a Q-normal linear derivation of $\Gamma \Rightarrow \Delta$. Then G_D is not self-referential and no semantic cycles occur in Γ.

Proof: Absence of self-reference follows from Theorem 5.3, which can trivially be seen to hold also for linear derivations.
Now suppose G_D contained a semantic cycle. Then the formulas of Γ would contain pairs of quantifiers $< Qx_i, Qy_i >$, $i = 1, \ldots, n$, with Qx_i positive, Qy_i negative and within the scope of Qx_i, as well as derivations $x_1 \to\to f_1(\ldots y_2 \ldots), x_2 \to\to f_2(\ldots y_3 \ldots), \ldots, x_n \to\to f_n(\ldots y_1 \ldots)$ belonging to G_D. From the latter and Lemma 5.3 it follows that Qx_i, $i = 1, \ldots, n-1$ occurs above Qy_{i+1} and that Qx_n occurs above Qy_1. We also know that, for all i, Qy_i occurs above Qx_i (since the former falls within the scope of the latter and must hence be introduced earlier). Taken together, this means that there would have to be a branch of D in which there occurs the following sequence of quantifier introductions (in ascending order): Qx_1, Qy_2, Qx_2, $\ldots$, Qy_n, Qx_n, Qy_1, Qx_1. This is impossible, as Qx_1 would be introduced twice. Therefore G_D cannot contain a semantic cycle. □

Definition 5.19 ($Qx_2 <_G Qx_1$)
Let $\Gamma \Rightarrow \Delta$ be a sequent and G a reference grammar in the same language. Let Qx_1 and Qx_2 be two positive quantifiers from (possibly different) formulas of Γ. We then say $Qx_2 <_G Qx_1$ if:
1. Qx_2 falls within the scope of Qx_1 or
2. there is a G-derivation of the form $x_2 \to\to f(\ldots y_1 \ldots)$, where Qy_1 is a negative quantifier within the scope of Qx_1.

Furthermore, $\ll_G$ is the transitive closure of $<_G$ (G may be omitted if it is obvious from the context).

Example 5.8 Consider the following derivation of

$$\forall x_1 \exists y_1[R(x_1, y_1)], \forall x_2 \exists y_2[R(x_2, y_2)] \Rightarrow \forall v \exists z \exists w[R(v, z) \land R(z, w)]$$

with reference grammar

$$G = \{w \to d, z \to b, d \to y_2, x_2 \to b, b \to y_1, x_1 \to a, a \to v\}.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R(a,b) \Rightarrow R(a,b) \qquad R(b,d) \Rightarrow R(b,d)}{R(a,b), R(b,d) \Rightarrow R(a,b) \land R(b,d)}\land R}{R(a,b), R(b,d) \Rightarrow \exists z \exists w[R(a,z) \land R(z,w)]\ \ [w \to d, z \to b]}\exists R}{R(a,b), \exists y_2[R(b,y_2)] \Rightarrow \exists z \exists w[R(a,z) \land R(z,w)]\ \ [d \to y_2]}\exists L}{R(a,b), \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \land R(z,w)]\ \ [x_2 \to b]}\forall L}{\exists y_1[R(a,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \land R(z,w)]\ \ [b \to y_1]}\exists L}{\forall x_1 \exists y_1[R(x_1,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \exists z \exists w[R(a,z) \land R(z,w)]\ \ [x_1 \to a]}\forall L}{\forall x_1 \exists y_1[R(x_1,y_1)], \forall x_2 \exists y_2[R(x_2,y_2)] \Rightarrow \forall v \exists z \exists w[R(v,z) \land R(z,w)]\ \ [a \to v]}\forall R$$

$\forall x_1$ and $\forall x_2$ are two positive quantifiers from different formulas in $\Gamma = \forall x_1 \exists y_1[R(x_1, y_1)], \forall x_2 \exists y_2[R(x_2, y_2)]$.

As one can see in the derivation, $\forall x_2$ is introduced before $\forall x_1$. That is, $\forall x_2 <_G \forall x_1$, because there is a G-derivation $x_2 \to b \to y_1$ with $\exists y_1$ a negative quantifier within the scope of $\forall x_1$.

Notice that one more application of (CL) would yield a LK derivation of $\forall x \exists y[R(x, y)] \Rightarrow \forall v \exists z \exists w[R(v, z) \land R(z, w)]$ with the extra references $x \to x_1, x \to x_2, y_1 \to y, y_2 \to y$ and hence, together with $x_2 \to b$ and $b \to y_1$, would yield a semantic cycle $x \to x_2 \to b \to y_1 \to y$ in $\Gamma = \forall x \exists y[R(x, y)]$.

Theorem 5.4
If D is a Q-normal linear derivation of $\Gamma \Rightarrow \Delta$ with reference grammar G , then $\ll_G$ is irreflexive.

Proof sketch: By reductio ad absurdum. If there are quantifiers for which $<<$ is reflexive, then this will give rise to a semantic cycle (as could be expected). This in turn contradicts Corollary 5.3.
The full proof, which turns out to be somewhat laborious, can be found in prof. Isles' article [13].

Definition 5.20 ($d(Qx)$)
1. For any positive quantifier Qx_i in Γ, given a Q-normal linear derivation of $\Gamma \Rightarrow \Delta$, $d(Qx_i)$ is the length of the longest ascending $<<$-chain starting at Qx_i. By definition, $d(Qx_i) \geq 1$.
2. For any negative quantifier Qy_j in Γ, given a Q-normal linear derivation of $\Gamma \Rightarrow \Delta$, $d(Qy_j)$ is either $d(Qx)$, where Qx the closest universal quantifier within whose scope Qy_j lies, or 0 if Qy_j is not within the scope of any Qx_i.

Note that this definition uses the fact that $<<$ is irreflexive.

Example 5.9 In Example 5.8 $d(\forall x_1)$ is the length of the longest ascending chain $\forall x_2 <_G \forall x_1$ starting at $\forall x_1$, that is, $d(\forall x_1) = 2$. Similarly, $d(\forall x_2) = 1$.
For the negative quantifier $\exists y_2$ in Γ, by definition, $d(\exists y_2) = d(\forall x_2) = 1$ and $d(\exists y_1) = d(\forall x_1) = 2$.

Theorem 5.5 (Finite Analog of Löwenheim-Skolem)
If a sequence of formulas Γ has a Tarski model M, then any Q-normal linear derivation D of $\Gamma \Rightarrow \Delta$ has a finite soul M'_r which is a BV substructure of M.

Notice that the derivations given in Chapter 3, proving the totality of Addition, Multiplication and Exponentiation, although linear, are not Q-normal since they use quantified formulas in the application of the Cut rule.

Proof: Let Γ and Δ contain only sentences. Let $\{Qx_i\}$ and $\{Qy_j\}$ be respectively the sets of positive and negative quantifiers in Γ. Let $\{Qz_k\}$ and $\{Qw_l\}$ be respectively the sets of positive and negative quantifiers in Δ. It follows from lemma 5.2 that in G_D, $\{z_k\}$ and $\{y_j\}$ are not source variables, and $\{x_i\}$ and $\{w_l\}$ are not target variables. We also know G_D is free from semantic cycles (corollary 5.3 and the fact that D is a linear Q-normal derivation). So, for q any source variable or parameter in G_D, we know that $G_D(q)$ is finite and consists of pseudo-terms $t(z_1, \ldots, z_k, y_1, \ldots, y_j, \rho_1, \ldots, \rho_m)$ (with each ρ_i being a non-source variable or parameter of D not occurring in $\Gamma \Rightarrow \Delta$). All of the variables and parameters of these pseudo-terms are included in the set $\{z_k\} \cup \{y_j\} \cup \{\rho_1, \dots, \rho_m\}$.

