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Continuum of Zero Points of a Mapping on a Compact, Convex Set [1]

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Abstract: Let X be a convex, compact, nonempty set in $\mathbb{R}^n$ and ϕ an upper semicontinuous mapping from X to the collection of nonempty, convex, compact subsets of $\mathbb{R}^n$. We show that with respect to any nonzero vector c in $\mathbb{R}^n$ there exists a connected set of stationary points of ϕ on X containing a point in the boundary of X at which $c \cdot x$ is minimized on X and another point in the boundary of X at which $c \cdot x$ is maximized on X. We give several conditions on ϕ, under which there exists a continuum of zero points of ϕ connecting two such points in the boundary of X. We also propose a simplicial variable dimension algorithm to approximate the connected set. The algorithm traces a piecewise linear path of points in X by making a seqence of linear programming pivoting steps in a system of $n+1$ equations. Each point on the path is a stationary point for the vector c of a piecewise linear approximation of the mapping with respect to the underlying simplicial subdivision of a convex polyhedron containing the set X.

Keywords: stationary point, continuum of zero points, simplicial algorithm, system of nonlinear equations, variational inequality.

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1 Introduction

Whenever a mathematical model of some phenomenon is constructed, for instance in engineering or in economics, the first question to ask is whether a solution to the model exists. A very powerful tool that is used to this end is Brouwer's fixed point theorem; see Brouwer (1912). When the model is not a system of equations but a system of correspondences, Kakutani's fixed point theorem (1941) is invoked. An alternative to fixed point theorems consists of using intersection theorems on polytopes, with the KKM theorem of Knaster, Kuratowski and Mazurkiewicz (1929) perhaps the most prominent example. It is well-known that there is a close relationship between fixed point theorems and intersection theorems. Yet another alternative consists of results that claim the existence of solutions to variational inequality problems, the existence of stationary points, or the existence of zero points.

For certain models, it is not only important to know that there exists at least one solution, but one would like to show the existence of a continuum of solutions. In economics the existence of a continuum of solutions leads to difficulties in expectation formation of agents, and as a consequence provides scope for endogenously generated fluctuations. A particular example comes from general equilibrium theory with price rigidities, where a continuum of solutions on the unit cube as a polytope is shown to exist in Herings (1998). It is therefore important to have generally applicable tools that guarantee the existence of a continuum of zero points to a certain system of equations.

This leads us to the following problem: Given a point-to-set mapping $\varphi : X \to \mathbb{R}^n$, with X an arbitrary nonempty, convex, compact set, *what reasonable conditions can guarantee the existence of a continuum of solutions to the system*

$$0 \in \varphi(x).$$

Our approach to show the existence of a continuum of solutions is to show that there is a connected subset of solutions that links together at least two distinct points in X, thereby guaranteeing the continuum. In particular, we will show that any upper semi-continuous point-to-set mapping with some mild (boundary) conditions will have a connected set of zero points linking together two distinct points in the boundary of X.

It is well known that under certain conditions a point-to-set mapping defined on a nonempty, convex, compact set has a solution to the variational inequality problem. We generalize the variational inequality problem and define a parametric variational inequality problem. In this paper we show that under similar conditions a point-to-set mapping defined on a nonempty, convex, compact set has a connected set of solutions to the parametric variational inequality problem, called parametrized stationary points. The set of parametrized stationary points connects two distinct points in the boundary of X. With respect to some given nonzero vector c, at one of these points, denoted by x^-, the value $c^\top x$ is minimized for $x \in X$, while at the other point, denoted by x^+, the value $c^\top x$ is maximized for $x \in X$. A special case occurs when both x^- and x^+ are extreme points of X and the set of parametrized stationary points always contains both these points. We give several conditions under which the set of stationary points is a connected set of zero points linking the two distinct boundary points x^- and x^+ of X.

We aso propose a simplicial variable dimension algorithm to approximate a connected set of stationary points on a comapact, convex set. This type of algorithm was initiated by Scarf (1967). Simplicial homotopy methods were developed by Eaves (1972). The simplicial variable dimension restart algorithm was introduced by van der Laan and Talman (1979) to compute a fixed point of a continuous function from the unit simplex into itself. Such an algorithm generates a unique sequence of simplices of varying dimension in a simplicial subdivision of the set and connects the arbitrarily chosen starting point with an approximate solution. For other recent developments, we refer to Talman and Yamamoto (1989), Yamamoto (1993), Brown, DeMarzo and Eaves (1996), DeMarzo and Eaves (1996), Yang (1996, 1999), and van der Laan, Talman and Yang (1998). Allgower and Georg (1990), Todd (1976), and Yang (1999) provide comprehensive treatments of simplicial algorithms.

In this paper we use the simplicial algorithm of Herings, Talman and Yang (2001) on polytopes to generate within a simplicial subdivision of a polytope containing the set X a finite sequence of simplices of varying dimension. This sequence connects two different simplices, one simplex in a face of the polytope containing x^- and the other one in a face of the polytope containing x^+. The sequence of simplices connecting these two simplices is generated by the algorithm by making a sequence of semi-lexicographic pivot steps in a linear system of equations. In case the face is not a vertex of the polytope, the algorithm starts by finding a suitable simplex in the face. Induced by the sequence of adjacent simplices, the algorithm yields a piecewice linear path of parametrized stationary points of a piecewise linear approximation of the underlying mapping. When the mesh size of the simplicial subdivision of the polytope goes to zero and the sequence of polytopes converges to X, the sequence (or at least a subsequence) of piecewise linear paths converges to a connected set of parametrized stationary points of the original mapping.

The results in the paper generalize earlier results of Browder (1960), Mas-Colell (1974), and of Herings, Talman and Yang (2001). In case of Browder's theorem the polytope is the cartesian product of a polytope of one dimension less and the unit interval $[0, 1]$, while c is the unit vector with the one on the last position. Mas-Colell's result is an extension of Browder's result to deal with correspondences. Both Browder and Mas-Colell proved their results via a rather sophisticated machinery. Since our approach here is constructive, we therefore also obtain an alternative but constructive proof for their results. In Herings, Talman and Yang (1996) the compact, convex set is a poytope. In Browder's theorem and in Mas-Colell's theorem a connected set of fixed points is obtained connecting the levels 0 and 1, whereas the result on the polytope yields a connected set of zero points connecting two different faces of the polytope.

Intersection theorems with a continuum of intersection points can be found in Freidenfelds (1974) and Herings and Talman (1998). Although Freidenfelds' intersection theorem typically has a continuum of intersection points, this is not necessarily the case. Freidenfelds' result generalizes the intersection theorem of Scarf (1967). Herings and Talman's results generalize a number of intersection theorem on the unit simplex, including the Scarf-result and the KKM-result, to intersection theorems on the unit cube and show the existence of a continuum of intersection points. The reader should be aware that compared with a large amount of existence results for a single fixed or zero point, existence results for a continuum of fixed or zero points are very rare.

This paper is organized as follows. In Section 2 we state the problem and in Section 3 we give three sufficient conditions for the existence of a connected set of zero points of an upper semi-continuous point-to-set mapping over an arbitrary polytope which link together two distinct faces of the polytope. In Section 4 we introduce the algorithm and prove its convergence. In Section 5 we analyse the accuracy of the approximation of zero points and prove the existence theorems. In Section 6 we discuss a more general case. In Section 7 we derive as special cases Browder's and Mas-Colell's theorems and an earlier result of the authors on the unit cube, and we give an economic application.