Now let M be a Tarski model of Γ. If we assign finite subsets $\{r(z_k)\}$, $\{r(y_j)\}$, $\{r(\rho_1), \dots, r(\rho_m)\}$ of $|M|$ to $z_k, y_l, \rho_1, \dots, \rho_m$ respectively, we shall determine finite ranges for the other variables and parameters in D to obtain a finite BV interpretation M'_r which is a BV substructure of M and compatible with the reference grammar of D.

We now choose the ranges for $\{z_k\} \cup \{y_j\} \cup \{\rho_m\}$ in such a way that $M'_r \models \Gamma$. For each $v \in \{z_k\} \cup \{\rho_m\}$ let $r(v) \subseteq |M|$ be finite.

We now define $r(v)$, $v \in \{x_i\} \cup \{y_j\}$ by induction on $d(Qv)$.

Base: $d(Qv) = 0$. Then by definition 5.20, $v = y$ and Qy is not within the scope of any Qx_i. Let $r(v)$ be any finite subset of $|M|$.

Step: $d(Qv) = n + 1$.

Case 1: $v = x_i$

Suppose $t \in G_D(x_i)$. If any variable y_j occurs in t, we know that $d(Qy_j) \leq n < d(Qx_i)$. For if not, then y_j would be within the scope of some Qx^*, with $n + 1 \leq d(Qx^*)$. But then, by the definition of $<<$ and the fact that $t \in G_D(x_i)$, $n + 1 \leq d(Qx^*) < d(Qx_i)$, which would contradict the induction step assumption. Thus, if y_j occurs in any $t \in G_D(x_i)$, $r(y_j)$ has already been defined as a finite set. By $|t|$ we denote the value of t in M.

$$\text{Let } r(x_i) = \{|t| \mid t \in G_D(x_i) \\ \text{as } \{z_k\}, \{y_j\}, \{\rho_m\} \text{ taking values in their respective domains}\}.$$

Case 2: $v = y_j$

Then we know that for all positive Qx_i that have Qy_j within their scope, $r(x_i)$ has been defined. Assume without loss of generality that $\{Qx_1, ..., Qx_n\}$ are these universal quantifier occurrences. Then we can use the fact that $M \models \Gamma$ and the appropriate Skolem-function s_j to define

$$r(y_j) = \{s_j(p_1, ..., p_n) | p_i \in r(x_i), i = 1, ..., n\}.$$

Finally, because all of $r(z)$, $r(y)$ and $r(\rho)$ have already been defined, we let $r(w_l) :=$

$\{|t| \mid t \in G_D(w_l)$ as $\{z_i\}, \{y_j\}, \{\rho_n\}$ take values in their respective ranges$\}$.

This defines the function r for M'_r. $\square$

Example 5.10 We shall illustrate the proof of Theorem 5.5 for the following Q-normal linear derivation D of

$$\forall x_2 \exists y_2[R(x_2, y_2)] \wedge \forall x_1 \exists y_1[R(x_1, y_1)] \Rightarrow \forall z \exists u \exists w[R(z, u) \wedge R(u, w)]$$

with reference grammar

$$G = \{w \to d,\ u \to b,\ d \to y_1,\ x_1 \to b,\ b \to y_2,\ x_2 \to a,\ a \to z\}.$$

$$\dfrac{R(a,b) \Rightarrow R(a,b) \qquad R(b,d) \Rightarrow R(b,d)}{R(a,b), R(b,d) \Rightarrow R(a,b) \wedge R(b,d)}\ \wedge R$$

$$\overline{R(a,b), R(b,d) \Rightarrow \exists u \exists w[R(a,u) \wedge R(u,w)] \ \ [w \to d, u \to b]}\ \exists R$$

$$\overline{R(a,b), \exists y_1[R(b,y_1)] \Rightarrow \exists u \exists w[R(a,u) \wedge R(u,w)] \ \ [d \to y_1]}\ \exists L$$

$$\overline{R(a,b), \forall x_1 \exists y_1[R(x_1,y_1)] \Rightarrow \exists u \exists w[R(a,u) \wedge R(u,w)] \ \ [x_1 \to b]}\ \forall L$$

$$\overline{\exists y_2[R(a,y_2)], \forall x_1 \exists y_1 R(x_1,y_1)] \Rightarrow \exists u \exists w[R(a,u) \wedge R(u,w)] \ \ [b \to y_2]}\ \exists L$$

$$\overline{\forall x_2 \exists y_2[R(x_2,y_2)], \forall x_1 \exists y_1 R(x_1,y_1)] \Rightarrow \exists u \exists w[R(a,u) \wedge R(u,w)] \ \ [x_2 \to a]}\ \forall L$$

$$\overline{\forall x_2 \exists y_2[R(x_2,y_2)], \forall x_1 \exists y_1[R(x_1,y_1)] \Rightarrow \forall z \exists u \exists w[R(z,u) \wedge R(u,w)] \ \ [a \to z]}\ \forall R$$

$$\overline{\forall x_2 \exists y_2[R(x_2,y_2)] \wedge \forall x_1 \exists y_1[R(x_1,y_1)] \Rightarrow \forall z \exists u \exists w[R(z,u) \wedge R(u,w)]}\ CJL$$

$\forall x_1$ and $\forall x_2$ are two positive quantifiers in $\Gamma = \forall x_2 \exists y_2[R(x_2, y_2)] \wedge \forall x_1 \exists y_1[R(x_1, y_1)]$.

As one can see in the derivation, $\forall x_1$ is introduced before $\forall x_2$. That is, $\forall x_1 <_G \forall x_2$, because there is a G-derivation $x_1 \to b \to y_2$ with $\exists y_2$ a negative quantifier within the scope of $\forall x_2$.

$d(\forall x)$ is by definition the longest ascending $<$-chain starting at $\forall x$. So, $d(\forall x_1) = 1$ and $d(\forall x_2) = 2$, since $\forall x_1 <_G \forall x_2$.

$\forall x_1$ and $\forall x_2$ are the positive quantifiers in Γ, while $\exists y_1$ and $\exists y_2$ are the negative quantifiers in Γ. $\forall z$ is the positive quantifier in Δ, while $\exists u$ and $\exists w$ are the negative quantfiers in Δ. So, from Lemma 5.2 it follows that z, y_1 and y_2 are not source variables and that x_1, x_2, u and w are not target variables.

Because derivation D is Q-normal, we know and see that the reference grammar G_D of D is cycle-free and if q is any source variable or parameter in G_D, then $G_D(q)$, i.e., the set of terminal strings derivable from q, is finite. In our case, we have $x_1 \to b \to y_2$, $x_2 \to a \to z$, $u \to b \to y_2$ and $w \to d \to y_1$. So, $G_D(x_1) = \{y_2\}$, $G_D(x_2) = \{z\}$, $G_D(u) = \{y_2\}$ and $G_D(w) = \{y_1\}$.

a, b and d are parameters not occurring in the conclusion of D.

Let $M = \langle \mathbb{N}, < \rangle$ be a model of Γ. We have to define $r(x_1)$, $r(x_2)$, $r(u)$ and $r(w)$ in such a way that $M'_r \models \forall x_2 \exists y_2[R(x_2, y_2)] \wedge \forall x_1 \exists y_1[R(x_1, y_1)]$ and $M'_r \models \forall z \exists u \exists w[R(z, u) \wedge R(u, w)]$.

We start with assigning an arbitrary finite subset of $\mathbb{N}$ to z, say $r(z) = \{1, 2, 3\}$. Because $a \to z$, we choose $r(a) = \{1, 2, 3\}$ too. Next we define $r(x_2)$, $r(y_2)$, $r(x_1)$ and $r(y_1)$ in such a way that M'_r is a finite BV interpretation which is compatible with the reference grammar G_D of D and $M'_r \models \forall x_2 \exists y_2[R(x_2, y_2)] \wedge \forall x_1 \exists y_1[R(x_1, y_1)]$.

$G_D(x_2) = \{z\}$ and $r(z) = \{1, 2, 3\}$ has already been defined. So, because of $x_2 \to a \to z$, we define $r(x_2) = r(a) = r(z) = \{1, 2, 3\}$.

Next we determine $r(y_2)$. We have assumed that $M \models \forall x_2 \exists y_2[R(x_2, y_2)]$. So, there is a Skolem function s such that $M \models \forall x_2[R(x_2, s(x_2))]$, say $s(1) = 2$, $s(2) = 3$ and $s(3) = 4$. Therefore we choose $r(y_2) = \{2, 3, 4\}$. Because $u \to b \to y_2$, we choose $r(u) = r(b) = r(y_2) = \{2, 3, 4\}$.

Next we determine $r(x_1)$. $G_D(x_1) = \{y_2\}$ and $r(y_2) = \{2, 3, 4\}$ has already been defined. So, because of $x_1 \to b \to y_2$, we define $r(x_1) = \{2, 3, 4\}$.

Next we determine $r(y_1)$. We have assumed that $M \models \forall x_1 \exists y_1[R(x_1, y_1)]$. So, there is a Skolem function s' such that $M \models \forall x_1[R(x_1, s'(x_1))]$, say $s'(2) = 3$, $s'(3) = 4$ and $s'(4) = 5$. Therefore we choose $r(y_1) = \{3, 4, 5\}$. Because $w \to d \to y_1$, we choose $r(w) = r(d) = r(y_1) = \{3, 4, 5\}$.

Clearly, given model $M = \langle \mathbb{N}, < \rangle$ of $\Gamma = \forall x_2 \exists y_2[R(x_2, y_2)] \wedge \forall x_1 \exists y_1[R(x_1, y_1)]$, and given a Q-normal linear derivation D of $\Gamma \Rightarrow \Delta$, M'_r is a finite soul of D, i.e., M'_r is a BV interpretation compatible with D such that $M'_r \models \Gamma$ and $M'_r \models \Delta$ with $\Delta = \forall z \exists u \exists w[R(z, u) \wedge R(u, w)]$.

5.2 Lőwenheim-Skolem in BV

The primary differences between the system considered in this thesis and the system considered in article [13] by prof. Isles are the absence of Induction in the latter and the notion of functional numerical domains in the former. The implications of these distinctions need to be explored.

5.2.1 Induction

There are some preliminary remarks to be made. Given the large impact of the Induction rule on the reference grammar, identification of domains of bound variables, introduction of a new variable (y) that has N as domain, necessitating that the parameter (a) in the induction formulae has the domain of y as a subset, it seems unlikely that the results obtained in prof. Isles' article will continue to hold good under the introduction of the Induction rule. Particularly the fact that y has N as domain and that $a \to y$ figures in the reference grammar seems problematic with respect to prof. Isles' finitistic results. However, the goal of this thesis was not finitism, but a consideration of what a purely syntactic logic would look like. So if the finitist viewpoint is abandoned, it seems that most of the results in the article by prof. Isles, albeit without reference to the existence of finite models but merely concerning the existence of models (finite or infinite), could still apply to the BV system under consideration in this thesis.

This is all the more likely given the Cut-elimination proof that has been given for the BV system minus Induction (see Section 4.1), which means that Q-normality is guaranteed in all cases except where Cut is applied to a formula supplied by Induction, and does not, therefore, pose a restriction on the theorems that can be proved, provided these cases (of Cut applied to a formula supplied by Induction) can be accounted for. This alone would guarantee the absence of self-referential reference grammars (see Theorem 5.3). Therefore, the only extra condition used in Isles' article [13] is linearity of derivation (see Definition 5.18), to guarantee the absence of semantic cycles. Given that linearity of derivation does not preclude the proof of any theorem, it merely necessitates the conjunctive joining of equiform formulae rather than the reduction of these formulae to one equiform formula, this does not seem to be a serious limitation at all, all the more so as it only applies to formulae that contain quantifiers (all others can still be eliminated by use of the contraction rule). And since the new conjunction rules for linearity do not add to the reference grammar, there is no possibility that they will cause inconsistency in the assignment of domains to the variables. Thus it seems that the BV system might meet all the criteria that are required in prof. Isles' article. And since Induction will only play a part in the finitistic aspect of his results, they should be applicable to the BV system if this consideration of finitism is removed.

There is however still the matter that the definition of the LK system considered by prof. Isles differs slightly from BV at the Cut and Contraction rules (particularly, in the way these add to the reference grammar). In BV we have:

Cut rule (C)
$$\frac{\Gamma \Rightarrow \Delta, A_1 \;\; [G_1] \qquad A_2, \Theta \Rightarrow \Sigma \;\; [G_2]}{\Gamma, \Theta \Rightarrow \Delta, \Sigma \;\; [G_1 \cup G_2 \cup \{[x_{i_1}, x_{i_2}]\}]}$$

Here, A_1 and A_2 are equiform, x_{i_1} is the set of bound variables in A_1, x_{i_2} is the set of bound variables in A_2. All bound variables in one subderivation are distinct from all of those in the other and no free variable in one subderivation is an eigenparameter in the other.

In LK the reference grammar is added to the derivation in a different way. Using x for the positive and y for the negative occurrences of variables (with the appropriate subscripts added), the new reference grammar becomes $G_1 \cup G_2 \cup \{x_{i_1} \to x_{i_2}\} \cup \{y_{j_2} \to y_{j_1}\}$. This omits the inverse arrows that would have been present in the BV version.

Similarly for contraction (keeping the naming conventions the same):

Contraction
$$\text{(CL)} \frac{A_1, A_2, \Gamma \Rightarrow \Delta \;\; [G]}{A, \Gamma \Rightarrow \Delta \;\; [G \cup \{[x_i, x_{i_1}, x_{i_2}]\}]} \qquad \text{(CR)} \frac{\Gamma \Rightarrow \Delta, A_1, A_2 \;\; [G]}{\Gamma \Rightarrow \Delta, A \;\; [G \cup \{[x_i, x_{i_1}, x_{i_2}]\}]}$$

A_1, A_2 and A are equiform and all bound variables in A are distinct from all bound variables in the derivation to this point.

For CL the reference grammar is $G \cup \{x_i \to x_{i_1}, x_i \to x_{i_2}\} \cup \{y_{j_1} \to y_j, y_{j_2} \to y_j\}$
For CR the reference grammar is $G \cup \{x_{i_1} \to x_i, x_{i_2} \to x_i\} \cup \{y_j \to y_{j_1}, y_j \to y_{j_2}\}$
Here also the inverse arrows that would have been present in BV are omitted.

The extra reference arrows in the BV system do not seem to pose a problem for the arguments and proofs given by prof. Isles in his article [13]. BV equates the domains of all bound variables, in LK they are not equated but subsumed according to their status as positively or negatively occurring. The extra arrows, however, do not create conditions of possible inconsistency (given the equiformity of the formulae in question), nor do they affect the ‘backtracking’ occasionally used by prof. Isles in his proofs (some new ‘horizontal’ links are created, but the ‘vertical’ linkage is not affected).