2 The Problem

Let I_m denote the set of positive integers $\{ 1, \cdots, m \}$. Consider an arbitrary full-dimensional, convex, compact set X in $\mathbb{R}^n$ and let c be an arbitrary nonzero vector in $\mathbb{R}^n$. Without loss generality we assume that $\| c \|_2 = 1$, where $\| \cdot \|_2$ denotes the Euclidean norm. Let X^- and X^+ be defined by

$$X^- = \{x^- \in \mathbb{R}^n \mid c^\top x^- = \min_{x \in X} c^\top x\}$$

and

$$X^+ = \{x^+ \in \mathbb{R}^n \mid c^\top x^+ = \max_{x \in X} c^\top x\}.$$

Since X is compact, convex and full-dimensional, X^- and X^+ are disjoint and both sets are nonempty, convex and compact. Moreover, for any $\alpha \in \mathbb{R}$, let $X_\alpha = \{x \in X \mid c^\top x = \alpha\}$, then there exist α^- and α^+ with $\alpha^+ > \alpha^-$ such that $X_{\alpha^-} = X^-$ and $X_{\alpha^+} = X^+$. Without loss of generality we assume that $\alpha^- = 0$ and $\alpha^+ = 1$, i.e., $X_0 = X^-$ and $X_1 = X^+$. For any α, $0 \le \alpha \le 1$, the set X_α is a nonempty, convex, compact set. For $x \in X$, with $\alpha = c^\top x$, let $G(x)$ be the subgradient of X_α at x, i.e.,

$$G(x) = \{y \in \mathbb{R}^n \mid x'^\top y \le x^\top y \text{ for any } x' \in X_\alpha\}.$$

Since X is compact and convex, G is upper semicontinuous and for any $x \in X$ the set $G(x)$ is a nonempty convex cone. Moreover we have the next lemma.

Lemma 2.1 *If $y \in G(x)$, then also $y + \beta c \in G(x)$ for any $\beta \in \mathbb{R}$.*

Proof: Take any $y \in G(x)$ and let $\alpha = c^\top x$. By definition $x'^\top y \le x^\top y$ for any $x' \in X_\alpha$. Hence, for any $\beta \in \mathbb{R}$,

$$x'^\top(y + \beta c) = x'^\top y + \beta x'^\top c = x'^\top y + \beta\alpha \le x^\top y + \beta x^\top c = x^\top(y + \beta c),$$

for any $x' \in X_\alpha$. Therefore, $y + \beta c$ lies in $G(x)$. □

Next, let ϕ be an upper semicontinuous, bounded point-to-set mapping from X to the collection of nonempty, convex, compact subsets of $\mathbb{R}^n$. A stationary point of ϕ on X with respect to c is defined as follows.

Definition 2.2 A point $x^* \in X$ is a stationary point of the correspondence ϕ on X with respect to the vector c if there exists $f \in \phi(x^*)$ such that $(x^* - x)^\top f \ge 0$ for all $x \in X_\alpha$, where $\alpha = c^\top x^*$, i.e., $\phi(x^*) \cap G(x^*) \ne \emptyset$. A point $x^* \in X$ is a zero point of ϕ if $0^n \in \phi(x^*)$.

Clearly, a zero point of ϕ in X is a stationary point of ϕ on X with respect to any nonzero vector c. We call the problem of finding a stationary point with respect to a nonzero vector c a *parametric variational inequality problem.* A solution to it is also called a *parametrized stationary point.* Clearly, x^* with $\alpha = c^\top x^*$ is a stationary point of ϕ on X with respect to c if and only if x^* is a stationary point of ϕ on the set X_α. Since the set X_α is nonempty, convex and compact, for any α, $0 \le \alpha \le 1$, there exists a stationary point of ϕ on X_α and therefore there exists a stationary point of ϕ on X with respect to c satisfying $c^\top x^* = \alpha$. When varying α from 0 to 1, we want to show that there exists a connected set of such points, having a nonempty intersection with both X^- and X^+ and give conditions on ϕ under which there exists a connected set of zero points containing points in both these sets.

To show that there always exists a connected set of stationary points with respect to the vector c, for let the hyperplane $H(\alpha)$ be defined by $H(\alpha) = \{x \in \mathbb{R}^n \mid c^\top x = \alpha\}$, for any $\alpha, 0 \le \alpha \le 1$,. Clearly, $X_\alpha = X \cap H(\alpha)$. We define $H = \{h \in H(\alpha) \mid 0 \le \alpha \le 1\}$. For $x \in H$ with $c^\top x = \alpha$ we denote by $p(x)$ the orthogonal projection of x on X_α. Clearly, $x - p(x) \in G(p(x))$ for any $x \in H$. The next lemma shows that p is a continuous function.

Lemma 2.3 *The function p from H to X is a continuous function.*

Proof: We show that the mapping S from H to X defined by

$$S(x) = X \cap H(c^\top x)$$

is continuous. From Corollary 8.1 of Hogan (1973) it follows then that p is a continuous function on H. To prove that S is a continuous mapping we prove that S is both upper semicontinuous and lower semicontinuous on H. Let $(x^k)_{k \in \mathbb{N}}$ be a sequence of points in H converging to $\overline{x} \in H$ and let $(y^k)_{k \in \mathbb{N}}$ be a sequence of points converging to some $\overline{y}$ such that $y^k \in S(x^k)$ for any $k \in \mathbb{N}$. Clearly, $\overline{y} \in X$ and $c^\top x^k \to c^\top \overline{x}$. Since $c^\top x^k = c^\top y^k$, for all $k \in \mathbb{N}$, we also have that $c^\top \overline{x} = c^\top \overline{y}$. Hence, $\overline{y} \in H(c^\top x)$ and S is upper semicontinuous on H.

To prove lower semicontinuity on H, take any sequence $(x^k)_{k \in \mathbb{N}}$ in H converging to some $\overline{x} \in H$ and take any $\overline{y} \in S(\overline{x})$. We have to construct a sequence $(y^k)_{k \in \mathbb{N}}$ in X such that y^k converges to $\overline{y}$ and $y^k \in S(x^k)$ for any $k \in \mathbb{N}$. Take any y^- in X^- and y^+ in X^+. For $k \in \mathbb{N}$, let

$$\lambda^k = (c^\top \overline{y} - c^\top x^k)/c^\top \overline{y} \text{ if } c^\top x^k < c^\top \overline{y} \text{ or } c^\top \overline{y} = 1$$

and

$$\lambda^k = (c^\top x^k - c^\top \overline{y})/(1 - c^\top \overline{y}) \text{ if } c^\top x^k \geq c^\top \overline{y} \text{ and } c^\top \overline{y} < 1.$$

Next, for $k \in \mathbb{N}$, let

$$y^k = (1 - \lambda^k)\overline{y} + \lambda^k y^- \text{ if } c^\top x^k < c^\top \overline{y} \text{ or } c^\top \overline{y} = 1$$

and

$$y^k = (1 - \lambda^k)\overline{y} + \lambda^k y^+ \text{ if } c^\top x^k \geq c^\top \overline{y} \text{ and } c^\top \overline{y} < 1.$$

Clearly, $\lambda_j \geq 0$ for any $k \in \mathbb{N}$, and, since $c^\top x^k \to c^\top \overline{y}$, we have that $\lambda^k \to 0$. So, for sufficiently large k, $0 \leq \lambda^k \leq 1$ and therefore $y^k \in X$. Moreover, $c^\top y^k = c^\top x^k$ for all $k \in \mathbb{N}$. Hence, $y^k \to \overline{y}$ and for sufficiently large k it holds that $y^k \in X \cap H(c^\top x^k) = S(x^k)$. $\square$

Using the two lemmas above we are able to prove the main result.