Therefore the difference between the systems seems to be superficial, provided the cases where Cut is applied to a formula supplied by Induction can be accounted for, and gives no rise to a need to assume the results no longer hold. But let us consider the system BV_{pn}, which is BV with the derivation rules adding to the reference grammar according to the distinction between positive and negative quantifiers (instead of equating domains, as happens in BV). So BV_{pn} is LK with Induction added to it.

Theorem 5.6 (Lőwenheim-Skolem theorem for BV_{pn})
If a sequence of formulas Γ has a Tarski model M, then any valid linear derivation D of $\Gamma \Rightarrow \Delta$ in BV_{pn} has a denumerable soul M'_r, provided that all induction formulas F are quantifier-free.

Notice that the denumerable soul M'_r in general will not be a substructure of the original model M, because it may contain the set $N = \{0, s0, ss0, \ldots\}$.

Proof: Note that the only difference between LK and BV_{pn} is the presence of the Induction rule. If the Induction rule does not invalidate the proofs given by prof. Isles, then we have a proof of this theorem. Induction would introduce a denumerable domain N, but all other domains would still be either finite or, at most, if so required by the reference grammar, denumerable.

Note that the Induction rule does not remove a quantified variable from the derivation, only cut and contraction rules can do this. Therefore the Induction rule does not present a problem for the 'backtracking' of quantification variables.

Since Cut-elimination has been proved for BV without Induction (Section 4.1), Q-normality can be guaranteed for any derivation without Induction. Linearity removes Contraction from the derivation, hence there are no derivation rules that can remove a quantification variable, save Cut applied to a formula supplied by Induction. Let us now consider these cases.

(**Induction rule (I)**)

$$\frac{F_1[a], \Gamma \Rightarrow \Delta, F_2[a/s(a)] \;\; [G]}{F_1[a/0], \Gamma \Rightarrow \Delta, \forall y F_2[a/y] \;\; [G \cup \{[x_{i_1}, x_{i_2}]\} \cup \{Dom(y) = N\} \cup \{a \to y\}]}$$

with F_1 and F_2 equiform, y is new.

(**Cut rule (C)**)

$$\frac{\Gamma \Rightarrow \Delta, A_1 \;\; [G_1] \qquad A_2, \Theta \Rightarrow \Sigma \;\; [G_2]}{\Gamma, \Theta \Rightarrow \Delta, \Sigma \;\; [G_1 \cup G_2 \cup \{[x_{i_1}, x_{i_2}]\}]}$$

Conditions as in $(\vee L)$; A_1 and A_2 equiform, x_{i_1} set of bound variables in A_1, x_{i_2} set of bound variables in A_2.

The case where Induction supplies the cut-formula on the right-hand side is fairly straightforward. In this case the cut-formula is $F_1[a/0]$ (and the equiform formula on the left-hand side), which is quantifier-free. So in this case Q-normality is guaranteed and therefore prof. Isles' proof applies. Induction I may supply the cut-formula on the righthand side with cut-formula $F(a/0)$:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, F_1(a/0) \qquad F_2(a/0), \Gamma_2 \Rightarrow \forall z[A(z)], \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \forall z[A(z)]}$$

or Induction I may supply the cut-formula on the lefthand side with cut-formula $\forall z[A(z)]$:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \forall y[A(y)] \qquad \forall z[A(z)], \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Now for the case where Induction supplies the cut-formula on the left-hand side. In this case the cut-formulae are $\forall y F_2[a/y]$ on the left and an equiform formula, say $\forall z F_2[a/z]$, on the right. The derivation is now no longer Q-normal. Application of Cut will remove both formulae, and require in the reference grammar that their domains be identical. However, induction variable y is a brand new variable (it has, by hypothesis, just been introduced by the Induction rule) that immediately gets eliminated from the derivation. It has domain N but can no longer affect any other variables, nor does it have pre-existing conditions (it's new). So y poses no threat to the denumarability of the soul of the derivation. z is also eliminated from the derivation (and can hence contract no new conditions) and assigned domain N. However, z is not necessarily new, so we must account for possible pre-existing conditions (in the RG) pertaining to z.

Let D_r be the right-hand side derivation that gives rise to $\forall z F_2[a/z]$.

Suppose D_r is Q-normal. Then prof. Isles' proof applies to D_r, so all variables in D_r must have a finite domain or, at most, the denumerable domain N (as a consequence of Induction, as is the case with z). Therefore, the theorem holds in this case.

Suppose D_r is not Q-normal. The only way this can happen is by occurrence(s) of exactly the case we are considering, namely an application of the Cut rule to a cut-formula supplied by Induction on the left-hand side (from now on: lIC). But now we can give an induction argument on the number of occurrences of lIC. If lIC does not occur, we have the situation where D_r is Q-normal, which has been discussed above. So the base case holds. Now for the induction step. Assume the theorem holds for n occurrences of lIC in D_r. It remains to prove that it holds for $n+1$ occurrences in D_r. By the induction hypothesis, the section of D_r up to but not including the last occurrence of lIC in D_r, let us call this $D_{r,n}$, satisfies the condition that it has a denumerable soul. But if $D_{r,n}$ has a denumerable soul, then so must D_r itself because the last occurrence of lIC will yield at most denumerable (via the cut-formula and induction variable) additions to the soul of D_r (analogous to the base case, but instead of knowing D_r to be Q-normal we now know $D_{r,n}$ to satisfy the theorem). Therefore, the theorem also holds if D_r is not Q-normal.

The Induction rule introduces a universal quantifier, and the proofs given by prof. Isles rely crucially on the ability to 'backtrack' any quantifier to the rule that originated it (see Lemma 5.3). However, since the reference grammar rule $a \to y$ introduced by Induction (y being the induction variable) has the property that a is eliminated from the derivation (this is exactly what the Induction rule does, it eliminates a in favour of a universal quantification over y), and hence that a behaves like an eigenvariable in this respect, the proof of Lemma 5.3 holds under the introduction of the Induction rule. This means that the definition of $<<$, Theorem 5.4 and Definition 5.20 also hold for BV_{pn}. And then the proof of Theorem 5.5 also holds for BV_{pn} as this proof does not depend directly on the derivation rules but rather uses the relations between the quantifier variables (and, as we have just argued, these are not materially altered by the introduction of Induction).

Given Theorem 5.5, it follows that there must exist a model M'_r such that every variable can be assigned a finite soul, except those variables, if any, for which the reference grammar demands that the domain of the induction variable y (i.e., N) be a subset of their domain. But for those variables the domain would still be denumerable, as it consists at most of the union between a denumerable and a finite domain, which cannot give rise to a non-denumerable domain.

This concludes the proof. □

Note that the reference grammar of D may now become inconsistent, but this does not contradict the theorem as D would in this case not be a valid BV_{pn} derivation.

The only difference between BV and BV_{pn} is that there are some extra reference arrows between quantified variables in equiform formulae, or, put otherwise, domains of these variables are identified rather than subsuming one into the other. This does not interfere with the 'backtracking' of these variables, so it is not that surprising that the theorem also holds for BV itself.

If F is not quantifier-free, Q-normality cannot be guaranteed. However, Q-normality served only to avoid self-reference (see Theorem 5.3) and self-reference was to be avoided only because it gave rise to infinite domains. Given that we are not considering finitism here, it can be argued that we should not be concerned about the absence of Q-normality (or presence of self-reference). But self-reference also has consequences when considering the functional numerical domains. So, let us now turn to their consideration.

5.2.2 Functional Numerical Domains

Let us remind the reader that in BV the domains are numerical expressions, such as the natural numericals $0, s0, ss0, \ldots$, and functional numerical expressions such as $a + b$, $a \cdot b$ and a^b. Therefore, domains in BV can increase both in size and in complexity.

In general, BV domains increase in size faster than (standard) LK domains when their complexity increases. For instance, suppose the reference grammar contains $x \twoheadrightarrow a + b$. Let domain$(a) = \{1, 2\}$ and domain$(b) = \{2, 3\}$. Then in LK $\{3, 4, 5\}$ of size 3 is added to the domain of x, and in BV $\{1 + 2, 1 + 3, 2 + 2, 2 + 3\}$ of size 4 is added to the domain of x, while the complexity goes from N to AN. This necessitates expansion of the notion of self-reference.

In the system LK of [13] the only self-referential expressions are of the form $x \twoheadrightarrow t(x)$, but in BV there are three ways for an expression of the form $x \twoheadrightarrow t(y_1, \ldots, y_n)$ to be functionally self-referential, i.e., giving rise to an infinite domain of x through the interactions of the variables used in the functional expressions:
1) $y_i \twoheadrightarrow t(x)$; this does not reduce to $x \twoheadrightarrow t(x)$.
2) $y_i \twoheadrightarrow t(y_j)$ and $y_j \twoheadrightarrow t'(y_i)$ or any cyclical expansion thereof.
3) $t(y_1, \ldots, y_n) \twoheadrightarrow t(x)$ which reduces to $x \twoheadrightarrow t(x)$.

Example 5.11 Assume $x \twoheadrightarrow a + b$.
ad 1) Suppose $a \twoheadrightarrow s(x)$ This adds all expressions of the form $s(x)$ to the domain of a and next all expressions of the form $a + b$ are added to the domain of x. This process is repeated ad infinitum.

ad 2) Suppose $a \to\to b + s0$; this adds all expressions of the form $b + s0$ to the domain of a, and
$b \to\to a + 0$; this adds all expressions of the form $a + 0$ to the domain of b. Again, this process is repeated ad infinitum.
ad 3) Suppose $a + b \to\to s(x)$. Together with the assumption $x \to\to a + b$, this yields $x \to\to s(x)$ which is of the type $x \to\to t(x)$.

This constitutes a marked difference between BV and traditional logic, as it gives a number of conditions, notably 1) and 2), in which BV models become infinite, while traditional models may remain finite.

Concerning case 2) notice that if in $y_i \to\to t(y_j)$, $t(y_j) = y_j$, then this case would reduce to the form $y_i \to\to t'(y_i)$, which is traditional self-reference.

Without induction, professor Isles' result for Q-normal linear derivations (see Theorem 5.5) still holds provided the reference grammar does not contain functional self-reference. In LK with the domains as in BV the conditions for self-reference can be strengthened with the addition of functional self-reference. Provided the reference grammar does not contain self-reference of any kind, prof Isles' result still holds and it should be possible to give finite domains to all variables. Also, in the absence of Induction there is no way for the reference grammar to become inconsistent; hence, all derivations in LK will remain valid in BV.

With induction the result from Theorem 5.6 still holds and functional self-reference means only that domains other than N are infinite. But there is also the possibility that the reference grammar becomes inconsistent; see, for instance, Euclid's lemma. Hence, there is no guarantee that traditionally valid derivations will remain valid in BV. The absence of functional self-reference would mean that models exist in which only finite portions of functional domains have to be added to N in order to provide suitable domains for all variables, provided the derivation is BV-valid.

Chapter 6

Philosophical Considerations

After all the formalism of the previous chapters, it is now time to consider some philosophical points concerning the BV system. The object of this chapter is twofold. On the one hand justification and elucidation of the choices made in the construction of the BV system. On the other hand clarification of the philosophical significance of the results obtained in the previous chapters. There is some natural overlap between these two goals; clarification of philosophical significance can sometimes also serve as justification, and justification and elucidation can sometimes serve as the basis of clarification of philosophical significance. Hence, in general, no effort shall be made to preserve the somewhat artificial distinction between these two goals.

An attempt will be made to discuss matters thematically, so as to not jumble everything together in a large opaque mass. There are however severe limits to this, as many of the themes touch upon each other. So, it is to be kept in mind while reading that the sections of this chapter are not primarily intended as separate entities of argumentation, but only receive their full import in the context of the chapter as a whole. The first sections, on reduction versus reference, nominalism versus realism, and the interpretation of induction, are a good example of this interconnectedness and should not be evaluated on their own but in relation to each other.

6.1 Reduction vs Reference

At the heart of the entire project lies the notion of considering logic as a fully syntactic discipline. The first step to take is then to remove all, or at least the maximum possible, semantic content from the formalism. To that end the notion of reference (i.e., every numerical notation refers to a number) is replaced by that of reduction (i.e., every numerical notation either is a numeral numerical notation or reduces to a numeral numerical notation). The notions of numerical notation and of reduction are made exact in the first two sections of Chapter 2.

Clearly the choice to replace reference by reduction has consequences for the question whether one is a realist or nominalist with regard to mathematical entities. This matter will be expounded on in Section 6.2 of this chapter. Somewhat less obvious, but crucially important, the notion of induction is also affected. Induction as a logical rule strictly speaking only applies to numeral numerical notation. This is the reason for the restriction of the domain of an induction variable to N. Without this restriction, no contradiction would ever occur within a reference grammar, and hence the entire project of BV derivation would be meaningless as it would, at best, be distinction without a difference. With this restriction, however, BV differs significantly from traditional logic, as was shown in Chapter 3. More will be said on this interpretation of induction in Section 6.3.

The previous paragraph glosses over a significant point. There can only be a contradiction in a reference grammar when there is, in fact, a reference grammar. If one does not accept the notion of a reference grammar, then the restriction of the induction variable to N (which is in itself also an expression in a reference grammar) has no consequences, and cannot even be meaningfully expressed within the formal system. But why would one not accept reference grammars? All they do is elicit extra information concerning their derivation. At the very worst, the extra information could be irrelevant and thus an unnecessary burden to the logician. I claim that the previous chapters have shown this is not the case. Therefore one must conclude that reference grammars contain relevant information that traditional logic ignores (by way of fixing the domains of variables in advance and choosing reference over reduction, thus obviating the need to keep track of the interdependencies between the domains of variables, as then no contradictions can occur given that induction becomes unproblematic). Regardless of what choices one makes as a logician in these matters, one can no longer claim that the notion of a reference grammar is meaningless and should be discarded out of hand. Later on, in Section 6.4, some attention will be given to an explication of the meaning and function of reference grammars.

Note that it is not the intention to state that there is a fundamental problem with reference. Outside mathematics, specifically in philosophy of language, it would be difficult to see how we could get by without the notion of reference. And even within mathematics and logic the notion need not necessarily be discarded. The only point set out to be made here is that the choice for reference over reduction in mathematics is not a neutral one. There is a real difference between the logical systems associated with both reference and reduction, and therefore one should not treat the choice as merely cosmetic. Since reference is so naturally related to realism and reduction likewise to nominalism, this tentatively amounts to saying that the choice between realism and nominalism is, even from a purely formal point of view, not a neutral one.

This latter point runs contrary to accepted wisdom and may hence be difficult to accept. Against this reluctance to accept the formal importance of the choice between nominalism and realism, I would like to argue that the results from the previous chapters show that the difference between the formal systems is real, and that BV is acceptable as a first-order logic (or at least that there are no formal reasons to exclude BV from consideration). Furthermore, the metaphysical importance of the distinction between nominalism and realism is enormous. Why should it be so surprising that a significant metaphysical difference results in a significant logical difference? It seems to me that the absence of a significant logical difference should be considered surprising, not its presence. After all, logic does not exist in a vacuum. Conservation of truth and correctness of argumentation are not matters unrelated to the world, so any choice that has significant metaphysical consequences might, ceteris paribus, be expected to have consequences for logic too.