Theorem 2.4 *Let X be a full-dimensional, compact, convex set in $\mathbb{R}^n$, let ϕ be an upper semicontinuous, nonempty-, convex-, compact-valued point-to-set mapping from X to $\mathbb{R}^n$, and let c be an arbitrary nonzero vector in $\mathbb{R}^n$. Then there exists a connected set C of stationary points of ϕ on X with respect to c such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.*

Proof: Let r be the orthogonal projection from $\mathbb{R}^n$ to $H(0)$, so r is a linear function and for any $y \in \mathbb{R}^n$ it holds that $r(y) = y - (c^\top y)c$. Since X is bounded and ϕ is bounded, the set

$$Y = \{y \in H(0) \mid y = r(x + f), f \in \phi(x), x \in X\}$$

is a bounded set in $\mathbb{R}^n$. Let S be a compact, convex set of $H(0)$ containing Y in its relative interior and let the mapping $\psi: S \times [0, 1] \to \mathbb{R}^n$ be defined by

$$\psi(y, \alpha) = \{z \in \mathbb{R}^n \mid z = r(p(y + \alpha c) + f), f \in \phi(p(y + \alpha c))\}.$$

Since p is a continuous function, r is a linear function and given the assumptions on ϕ, for any $(y, \alpha) \in S \times [0, 1]$ the set $\psi(y, \alpha)$ is a nonempty, convex and compact subset of S and ψ is upper semicontinuous and bounded on $S \times [0, 1]$. With S itself being a nonempty, convex, compact set, it follows from Mas-Colell (1975) that there exists a connected set $\overline{C}$ in $S \times [0, 1]$ of fixed points of ψ satisfying $\overline{C} \cap (S \times \{0\}) \neq \emptyset$ and$\overline{C} \cap (S \times \{1\}) \neq \emptyset$, where $(y, \alpha) \in S \times [0, 1]$ is called a fixed point of ψ if $y \in \psi(y, \alpha)$. Let $C' = \{z \in \mathbb{R}^n \mid z = y + \alpha c, (y, \alpha) \in \overline{C}\}$, then C' is a connected set in H satisfying $C \cap H(0) \neq \emptyset$ and $C \cap H(1) \neq \emptyset$ and $z \in C'$ implies $y = r(p(z) + f)$ for some $f \in \phi(p(z))$ with $y \in S$ given by $y = z - (c^\top z)z$. Therefore, there exists β satisfying

$$z - \alpha c = p(z) + f - \beta c.$$

Consequently,

$$f = z - p(z) + (\beta - \alpha)c.$$

Since $z - p(z) \in G(p(z))$, it follows from Lemma 2.1 that $f \in G(p(z))$ and so

$$f \in \phi(p(z)) \cap G(p(z)).$$

This implies that $x = p(z)$ is a stationary point of ϕ on X with respect to c. Finally, let the set C be defined by

$$C = \{x \in X \mid x = p(z), z \in C'\}.$$

Since p is a continuous function on H, we have that C is a connected set. Moreover, $C' \cap H(0) \neq \emptyset$ implies $C \cap X^- \neq \emptyset$ and $C' \cap H(1) \neq \emptyset$ implies $C \cap X^+ \neq \emptyset$. Also, $x \in C$ implies $x = p(z)$ for some $z \in C'$, i.e., x is a stationary point of ϕ on X with respect to c. $\square$

The theorem says that for any given nonzero vector c any Kakutani-type of mapping on a full-dimensional compact, convex set has a connected set of stationary points with respect to c having a nonempty intersection with both the (extreme) set of X at which $c^\top x$ is minimized on X and the (extreme) set of X on which $c^\top x$ is maximized on X. In the next section we discuss when a continuum of zero points exists.

3 Continuum of Zero Points

In this section we give sufficient conditions under which there exists a connected set of zero points of a mapping on a nonempty convex, compact set, connecting two different points in the boundary of the set. In the previous section we proved that there always exists a connected set of stationary points with repect to some nonzero vector. Clearly, if all these points are zero points of the mapping we obtain a continuum of zero points.

Theorem 3.1 *Let X, ϕ, c satisfy the conditions of Theorem 2.4. If for any $x \in X$ it holds that*

$$\phi(x) \cap G(x) \subset \{0^n\},$$

then there exists a connected set C of zero points of ϕ in X such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.

Proof: From Theorem 2.4 it follows that there exists a connected set C of stationary points of ϕ on X with respect to c satisfying $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$. Take any $x \in C$, then $G(x) \cap \phi(x) \neq \emptyset$. Since $G(x) \cap \phi(x) \subset \{0^n\}$ this implies $G(x) \cap \phi(x) = \{0^n\}$ and so $0^n \in \phi(x)$. Consequently, C is a connected set of zero points of ϕ in X having a nonempty intersection with both X^- and X^+. □

The condition in the theorem says that at any $x \in X$ no nonzero element of the image $\phi(x)$ is allowed to lie in the subgradient $G(x)$ of X_α. Although the condition itself is very weak it has to hold for every element in the image set. A sufficient condition that is much stronger but only has to hold for (at least) one element of the image set is the following one, where for $x \in X$ the set $G^*(x) = \{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for any } y \in G(x)\}$ denotes the polar cone of $G(x)$.

Theorem 3.2 *Let X, ϕ, c satisfy the conditions of Theorem 2.4. If for any $x \in X$ it holds that*

$$\phi(x) \cap G^*(x) \neq \emptyset,$$

then there exists a connected set C of zero points of ϕ in X such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.

The proof of this theorem does not follow immediately from Theorem 2.4, because the mapping G^* is in general not an upper semicontinuous mapping on X. To prove the theorem, let the set Q be defined by

$$Q = \{q \in \mathbb{R}^n \mid \| q - x \|_2 \leq 1, c^\top q = c^\top x, x \in X\}$$

and for α, $0 \leq \alpha \leq 1$, let

$$Q_\alpha = \{q \in Q \mid c^\top q = \alpha\}.$$

Lemma 3.3 *The set Q is a convex, compact and full-dimensional subset of H and for any $\alpha \in [0,1]$ the set Q_α contains X_α in its relative interior.*

Proof: To show convexity of Q, take any two points q^1 and q^2 in Q. Let $x^1 = p(q^1)$ and $x^2 = p(q^2)$. Take $q(\lambda) = \lambda q^1 + (1-\lambda)q^2$ for any $\lambda, 0 \leq \lambda \leq 1$. To show that $q(\lambda) \in Q$, let $x(\lambda) = \lambda x^1 + (1-\lambda)x^2$. Since X is convex, $x(\lambda)$ lies in X. Moreover,

$$c^\top q(\lambda) = \lambda c^\top q^1 + (1-\lambda)c^\top q^2 = \lambda c^\top x^1 + (1-\lambda)c^\top x^2 = c^\top x(\lambda)$$

and

$$\| q(\lambda) - x(\lambda) \|_2 = \| \lambda c^\top (q^1 - x^1) + (1-\lambda)c^\top (q^2 - x^2) \|_2$$

$$\leq \lambda \| q^1 - p(q^1) \|_2 + (1-\lambda) \| q^2 - p(q^2) \|_2 \leq 1.$$

Therefore, $q(\lambda) \in Q$. The other properties follow immediately from the definition of Q. $\square$

For $q \in Q$, let $v(q) = q - p(q)$. Since Q is a subset of H and, according to Lemma 2.3, the function p is continuous on H, the function v is continuous on Q. Also, $\| v(q) \|_2 = 1$ if and only if $q \in \text{bd}Q_\alpha$ with $\alpha = c^\top q$, $\| v(q) \|_2 = 0$ if and only if $q \in X$, and $c^\top v(q) = 0$ for any $q \in Q$. Furthermore, for $q \in Q$, let

$$B(q) = \{y \in \mathbb{R}^n \mid y = \mu v(q) + \beta c, \mu > 0, \beta \in \mathbb{R}\}$$

with polar cone

$$B^*(q) = \{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for all } y \in B(q)\},$$

and with $\alpha = c^\top q$ let $D(q)$ be the subgradient of Q_α at q.