Another consequence of replacing reference with reduction is that the notion of identity is affected, in the sense that there are now two notions of identity, reductive identity and syntactic identity. These are defined in Chapter 2, Definition 2.6. Informally, two expressions are reductively identical iff they reduce to the same numeral numerical notation and they are syntactically identical iff they are, symbol for symbol, the same. The notion of reductive identity corresponds to the traditional notion of identity, i.e. two expressions that are traditionally identical will also be reductively identical and vice versa. This follows from replacing reference with reduction. Where the traditional identity of a and b is to be read as 'a and b refer to the same number (or, more generally, the same mathematical object)', the reductive identity of a and b reads as 'a and b reduce to the same numeral numerical form'. Clearly, reducing to the same numeral numerical form implies traditional identity (referring to the same number), and equally clearly traditional identity implies reduction to the same numeral numerical form. Were this not the case, there would be a problem with the definition of reduction. No such problems occur in the definitions given in Chapter 2.

Syntactic identity has no real equivalent in traditional logic, it is simply subsumed in the notion of traditional identity. All objects that are syntactically identical are also traditionally (and reductively) identical, but not all traditionally identical objects are syntactically identical (2 + 2 and 4 are not syntactically identical). If we are to take the notion of logic as a purely syntactic discipline seriously, then it seems no more than to be expected that this should be reflected in the notion of identity, as indeed here it is. Syntactic identity holds only between objects that are symbol for symbol identical. This is a strict condition, to the point that, given a function f and two non-syntactically identical objects a and b (so a and b are referentially identical but not syntactically), $f(a, b)$ and $f(b, a)$ are not syntactically identical. Some examples have been given in Definition 2.6. The closest equivalent in traditional (meta-)mathematics to syntactic identity is the notion of two objects having the same Gödel number.

At the cost of stating the obvious, it must also be mentioned that the elements of the domain of a variable are of a different kind in a BV context than they are in a traditional context. Traditionally, the numbers themselves are taken to be the elements of a domain, not the expressions referring to these numbers. Thus, when two co-referential expressions occur within a domain, they would be considered to designate only one element of the domain, namely the number to which they both refer. In a BV context it is the numerical expressions themselves that are considered to be the elements of a domain. So two numerical expressions that reduce to the same normal form are still two distinct elements of the domain, in contrast to the analogous situation in the traditional context. This is entirely in conformity with the idea that logic is to be considered as a fully syntactical discipline, and hence should distinguish between different numerical expressions that nonetheless reduce to the same numeral numerical form (traditionally: have the same referent). This point will recur in the section on induction, as it is vital to rebutting a potentially serious objection to the interpretation of induction used in this thesis. First, however, some more will be said on the debate between nominalism and realism.

6.2 Nominalism vs Realism

This thesis is not primarily concerned with metaphysics of mathematics, so I intend to be relatively brief on the subject. It seems, however, that some remarks are in order.

As argued above, there seems to be a natural relation between, on the one hand the choice for reference and realism, and on the other hand the choice for reduction and nominalism. However, it would be far too strong a claim to say that these relations are necessary. At the very least it is possible to work within the framework of traditional logic, with its penchant towards reference, and still be, in varying degrees, of a nominalistic bent. We know this is possible, because it exists. And theoretically it is also possible to work within the BV framework and still be a realist up to some extent. For instance, it is conceivable that one believes in the existence of numbers while still being attached to the notion of logic as a purely syntactic discipline. However, it seems to me that this latter option would yield a somewhat uncomfortable position.

Suppose that one is a realist but still attached to the BV system. Then one would have to explain a number of points. I will name two, without any claim to completeness. Firstly, one would need to explain why one chooses reduction over reference. After all, if there is a metaphysical reality to numbers, it seems to make a lot of sense to assume that natural numerical expressions refer to this metaphysical reality. If they do not, then what is the use of the metaphysical reality? But if such a referential connection is assumed, one has stepped outside the BV project, specifically aimed at logic as a syntactic discipline and, therefore, the removal of all possible traces of semantic content. Secondly, and rather related to the first point, if one assumes a metaphysical reality of numbers, it becomes rather appealing to assume that all natural numerical expressions that relate to the same number (whatever the nature of this relation may be) must share the same properties. This would constitute an argument against the interpretation of induction as limiting the domain of the induction variable to N. After all, if all natural numerical expressions reducing to the same numeral numerical expression (which seems to be the most basic form of being related to the same number) must share the same properties, then there is no need to limit the domain of the induction variable. We will go into this in more detail in Section 6.3, but the reply to this must be that from a purely syntactic point of view, there are differences between the properties of co-reductive natural numerical expressions at least at the level of syntax. Of course one could claim that syntactic differences are not sufficiently real to constitute a genuine difference in properties, but that seems to be a claim at odds with the BV project. If one really believes this, it becomes hard to see why one would be interested in a purely syntactic approach to logic.

The opposite situation, where one works in traditional logic but is to some degree a nominalist, seems less problematic. There are various options to circumscribe one's willingness to assign reality to mathematical objects and it falls well outside the scope of this thesis to discuss them all (or indeed most of them). The only one I would like to question here is the dedicated nominalist that remains attached to the traditional framework of logic. If you truly believe mathematical objects have no more metaphysical reality than their expressed forms, it seems strange to me that one would not adhere to the BV framework. After all, there really is nothing for expressions to refer to when one holds this position, so it would seem to make sense that one would think in terms of reduction rather than reference. And once this step has been taken, the move from traditional logic to the BV framework seems like a natural transition. Again, of course, one could be attached to the idea that co-reductive natural numerical expressions must share all properties. A reply to this position has been hinted at above, and will now be given in more detail when considered in the context of the interpretation of induction (see the end of Section 6.3).

6.3 Induction

The induction rule is defined in Section 2.4. It has also been noted there that the condition $dom(y) = N$, with y the induction variable, is crucial for the BV system. It is the only occasion on which a direct limitation is put on the domain of a variable. In all other cases, the limitations put on the domain of a variable are always relative to the domain of another variable (reference arrow $v_1 \rightarrow v_2$ meaning that the domain of v_2 is a subset of the domain of v_1). Only the induction rule gives rise to a non-relative condition on the domains of variables in that the domain of the induction variable is limited to N. Further on in the derivation it is then possible that another variable receives this limitation due either to interaction with the induction variable, or to interaction with a variable that has already received the limitation earlier on in the derivation. See Chapter 3 for examples.

Were it not for the limitation of the domain of an induction variable to N, there would never occur inconsistency in a reference grammar. For if induction variables were not so limited and could not spread this limitation by interaction with other variables, there would only be relative limitations of the form: the domain of v_2 is a subset of the domain of v_1, or, the domain of $f(v_1, ..., v_n)$ is a subset of the domain of v_0. Both of these limiting conditions are met when one assigns the maximally large domain to all variables (note that the expressions formed using function f are included in the maximally large domain, as this is what it means to say that the domain is maximally large). In our case this would result in assigning the domain EN to all variables. Given that all functional expressions are contained within the maximally large domain, both types of restrictions (on simple variables or on functional expressions) are clearly met when all variables are assigned the maximally large domain.

But if no reference grammar would ever yield inconsistency, then there would be no use in employing the BV system. For the BV system would then amount to traditional logic with the added complication of keeping track of a reference grammar that would never make a difference. Thus for reasons of efficiency alone it would be expedient to ditch the whole notion of a reference grammar. Thus the limiting of the domain of an induction variable to N is rather crucial to our project, for if it were not present then the whole thing would collapse into redundancy.