Lemma 3.4 *The following properties hold:*

(i) $D(q) \subset B(q)$ *for any* q *in* Q.

(ii) $B(q) \subset G(p((q))$ *for any* q *in* Q.

(iii) The mapping B^* *is upper semicontinuous on* Q.

Proof: For $q \in \operatorname{int}Q_\alpha$ for some $\alpha \in [0,1]$ the subgradient $D(q)$ is equal to the set $\{y \in \mathbb{R}^n \mid y = \beta c, \beta \in \mathbb{R}\}$. Clearly, this set is contained in $B(q)$. For $q \in \operatorname{bd}Q_\alpha$ for some $\alpha \in [0,1]$, let the set $R(q)$ be defined by

$$R(q) = \{r \in R^n \mid c^\top r = \alpha, \| r - p(q) \|_2 \leq 1\}.$$

Then $R(q)$ is contained in Q_α and contains q. Hence, $v(q)$ is the only vector in $D(q)$ with length one and satisfying $c^\top v(q) = 0$. Now take any $w \in D(q)$. According to Lemma 2.3 also $v = w - (c^\top w)c$ lies in $D(q)$. Since $c^\top v = 0$, we must have $v = \mu v(q)$ for some $\mu \geq 0$. Therefore, $w = \mu v(q) + \beta c$ for some $\beta \in \mathbb{R}$, and so $w \in B(q)$. This proves property (i). Propery (ii) follows from Lemma 2.3 and from the fact that $v(q) \in G(p(q))$ for any $q \in Q$. To prove property (iii), notice that

$$B(q) = \{y \in \mathbb{R} \mid y = \beta c, \beta \in \mathbb{R}\}$$

if $q \in X$ and

$$B(q) = \{y \in \mathbb{R} \mid y = \mu v(q) + \beta c, \mu \geq 0, \beta \in \mathbb{R}\}$$

if $q \in Q \setminus X$. Since v is a continuous function on Q, we obtain that the mapping B^* is continuous at any point $q \notin \operatorname{bd}X$ and upper semicontinuous at any $q \in \operatorname{bd}X$. □

Next, consider the closure of the graph $\{(q, y) \mid y \in \phi(p(q)) \cap G^*(p(q)), q \in Q\}$ in $\mathbb{R}^n \times \mathbb{R}^n$ and let $\psi(q)$ be the convex hull of the image of q in this graph. Then ψ is an upper semicontinuous, bounded point-to-set mapping from the full-dimensional, compact, convex set Q in $\mathbb{R}^n$ to the collection of nonempty, conpact, convex subsets of $\mathbb{R}^n$. From Theorem 2.4 it follows that there exists a connected set $\overline{C}$ of stationary points of ψ on Q with respect to the vector c.

Lemma 3.5 *Let q^* be a stationary point pf ψ on Q with respect to c, then $x^* = p(q^*)$ is a zero point of ϕ in X.*

Proof: Since q^* is a stationary point of ψ on Q with respect to c, there exists $f \in \psi(q^*)$ satisfying $f \in D(q^*)$. The first property implies there exist f^i and nonnegative λ_i, $i \in I_{n+1}$, with $\sum_{i=1}^{n+1} \lambda_i = 1$, such that $f = \sum_{i=1}^{n+1} \lambda_i f^i$ and for every $i \in I_{n+1}$ it holds that $f^i = \lim_k f^{i,k}$ for certain $f^{i,k} \in \phi(p(q^{i,k})) \cap G^*(p(q^{i,k}))$, for all $k \in \mathbb{N}$, where $(q^{i,k})_{k \in \mathbb{N}}$ is a sequence of points in Q converging to q^*. Since p is a continuous funtion and ϕ is upper semicontinuous, it follows that $f^i \in \phi(x^*)$ for all $i \in I_{n+1}$, and, since ϕ is convex-valued, also that $f \in \phi(x^*)$. On the other hand, from property (ii) of Lemma 3.4 it follows for all i and k

$$B(q^{i,k} \subset G(p(q^{i,k}))$$

and therefore

$$G^*(p(q^{i,k})) \subset B^*(q^{i,k}).$$

Hence, $f^{i,k} \in B^*(q^{i,k})$ for all i and k. Since B^* is upper semicontinuous according to property (iii) of Lemma 3.4, we obtain that $f^i \in B^*(q^*)$ for all $i \in I_{n+1}$, and so, since $B^*(q^*)$ is convex, also that $f \in B^*(q^*)$. Finally, property (i) of Lemma 3.4 together with $f \in D(q^*)$ implies that $f \in B(q^*)$. Therefore, $f \in B(q^*) \cap B^*(q^*)$. Since the latter set only contains the n-vector of zeros, it follows that $f = 0^n$ and so $0^n \in \phi(x^*)$. □

PROOF OF THEOREM 5.3: From Lemma 3.5 it follows that there exists a coonected set C' of points in Q satisfying $C' \cap Q_0 \neq \emptyset$ and $C' \cap Q_1 \neq \emptyset$ and $q \in C'$ implies $0^n \in \phi(p(q))$. Now, let $C = \{x \in X \mid x = p(q), q \in C'\}$. Since p is a continuous function on X, the set C is a connected set in X. Moreover, $C' \cap Q_0 \neq \emptyset$ implies $C \cap X^- \neq \emptyset$ and $C' \cap Q_1 \neq \emptyset$ implies $C \cap X^+ \neq \emptyset$, whereas $x \in C$ implies $0^n \in \phi(x)$. Hence, C is a connected set of zero points of ϕ in X having a nonempty intersection with both X^- and X^+. □

The next theorem is a combination of the two latter theorems. It relaxes the rather strong condition of Theorem 5.3 to hold for at least one element of every image set and adds a condition for all elements in every image set, which is a weaker condition than the one in Theorem 5.2. For $x \in X$ with $\alpha = c^\top x$, let

$$G_0(x) = \{y \in \mathbb{R}^n \mid c^\top y = 0, x'^\top y \leq x^\top y \text{ for all } x' \in X_\alpha\}$$

and

$$G_0^*(x) = \{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for all } y \in G_0(x)\}.$$

Theorem 3.6 *Let X, ϕ, c satisfy the conditions of Theorem 2.4. If for any $x \in X$ it holds that*

(i) $\phi(x) \cap (G_0^* \cap G(x)) \subset \{0^n\}$,

(ii) $\phi(x) \cap G_0^* \neq \emptyset$,

then there exists a connected set C of zero points of ϕ such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.

Proof: Define the mapping ϕ' on X by

$$\phi'(x) = \{y \in \mathbb{R}^n \mid c^\top y = 0.y + \beta c \in \phi(x) \text{ for some } \beta \in \mathbb{R}\}.$$

Clearly, $\phi'(x)$ is upper semicontinuous and bounded on X and for every $x \in X$ the set $\phi'(x)$ is nonemty, convex and compact. We will show that ϕ' satisfies the condition of

Theorem 5.3. Condition (ii) implies that there exists $z \in \phi(x) \cap G^*(x)$. Let y be such that $y = z - \beta c \in \phi'(x)$ for some $\beta \in \mathbb{R}$, so $c^\top y = 0$. To prove that also $y \in G^*(x)$ take any $v \in G(x)$, then $w = v - (c^\top v)c$ lies in $G_0(x)$. Since $z \in G_0^*(x)$, we have that $z^\top w \leq 0$, and so $c^\top w \leq 0$ because $c^\top = 0$. On the other hand, $y^\top w \leq 0$ implies $y^\top v \leq 0$ because $c^\top y = 0$. Hence, $y^\top c \leq 0$ for any $v \in G(x)$, i.e. $y \in G^*(x)$. Consequently, for any $x \in X$ it holds that $\phi'(x) \cap G^*(x) \neq \emptyset$. From Theorem 5.3 it follows that there exists a connected set C of zero points of ϕ' in X having a nonempty intersection with both X^- and X^+. We now show that every $x \in C$ is also a zero point of ϕ. Take any $x \in C$, then $0^n \in \phi(x)$ and so there exists $\beta \in \mathbb{R}$ satisfying $\beta c \in \phi(x)$. We also have that $\mu c \in G_0^*(x) \cap G(x)$ for any $\mu \in \mathbb{R}$. Hence,

$$\beta c \in \phi(x) \cap (G_0^*(x) \cap G(x)).$$

Condition (i) implies that $\beta = 0$ and so $0^n \in \phi(x)$ for every $x \in C$. Therefore, C is a connected set of zero points of ϕ having a nonempty intersection with both X^- and X^+. $\square$

4 The Algorithm

To follow approximately a connected set of stationary points with respect to some given nonzero vector of a function or a mapping on a convex, compact set X, we approximate the latter set by a sequence of polytopes. Without loss of generality we assume again that the set X is full-dimensional. The sequence of polytopes $(P^r)_{r\in\mathbb{N}}$ is such that $X \subset P^r \subset P^{r-1}$ for all $r \in \mathbb{N}$ and $X = \lim_r P^r$. Let M^r be the set of all vectors $m \in \mathbb{Z}^n$ with $\sum_{i=1}^n |m_i| \leq r$, and let

$$A^r = \{a \in \mathbb{R}^n \mid a = m/ \parallel m \parallel_2, m \in M^r\} \cup \{-c, c\}, r \in \mathbb{N}.$$

The polytope P^r is now defined by

$$P^r = \{y \in \mathbb{R}^n \mid a^\top y \leq b(a), a \in A^r\}, r \in \mathbb{N},$$

where for $a \in \mathbb{R}^n$ the number $b(a)$ is given by

$$b(a) = \max\{a^\top x \mid x \in X\}.$$

Notice that $b(-c) = 0$, $b(c) = 1$, and $P^r \subset H$ for all $r \in \mathbb{N}$. For any $a \in \mathbb{R}^n \setminus \{0^n\}$, let the hyperplane $H(a)$ be given by $H(a) = \{y \in \mathbb{R}^n \mid a^\top y = b(a)\}$ and the halfspace $H^-(a)$ by $H^-(a) = \{y \in \mathbb{R}^n \mid a^\top y \leq b(a)\}$. Then, for every $r \in \mathbb{N}$,

$$P^r = \{y \in \mathbb{R}^n \mid y \in H^-(a) \text{ for all } a \in A^r\}$$

and $X \subset H^-(a)$ for any $a \in A^r$, and so $X \subset P^r$. Moreover, since $M^r \subset M^{r+1}$, we also have $P^{r+1} \subset P^r$ for all $r \in \mathbb{N}$.

Lemma 4.1 *The $lim_r P^r$ is well defined and is equal to the set X.*

Proof: Since for every $r \in \mathbb{N}$ it holds that $X \subset P^{r+1} \subset P^r$ and P^r is compact, the limit exists and contains X. Suppose now that there exists $y \notin X$ satisfying $y \in P^r$ for all $r \in \mathbb{N}$. Since $y \notin X$ and X is a compact, convex set in $\mathbb{R}^n$, there exists a hyperplane in $\mathbb{R}^n$ strictly seperating y and X. Let $H^* = \{z \in \mathbb{R}^n \mid h^\top z = d\}$ for some $h \neq 0$ and $d \in \mathbb{R}$ be such a hyperplane. Without loss of generality all components of h are integer, $h^\top y > d$, and $h \neq \beta c$ for any $\beta \in \mathbb{R}$. Let the vector a be defined by $a = h/ \parallel h \parallel_2$. Then a is an element of A^r for any $r \geq \sum_{i=1}^n |h_i|$. Let $e = d/ \parallel h \parallel_2$, then $H^* = \{z \in \mathbb{R}^n \mid a^\top z = e\}$ and so $a^\top y > e$ and $a^\top x < e$ for any $x \in X$. Consequently, $b(a) = \max\{a^\top x \mid x \in X\} < e$ and so $a^\top y > e > b(a)$. Hence, $y \notin H^-(a)$ while $ainA^r$ for any $r \geq \sum_{i=1}^n |h_i|$, i.e., $y \notin P^r$ for r large enough. This contradicts the fact that $y \in P^r$ for all $r \in \mathbb{N}$. □

For any $r \in \mathbb{N}$, let the faces P_0^r and P_1^r of P^r be defined by

$$P_0^r = \{y^0 \in \mathbb{R}^n \mid c^\top y^0 = \min_{y \in P^r} c^\top y\}$$

and

$$P_1^r = \{y^1 \in \mathbb{R}^n \mid c^\top y^1 = \max_{y \in P^r} c^\top y\}.$$

Since both c and $-c$ belong to A^r, it holds that $X^- \subset P_0^r$ and $X^+ \subset P_1^r$. Next take a sequence of simplicial subdivisions $(T^r)_{r \in \mathbb{N}}$ such that for every $r \in \mathbb{N}$ the subdivision T^r is a triangulation of P^r and mesh$T^r < r^{-1}$, i.e., every element of T^r is an n-dimensional simplex in P^r, being the convex hull of $n+1$ affinely independent points in P^r, with diameter less than r^{-1}, the union of all elements equals P^r, and the intersection of two elements is a common face of both. For triangulations satisfying these properties, we refer to Talman and Yamamoto (1989). Since for any given $r \in \mathbb{N}$, the set A^r is finite, we can index the elements of A^r from 1 up to, say, m_r and assume without loss of generality that P^r is simple and has no redundant constraints. So, the polytope P^r can be rewritten as

$$P^r = \{y \in \mathbb{R}^n \mid a^{i,r\top} y \leq b_i^r, \ i \in I_{m_r}\},$$

where $b_i^r = b(a^{i,r})$, $i \in I_{m_r}$, $r \in \mathbb{N}$.

For $I \subset I_{m_r}$ we define the face $F^r(I)$ of P^r by

$$F^r(I) = \{y \in P^r \mid a^{i,r\top} y = b_i^r, \ i \in I\},$$

if this set is nonempty. Due to the simpleness of P^r and because there are no redundant constraints, we have that the dimension of $F^r(I) = n - |I|$, where $|B|$ is the number of elements of a finite set B. For $r \in \mathbb{N}$, let $\mathcal{I} = \{\mathcal{I} \subset \mathcal{I}_{\Updownarrow\triangledown} \mid \mathcal{F}^\triangledown(\mathcal{I}) \neq \emptyset\}$ be the collection of index sets corresponding to the faces of P^r. For $I \subset \mathcal{I}^{\nabla}$ we also define the dual set $A^r(I)$ by

$$A^r(I) = \{y \in \mathbb{R}^n \mid y = \sum_{h \in I} \mu_h a^{i,r} + \beta c, \mu_i \geq 0, \beta \in \mathbb{R}\}.$$

Since T^r is a simplicial subdivision of P^r, $r \in \mathbb{N}$, every face $F^r(I)$, $I \in \mathcal{I}^{\nabla}$, of P^r is simplicially subdivided also. We denote the triangulation of a face $F^r(I)$ induced by T^r by $T^r(I)$. For any $r \in \mathbb{N}$ the mapping ϕ on X is now extended to the mapping ψ^r on P^r defined by

$$\psi^r(y) = \phi(p(y)), \; y \in P^r.$$

Finally, let f^r be the piecewise linear approximation of ψ^r on P^r with respect to the triangulation T^r, i.e., if $y = \sum_{j=1}^{n+1} \lambda_j y^j \in \sigma$ for some simplex $\sigma \in T^r$ with vertices y^j, $j \in I_{n+1}$, where all λ^js are nonnegative and sum up to one, then $f^r(y) = \sum_{j=1}^{n+1} \lambda_j f^j$, for some given $f^j \in \psi^r(y^j)$, $j \in I_{n+1}$. Since p is a continuous function, for every $r \in \mathbb{R}$, the mapping ψ^r is upper semicontinuous, bounded, nonempty-, convex- and compact-valued on P^r and the function f^r is continous on P^r and linear on every simplex of T^r.