The reason for the limitation of the domain of an induction variable to N is simple. Due to its definition, the induction rule only concerns numeral numerical expressions. It takes 0 as its starting point and then proves that the property under consideration holds for 0 and any natural numerical expression that can be constructed by means of placing any finite number of instances of the successor function s in front of 0, i.e., the property holds for exactly the numeral numerical expression (all numerals are considered and no functional expression comes under consideration). Thus the domain of the induction variable is limited to N.

Here we find a significant consequence of our project of taking the notion of logic as a purely syntactic discipline seriously. For traditional logic, the fact that induction is, strictly speaking, concerned only with numeral numerical notations is no reason to limit the reach of induction. After all, traditionally speaking, all co-referential expressions are considered to be interchangeable salva veritate. Hence, if a property holds for all numeral numerical expressions it is considered to hold for all expressions (or all numbers, as one would say in this context) because the numeral numerical expressions cover the whole set of natural numbers. In a BV context, however, the distinction between numeral numerical notation and natural numerical notation (which also contains the functional expressions) is considered to be significant. This has been elaborated on in Section 6.1. The domain of a variable is considered to contain natural numerical expressions under BV, as opposed to numbers under traditional logic. But this means that in the BV framework, the reach of induction is limited to the numeral numerical expressions, i.e., the subset of the natural numerical expressions actually subject to construction in the induction rule.

A possible objection to this interpretation of induction goes as follows. One could associate with every numeral numerical expression the class of natural numerical expressions that reduce to it. Call this the reduction class of the numeral. It would then make sense to consider induction as working on these reduction classes rather than merely on the numeral numerical expressions. After all, if expressions reduce to the same numeral form, then surely they must have all properties in common. And this amounts to saying that we really should do away with reduction and go back to reference, as the BV system has now become redundant (no direct limitation of domain of induction variable to N means no inconsistency in reference grammars, see above).

The problem with this objection is that it implicitly assumes the traditional point of view that co-referential expressions are interchangeable salva veritate. To wit, in the assumption that expressions reducing to the same numeral form must have all properties in common. But this does not hold under the BV interpretation, where syntactic differences are considered to be relevant. From this point of view, the assumption that expressions reducing to the same numeral form must have all properties in common is patently false. For instance, all numeral numerical expressions have the property of being constructed solely by means of the signs 0 and s, and this is a property that is not shared by the functional part of the natural numerical expressions, even though each of the latter will share a reduction class with a numeral numerical expression. Hence the objection is only valid from a traditional point of view (which is hardly surprising), but not from a BV point of view.

6.4 Reference Grammars

Next to replacing reference with reduction, another consequence of taking the notion of logic as a purely syntactic discipline seriously is the introduction of the notion of reference grammar. For conservation of truth in every step of a derivation it is not necessary that all variables be assigned the same domain (as is done traditionally), or even that all variables be assigned their, possibly different, domains previous to the derivation (many-sorted logic allows for different domains for variables, but still fixes them in advance of the derivation). For conservation of truth it is merely necessary that the restrictions that are placed on the relations between the domains of the variables in a derivation are respected. These restrictions can be found in the definition of the Buridan-Volpin calculus in Chapter 2. The reference grammar of a derivation is simply a list of all restrictions on relations between the domains of variables occurring in the derivation, if any. If the reference grammar yields no inconsistency, i.e., the restrictions on the domains are not in some way contradictory and hence satisfiable, the derivation is valid within BV. If, however, the reference grammar does yield inconsistency (see the proof of Euclid's Lemma at the end of Chapter 3), then the derivation is not BV valid as it is not possible to consistently assign domains to the variables. Thus the reference grammar elicits additional information and constitutes an additional condition on the validity of a derivation.

One might wonder why the introduction of reference grammars is to be seen as a consequence of taking the notion of logic as a purely syntactic discipline seriously, as there does not seem to be any specifically semantic aspect that is removed by it. However, as we have seen above, logic as a syntactic discipline required replacing reference with reduction. This in turn led to a new interpretation of induction as limited to the numeral numerical forms, which places a restriction (to N) on the domain of the induction variable. But if there is the possibility of restrictions on the domains of variables, then it becomes necessary to keep track of these restrictions in order to avoid unnoticed inconsistencies. Hence the need for reference grammars. Thus the introduction of reference grammars is a consequence of logic perceived as a syntactic discipline, albeit mediated by the replacement of reference by reduction.

This also explains why traditional logic is not plagued by unnoticed inconsistency due to its policy of assigning domains to variables prior to the derivation. When reference is not replaced by reduction, the interpretation of induction does not yield a restriction on the domain of the induction variable. Hence no restrictions on the assignment of domains to variables ever occur, and therefore there is no possibility of contradiction in the relations between the domains of the variables (without restriction, there can be no contradiction). When reference is primary, one treats all expressions that refer to the same number as if they were identical and hence interchangeable. Under these conditions it does not matter that induction works only on the numeral numerical expressions, as these are deemed interchangeable salva veritate with their syntactically more complex co-referents. This principle also obviates the need to distinguish between syntactically non-identical but co-referential expressions in the domains of the variables. At this cost, the need for reference grammars can be avoided.

There remains the question of the exact status of the conditions in the reference grammar. Are they akin to logical assumptions or something else entirely? This question has already been touched upon in Section 3.3, but warrants some further consideration here.

My contention is that the conditions in the reference grammar are not akin to logical assumptions at all. The only function of the reference grammar, and the conditions put forth in it, is to establish whether or not a consistent assignment of domains to the variables is possible (i.e., is the reference grammar consistent or inconsistent). If the reference grammar is inconsistent, it is impossible to consistently assign domains to the variables, which amounts to saying that the derivation is not valid. If the reference grammar is consistent, it is possible to consistently assign domains to the variables, and hence the derivation is valid. In the latter case, the reference grammar can be used to construct the class of all possible (i.e., consistent) assignments of domains to the variables. Nowhere is there a need to consider a condition in a reference grammar as having a function beyond this scope, their only function is to guarantee preservation of truth in every step of the derivation.

Chapter 7

Summary

Logic is traditionally considered to be a purely syntactic discipline, at least in principle. However, prof. David Isles has shown in [12], [13], [14] and [15] that this ideal is not yet met in traditional logic. Semantic residue is present in the assumption that the domain of a variable should be fixed in advance of a derivation, and also in the notion that a numerical notation must refer to a number rather than be considered a mathematical object in and of itself. Based on his work, the central question of this thesis is what kind of logic, if any, results from removing this semantic residue from traditional logic.

We differ from traditional logic in two significant ways. The first is that the assumption that a numerical notation must refer to a number is denied. Numerical notations are considered as mathematical objects in their own right, related to each other by means of rewrite rules. The traditional notion of reference is then replaced by the notion of reduction (by means of the rewrite rules) to a normal form. Two numerical notations that reduce to the same normal form would traditionally be considered identical, as they would refer to the same number, and hence they would be interchangeable salva veritate. In the new system, called Buridan-Volpin (BV), the numerical notations themselves are the elements of the domains of variables, and two numerical notations that reduce to the same normal form need not be interchangeable salva veritate, except when they are syntactically identical (i.e., have the same Gődel number).

The second is that we do away with the assumption that the domains of variables need to be fixed in advance of a derivation. Instead we focus on what is needed to guarantee preservation of truth in every step of a derivation.