According to the results in Herings, Talman, and Yang (2001) there exists a piecewise linear path $\pi_r([0,1])$ in P^rsatisfying $\pi^r(0) \in P^r_0$, $\pi^r(1) \in P^r_1$, and for any $y \in \pi([0,1])$ there exists $I^r \in \mathcal{I}^{\nabla}$ satisfying both $y \in F^r(I)$ and $f^r(y) \in A^r(I^r)$, i.e., y is a stationary point of the function f^r on P^r with respect to the vector c. Each linear piece of the path $\pi([0,1]))$ lies in a simplex of $T^r(I)$ for some I and can be generated by making a lexicographic linear programming pivot step in the following system of $n+1$ equations and $n+2$ variables

$$\sum_{j=1}^{t_r+1} \lambda_j \begin{pmatrix} 1 \\ -f(y^{j,r}) \end{pmatrix} + \sum_{h \in I^r} \mu_h \begin{pmatrix} 0 \\ a^{h,r} \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix}$$

satisfying $\lambda_j \geq 0$ for $j \in I_{t^r+1}$, $\mu_h \geq 0$ for $h \in I^r$, and $\beta \in \mathbb{R}$, where $y^{1,r}, \cdots, y^{t^r+1,r}$ are the vertices of a (t^r+1)-dimensional simplex of $T^r(I^r)$ and $t^r = n - |I^r|$. Since for any $r \in \mathbb{R}$ the triangulation T^r consists of a finite number of simplices and no simplex in some $T^r(I^r)$ can be visited more than once, the path $\pi^r([0,1])$ can be generated in this way within a finite number of iterations. Also the point $\pi^r(0)$ in P^r_0 can be found within a finite number of iterations, see Herings, Talman, and Yang (2001).

In order to show that that the piecewise linear paths $\pi^r([0,1])$, $r \in \mathbb{N}$, can be considered to be an approximation of a connected set of stationary points of ϕ on X with respect to c, we show that any convergent sequence $(x^r)_{r \in \mathbb{N}}$ of points, satisfying $x^r \in \pi^r([0,1])$ for all $r \in \mathbb{N}$, with limit x^* satisfies that the point x^* is a stationary point of ϕ on X with respect to c. From Lemma 4.1 it follows that the limit x^* belongs to X. Moreover, since $x^r \in \pi^r([0,1])$, for every $r \in \mathbb{R}$ there exists $I^r \in \mathcal{I}^{\nabla}$ satisfying both $x^r \in F^r(I^r)$ and

$f^r(x^r) \in A^r(I^r)$, i.e.,
(i) $x^r = \sum_{j=1}^{t_r+1} \lambda_j^r x^{j,r} \in \tau^r$ for some t_r-dimensional simplex $\tau^r \in T^r(I^r)$ with vertices $x^{j,r}$, $j \in I_{t^r+1}$, where $t_r = n - |I^r|$ and the λ_j^rs being nonnegative and sum up to one,
(ii) $f^r(x^r) = \sum_{h\in I^r} \mu_h^r a^{h,r} + \beta^r c$ for some $\mu_h^r \geq 0$, $h \in I^r$, and $\beta \in \mathbb{R}$.
Without loss of generality we assume that for every $r \in \mathbb{N}$ it holds that $I^r = I_k$ for some $k \in \{0\} \cup I_n$ (where $I_0 = \emptyset$). Since according to (i) $x^r \in F^r(I_k)$, for all $h \in I_k$ we have that ${a^{h,r}}^\top y \leq b_h^r$ for any $y \in P^r$, $r \in \mathbb{N}$, where $b_h^r = {a^{h,r}}^\top x^r$. For every $h \in I_k$, the $a^{h,r}$s lie in a compact set. Therefore, by taking subsequent subsequences if necessary, for every $h \in I_k$ the sequence $(a^{h,r})_{r\in\mathbb{N}}$ converges to some vector a^h satisfying $\| a^h \|_2 = 1$. Clearly, by taking limits, for every $h \in I_k$ it holds that ${a^h}^\top x \leq {a^h}^\top x^*$ for any $x \in X$, sothat $a^h \in G(x^*)$ for all $h \in I_k$. Since ϕ is bounded, the solutions of the system of equations on the sequence also ly in a compact set, i.e., by taking subsequences if necessary again, $\lambda_j^r \to \lambda_j \geq 0$, $j \in I_{t+1}$, with sum equal to 1, $\mu_h^r \to \mu_j \geq 0$, $h \in I_k$, $\beta^r \to \beta \in \mathbb{R}$, and $f^r(x^{j,r}) \to f^j$, $j \in I_{t+1}$. Since meshT^r goes to zero, for every $j \in I_{t+1}$ it holds that $x^{j,r} \to x^*$. Now, let $f = \sum_{j=1}^{t+1} \lambda_j f^j$. From the upper semicontinuity of ϕ it follows that $f^j \in \phi(x^*)$ for all j and from the convex-valuedness of ϕ that $f \in \phi(x^*)$. On the other hand, since $f^r(x^r) \to f$, it follows from (ii) that

$$f = \sum_{h\in I_k} \mu_h a^h + \beta c,$$

where $\mu_h \geq 0$, $h \in I_k$ and $\beta \in \mathbb{R}$. We already showed that $a^h \in G(x^*)$ for all $h \in I_k$. Hence, using Lemma 2.3 and the fact that $G(x^*)$ is a cone, we obtain that $f \in G(x^*)$. Consequently, x^* is a stationary point of ϕ on X with respect to c.

In case of Theorems 5.3 or 5.4 the piecewise linear approximation f^r must be chosen in such a way that in case y is a vertex of a simplex of the triangulation T^r of P^r the element $f^r(y)$ lies in the supposed nonempty intersection for $p(y)$. For these two theorems it is not necessary to approximate the bigger set Q by a sequence of polytopes. Approximating X by a sequence of polytopes suffices. For Theorem 5.2 any piecewise linear approximation of ϕ can be chosen. For all three theorems it holds that the limit point x^* is a zero point of ϕ in X. Notice that in this way we also obtain a constructive proof of the existence of a continuum of zero points on a compact, convex set.

For I of I_m, define

$$F(I) = \{x \in P \mid {a^i}^\top x = b_i,\ \forall i \in I\}.$$

Then $F(I)$ is called a *face* of P unless it is empty. Note that $F(\emptyset) = P$. Let

$$\mathcal{I} = \{I \subset I_m \mid F(I) \text{ is a face of } P\}.$$

A face F of the set $F(I)$ of dimension one lower than the dimension of $F(I)$ is called a *facet* of $F(I)$. The polytope P is said to be *simple* if the dimension of any face $F(I)$ of P is equal to $n - |I|$. Throughout the paper, whenever we use a polytope P, it is assumed that P is simple and that its representation as a polyhedron has no redundant constraints.

We have the following observations.

(i) For each face F of P, the set $I \in \mathcal{I}$ with $F = F(I)$ is unique and is given by the set $\{i \in I_m \mid a^{i\top} x = b_i, \ \forall x \in F\}$.

(ii) The set $F(I)$ is a vertex of P if and only if $I \in \mathcal{I}$ with $|I| = n$.

(iii) If $I \in \mathcal{I}$, then $I \setminus \{i\} \in \mathcal{I}$ for any $i \in I$.

(iv) For some $I \in \mathcal{I}$, F is a facet of $F(I)$ if and only if $F = F(I \cup \{i\})$ for some $i \notin I$ with $I \cup \{i\} \in \mathcal{I}$.

(v) For any $I \in \mathcal{I}$, the vectors a^i with $i \in I$ are linearly independent.

Let c be an arbitrary nonzero vector in $\mathbb{R}^n$. Then F^+ will denote the face of P such that for each $x^+ \in F^+$ it holds that $c^\top x^+ = \max_{x \in P} c^\top x$, and F^- will denote the face of P such that for each $x^- \in F^-$ it holds that $c^\top x^- = \min_{x \in P} c^\top x$. Let $t^+ = c^\top x^+$ for $x^+ \in F^+$ and $t^- = c^\top x^-$ for $x^- \in F^-$. Since P is full-dimensional, it follows that $t^- < t^+$ and therefore $F^- \cap F^+ = \emptyset$. We define

$$
\begin{aligned}
I^+ &= \{i \in I_m \mid a^{i\top} x = b_i, \ \forall x \in F^+\} \\
I^- &= \{i \in I_m \mid a^{i\top} x = b_i, \ \forall x \in F^-\}.
\end{aligned}
$$

So $F^+ = F(I^+)$ and $F^- = F(I^-)$.

We need some further notation. For each $I \in \mathcal{I}$, we define

$$
\begin{aligned}
A(I) &= \{y \in \mathbb{R}^n \mid y = \sum_{i \in I} \mu_i a^i + \beta c, \ \mu_i \geq 0, \forall i \in I, \text{ and } \beta \in \mathbb{R}\}, \\
A_0(I) &= \{y \in \mathbb{R}^n \mid y = \sum_{i \in I} \mu_i a^i, \ \mu_i \geq 0, \forall i \in I\}, \\
\\
A^*(I) &= \{x \in \mathbb{R}^n \mid x^\top y \leq 0, \ \forall y \in A(I)\}, \\
A_0^*(I) &= \{x \in \mathbb{R}^n \mid x^\top y \leq 0, \ \forall y \in A_0(I)\}.
\end{aligned}
$$

Note that $A(\emptyset) = \{y \in \mathbb{R}^n \mid y = \beta c, \ \beta \in \mathbb{R}\}$, $A_0(\emptyset) = \{0^n\}$, $A^*(\emptyset) = \{x \in \mathbb{R}^n \mid x^\top c = 0\}$, and $A_0^*(\emptyset) = \mathbb{R}^n$. Moreover, for any $I \in \mathcal{I}$ we have that $A_0(I) \subset A(I)$, $A^*(I) \subset A_0^*(I)$, $A^*(I) \cap A(I) = \{0^n\}$, and $A_0^*(I) \cap A_0(I) = \{0^n\}$.

Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be a correspondence that satisfies the following assumption.

Assumption 4.2 *The correspondence $\varphi : P \Longrightarrow \mathbb{R}^n$ is non-empty valued, compact valued, convex valued, and upper semi-continuous.*

For an arbitrary function $f : P \to \mathbb{R}^n$, the *stationary point* (or *variational inequality*) *problem* for f on the polytope P is to find a point $x^* \in P$ such that

$$(x^* - x)^\top f(x^*) \geq 0, \ \forall x \in P.$$

Such a point x^* is called a *stationary point* of f on P. It is well-known that a continuous function on a convex compact set has a stationary point; see Eaves (1971) and Hartman and Stampacchia (1966). Now we have the following simple but important observation.

Lemma 4.3 *A point $x^* \in P$ is a parametrized stationary point of the mapping φ on P with respect to c if and only if for some $I \in \mathcal{I}$ it holds that $x^* \in F(I)$ and $\varphi(x^*) \cap A(I) \neq \emptyset$.*

Proof: The result follows immediately from linear optimization. □

We show the existence of a continuum of parametrized stationary points of the correspondence φ with respect to the vector c. A point $x^* \in P$ is called a *zero point* of the mapping φ if $0^n \in \varphi(x^*)$, where 0^n denotes the n-vector of zeros. Any zero point of φ is a parametrized stationary point. The main purpose of this paper is to find conditions on φ which enable us to guarantee the existence and the computation of a continuum of zero points of a correspondence φ on P, such that it contains points of both F^- and F^+.

5 Existence Conditions

The first result is on the existence of parametrized stationary points.

Theorem 5.1 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be any correspondence satisfying Assumption 4.2 and let $c \in \mathbb{R}^n \setminus \{0\}$ be given. Then there exists a connected set C of parametrized stationary points of φ on P with respect to c, such that $C \cap F^- \neq \emptyset$ and $C \cap F^+ \neq \emptyset$.*

The theorem makes clear that there are many parametrized stationary points. To be more precise, there exists a continuum of them with a special topological structure. The set of parametrized stationary points has a connected subset that links the two faces F^- and F^+.

In order to obtain the existence of a continuum of zero points, we need to impose certain conditions on the correspondence φ. The following three results list sufficient conditions for the existence of a continuum of zero points of φ.

Theorem 5.2 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be any correspondence satisfying Assumption 4.2 and let $c \in \mathbb{R}^n \setminus \{0\}$. If for any $x \in F(I), I \in \mathcal{I}$, it holds that*

$$\varphi(x) \cap A(I) \subset \{0^n\},$$

then there exists a connected set C of zero points of φ such that $C \cap F^- \neq \emptyset$ and $C \cap F^+ \neq \emptyset$.

The condition in the theorem states that for any x in the face $F(I)$ of P the image $\varphi(x)$ may not have nonzero elements in common with $A(I)$.

Theorem 5.3 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be any correspondence satisfying Assumption 4.2 and let $c \in \mathbb{R}^n \setminus \{0\}$. If for any $x \in F(I), I \in \mathcal{I}$, it holds that*

(i) $\varphi(x) \cap A_0^*(I) \cap A(I) \subset \{0^n\}$;

(ii) $\varphi(x) \cap A_0^*(I) \neq \emptyset$,

then there exists a connected set C of zero points of φ such that $C \cap F^- \neq \emptyset$ and $C \cap F^+ \neq \emptyset$.

The conditions in the theorem imply that for any x in the face $F(I)$ of P the image $\varphi(x)$ may not have nonzero elements in common with $A(I) \cap A_0^*(I)$ and that at least one element of $\varphi(x)$ lies in $A_0^*(I)$.

Theorem 5.4 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be any correspondence satisfying Assumption 4.2 and let $c \in \mathbb{R}^n \setminus \{0\}$. If for any $x \in F(I), I \in \mathcal{I}$, it holds that*

$$\varphi(x) \cap A^*(I) \neq \emptyset,$$

then there exists a connected set C of zero points of φ such that $C \cap F^- \neq \emptyset$ and $C \cap F^+ \neq \emptyset$.

The condition in the theorem says that for any x in the face $F(I)$ of P at least one element of $\varphi(x)$ lies in $A^*(I)$.

The three theorems state different conditions for which a continuum of zero points can be shown to exist. Moreover, there is a logical order in these sufficient conditions. Theorem 5.2 states a weak condition, but one that holds for all elements of $\varphi(x)$. Theorem 5.4 gives a strong condition, but requires this condition to hold for only one element in $\varphi(x)$. Theorem 5.3 is in between: a very weak condition for all elements in $\varphi(x)$ together with a rather weak condition for some element in $\varphi(x)$.