These conditions on the domains of the variables, accumulated in the course of a derivation, are combined in a reference grammar. Whereas traditionally a derivation is considered valid when the conclusion follows from the premisses by way of the derivation rules (and possibly axioms), in the BV system a derivation must meet the extra condition that no inconsistency occurs within the reference grammar. For if the reference grammar were to give rise to inconsistency (i.e., it would be impossible to assign domains to all the variables without breaking at least one of the conditions placed on them in the reference grammar), there is no longer a guarantee that truth has been preserved in every step of the derivation, and hence the truth of the conclusion is not guaranteed by the derivation.

In Chapter 2 the BV system is introduced is some formal detail. Chapter 3 gives some examples of derivations, notably totality of addition, multiplication and exponentiation, as well as a lemma needed for the proof of Euclid's Theorem. These examples, taken from prof. Isles' [15], show that there is a real proof-theoretical difference between traditional logic and the BV system. Here we also find the first major point of departure between myself and prof. Isles, centered on the notion of inheritance of conditions in the reference grammar by way of lemmata. These different points of view are best illustrated in the sections on the totality of exponentiation and on Euclid's Lemma: prof. Isles maintains that the proof of totality of exponentiation is not BV valid, while I maintain that it is. But I do agree with him that the traditional proof of Euclid's Lemma is not BV valid. Chapter 6 also expands the arguments for my choice in this matter.

Now that it has been shown that there is a difference between traditional logic and BV, the properties of BV need to be examined. In Chapter 4 we give a proof of Cut-elimination for BV minus induction and the subformula property for BV, which allows us to prove the consistency of BV minus induction. We also expand on the reasons for excluding induction. In Chapter 5 we consider in detail the proof of a finite analog to the Lőwenheim-Skolem theorem given by prof. Isles in [13]. He proves that under certain conditions it is always possible, given the existence of a (possibly uncountable) model for a derivation, to give a finite model for this derivation. The system he considers deviates from BV as considered in this thesis in two significant ways: it does not contain the induction rule and the domains contain numbers instead of numerical notations. We then go on to show that it is possible to extend the result to include induction, in the sense that the existence of a possibly uncountable model for a derivation guarantees the existence of a model that is at most countable. We also consider the complications that arise from taking numerical expressions instead of numbers as the elements of domains.

Finally, in Chapter 6 we consider the philosophical consequences of the BV system, informed by the formal results from the previous chapters. In particular we discuss the relation between reduction and reference, the status of reference grammars, the notion of induction and its function within BV, and some brief considerations on the consequences of the BV system for the discussion regarding nominalism and realism with regard to mathematical objects. The object of the chapter is twofold. On the one hand applying the formal results to philosophical questions, on the other hand arguing that BV is not just a theoretically acceptable alternative to traditional logic, but is in fact deserving of further development and research into its properties. The latter will probably appeal most to those of a nominalist and/or finitist bent.

Chapter 8

Samenvatting (Dutch Summary)

Traditioneel wordt logica gezien als een louter syntactische discipline, maar dit is een reputatie die vooralsnog niet helemaal waar wordt gemaakt. In de voetsporen van prof. David Isles heb ik mij de vraag gesteld hoe een logica er uit zou komen te zien indien men het semantische aspect volledig, of alleszins zo volledig mogelijk, verwijdert.

Practisch gezien komt dit er op neer dat je de vooronderstelling dat er zoiets bestaat als de referentie naar een getal opgeeft, waardoor de notie van getal eigenlijk vervangen dient te worden door die van numerieke notatie. In wezen wordt dit dan een TRS (Term Rewrite System), waarbij referentie vervangen wordt door reductie tot een normaalvorm. Tevens geef je de vooronderstelling op dat de domeinen van variabelen vastgelegd dienen te worden voor aanvang van een afleiding. In plaats hiervan krijg je een collectie voorwaarden, verzameld in een referentiegrammatica, waaraan de domeinen van de variabelen dienen te voldoen. Waar traditioneel een afleiding geldig is wanneer de conclusie volgens de afleidingsregels en axiomata volgt uit de premissen, moet in het syntactische systeem (genaamd Buridan-Volpin, vanaf nu BV) aan de extra voorwaarde voldaan worden dat de referentiegrammatica consistent is, d.w.z. dat het mogelijk is op consistente manier domeinen aan de variabelen toe te kennen.

In hoofdstuk 2 wordt het formele systeem geïntroduceerd. In hoofdstuk 3 wordt middels enkele voorbeelden aangetoond dat er een werkelijk bewijstheoretisch verschil is tussen BV en traditionele eerste orde logica. Dit is in essentie een herhalen van het werk dat al is gedaan door prof. Isles, met dien verstande dat er een poging wordt gedaan om de formele grondslagen zo zuiver mogelijk naar voren te brengen (mogelijkerwijs tot op het pietluttige af). Tevens is er een punt waarop ik afwijk van prof. Isles, met name de overerving van voorwaarden uit de referentiegrammatica via het gebruik van lemmata. Dit verschil komt met name naar voren in de paragrafen over totaliteit van exponentiatie en het lemma van Euclides, en de argumentatie voor mijn keuze wordt verder uitgewerkt in hoofdstuk 6, waar ook verder ingegaan wordt op thema's als realisme versus nominalisme, referentie versus reductie, de interpretatie van inductie en referentiegrammatica's.

Het voorgaande werk was er allemaal op gericht aan te tonen dat BV niet louter een andere, omslachtigere, manier was om aan gewone eerste orde logica te doen, maar vanaf hoofdstuk 4 worden de eigenschappen van BV zelf onderzocht. Het project is om aan te tonen dat BV een levensvatbaar alternatief is voor traditionele eerste orde logica, en daartoe dient bewezen te worden dat BV een aantal basale eigenschappen heeft, zoals consistentie. Aangezien gewerkt wordt in een Gentzen afleidingssysteem wordt dan eerst een Snede-eliminatie bewijs gegeven, ietwat gecompliceerd door de aanwezigheid van referentiegrammatica's, om vervolgens via de deelformule eigenschap consistentie te bewijzen. Andere eigenschappen volgen dan.

Hoofdstuk 5 behandelt de Löwenheim-Skolem eigenschap. In het artikel [13] van prof. Isles bewijst hij dat het onder bepaalde voorwaarden steeds mogelijk is om, gegeven het bestaan van een (mogelijk overaftelbaar oneindig) model voor een afleiding, een eindig model te geven voor deze afleiding. Het systeem waarvoor prof. Isles dit bewijst wijkt enigermate af van BV zoals beschouwd in deze thesis, met name ontbreekt de afleidingsregel voor Inductie en bestaan de domeinen uit getallen in plaats van getalsmatige uitdrukkingen (numeralia en functionele uitdrukkingen). Eerst wordt het bewijs van prof. Isles in detail gerecapituleerd, daarna wordt aangetoond dat het toevoegen van de Inductieregel het bewijs grotendeels intact laat. Met enige aanpassingen kan bewezen worden dat, gegeven het bestaan van een (mogelijk overaftelbaar oneindig) model voor een afleiding, het steeds mogelijk is een (hoogstens) aftelbaar oneindig model te construeren voor deze afleiding.

Hoofdstuk 6 heeft dan twee doeleinden. Ten eerste een aantal van de filosofische thema's uitdiepen, mede op basis van de behaalde formele resultaten. Ten tweede een argumentatie opzetten die aantoont dat BV niet alleen louter theoretisch een acceptabel alternatief is voor traditionele eerste orde logica, maar het ook verdient om verder uitgewerkt te worden. Dit laatste zal waarschijnlijk met name de nominalistisch of finitistisch geïnteresseerden onder ons aanspreken.

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Chapter 9

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Index