The following result claims that for the special case where φ is a function f, Theorem 5.2 is the strongest and Theorem 5.4 the weakest. Note that Assumption 4.2 implies that the function f is continuous.

Theorem 5.5 *Let P be any polytope and let $c \in \mathbb{R}^n \setminus \{0\}$. The collection of functions satisfying the conditions of Theorem 5.2 contains the collection of functions satisfying the conditions of Theorem 5.3, which contains the collection of functions satisfying the conditions of Theorem 5.4.*

Proof: Suppose that a function f from P to $\mathbb{R}^n$ satisfies the conditions of Theorem 5.4. Take any x in $F(I)$, so $f(x) \in A^*(I)$. Since $A^*(I) \subset A_0^*(I)$ and $A(I) \cap A^*(I) = \{0^n\}$ we obtain that $f(x) \in A_0^*(I)$ and $f(x)$ not in $A(I)$ unless $f(x) = 0^n$. Hence the conditions of Theorem 5.3 are satisfied. Suppose now that a function satisfies the conditions of Theorem 5.3. Again take any x in $F(I)$, so $f(x) \in A_0^*(I)$ and $f(x)$ not in $A(I) \cap A_0^*(I)$ unless $f(x) = 0^n$. Hence $f(x)$ not in $A(I)$ unless $f(x) = 0^n$. □

That Theorems 5.2, 5.3, and 5.4 are mutually exclusive for correspondences follows from the fact that in case of correspondences the image $\varphi(x)$ of any point x in P might consist of more than one element. For example for a point x in the face $F(I)$ of P it is required in Theorem 5.2 that no nonzero element of $\varphi(x)$ lies in $A(I)$ which does not imply that at least one such element should lie in $A^*(I)$ as required in the conditions of Theorem 5.4. On the other hand if for an x in $F(I)$ it holds that some nonzero element $f \in \varphi(x)$ lies in $A^*(I)$, as in the conditions of Theorem 5.4, then this implies that f indeed does not lie in $A(I)$ but not necessarily that all the other elements of $\varphi(x)$ also do not lie in $A(I)$ as required in Theorem 5.2. Similar remarks can be made when comparing the conditions of Theorem 5.3 with the conditions in the other two theorems.

The theorems above will be proved in a constructive manner in the following sections. It is worth pointing out that the above results can be extended to a nonempty convex compact set by using the subgradient set of a convex function since every convex set can be described by means of a convex function.

6 The Algorithm and Its Convergence Proof

In this section we describe an algorithm on the polytope P that generates a piecewise linear path of parametrized stationary points, with respect to the given nonzero vector c, of a piecewise linear approximation of the mapping φ. The piecewise linear approximation is taken with respect to some simplicial subdivision of the set P. In this and the next sections we assume that the faces F^- and F^+ are vertices of P, denoted by x^- and x^+, respectively. In that case the piecewise linear path generated by the algorithm contains both x^- and x^+ and can be traced by a sequence of semi-lexicographic pivot steps in a system of linear equations. The general case is discussed in Section 6.

For a nonnegative integer t, a t-dimensional simplex or t-simplex, denoted by σ, is defined by the convex hull of $t + 1$ affinely independent points $x^1, \cdots, x^{t+1}$ in $\mathbb{R}^n$. We often write $\sigma = \sigma(x^1, \cdots, x^{t+1})$ and call $x^1, \cdots, x^{t+1}$ the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of σ is said to be a facet of σ. The facet $\tau(x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^{t+1})$ is called the facet of $\sigma(x^1, \cdots, x^{t+1})$ opposite to the vertex x^i. For k, $0 \leq k \leq t$, a k-simplex being the convex hull of $k + 1$ vertices of σ is said to be a k-face or face of σ. A finite collection $\mathcal{T}$ of n-simplices is a triangulation of the polytope P if

(i) P is the union of all simplices in $\mathcal{T}$;

(ii) The intersection of any two simplices of $\mathcal{T}$ is either the empty set or a common face of both.

Let $\mathcal{T}$ be any triangulation of P. Then every face $F(I)$ of P is subdivided into t-simplices, where $t = n - |I|$. For example we can take the V-triangulation of Talman and Yamamoto (1989). Since $\mathcal{T}$ is finite and P is compact, every facet τ of an $(n - |I|)$-simplex σ in $F(I)$ either lies in the boundary of $F(I)$ and is only a facet of σ or is a facet of exactly one other $(n - |I|)$-simplex in $F(I)$. Let f be a simplicial approximation of φ with respect to $\mathcal{T}$. This means that $f(x) \in \varphi(x)$ for each vertex of $\mathcal{T}$ and f is affine on each simplex of $\mathcal{T}$.

A row vector is *lexicopositive* if it is a nonzero vector and its first nonzero entry is positive. A matrix is said to be *lexicopositive* if all its rows are lexicopositive. A matrix is said to be *semi-lexicopositive* if each row except possibly the last row is lexicopositive.

Definition 6.1 *Let $\tau(x^1, \cdots, x^t)$ be a $(t-1)$-simplex in $F(I)$ where $I \in \mathcal{I}$ with $I = \{i_{t+1}, \cdots, i_n\}$, $t = n - |I|$. The $(n+1) \times (n+1)$ matrix*

$$A_{\tau,I} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ -f(x^1) & \cdots & -f(x^t) & a^{i_{t+1}} & \cdots & a^{i_n} & c \end{bmatrix}$$

is the label matrix of τ with respect to I. The simplex τ is I-complete if $A_{\tau,I}^{-1}$ exists and is semi-lexicopositive, i.e. the first nonzero entry in every row, except possibly the last one, is positive.

Notice that if for an I-complete simplex τ we change the ordering of the first n columns of matrix $A_{\tau,I}$, the inverse of the resulting matrix still exists and is semi-lexicopositive. Clearly, if, for some $I \in \mathcal{I}$, a $(t-1)$-simplex $\tau(x^1, \ldots, x^t)$ is an I-complete facet of a simplex $\sigma(x^1, \ldots, x^{t+1})$ in some face $F(I)$, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{j=1}^{t+1} \lambda_j \begin{pmatrix} 1 \\ -f(x^j) \end{pmatrix} + \sum_{i \in I} \mu_i \begin{pmatrix} 0 \\ a^i \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (*)$$

has a solution $(\lambda, \mu, \beta) = (\lambda_1, \ldots, \lambda_{t+1}, (\mu_i)_{i \in I}, \beta)$ satisfying $\lambda_j \geq 0$ for $j \in I_{t+1}$ and $\mu_i \geq 0$ for $i \in I$, with $\lambda_{t+1} = 0$. Let x be defined by $x = \sum_{j=1}^{t+1} \lambda_j x^j$ at a solution (λ, μ, β) of $(*)$, then x lies in σ and is a parametrized stationary point of f with respect to c.

The following result is a special case of Theorem 2.6 in Fujishige and Yang (1998) and will be used later on. This result has been proved in a constructive way.

Theorem 6.2 *Consider any polytope Q given by $\mathrm{Q} = \{x \in \mathbb{R}^n \mid c^{i\top} x \leq d_i,\ i \in I_n$ and $c^{0\top} x = d_0\}$. Assume that Q is an $(n-1)$-dimensional simple polytope with no* *redundant constraints. For any $g \in \mathbb{R}^n$, there exists a unique subset $I = \{j_1, \cdots, j_{n-1}\}$ of I_n with $|I| = n-1$ such that the following matrix*

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ g & c^{j_1} & c^{j_2} & \cdots & c^{j_{n-1}} & c^0 \end{bmatrix}^{-1}$$

exists and is semi-lexicopositive.

We now show that each of the two 0-simplices $\{x^-\}$ and $\{x^+\}$ is I-complete in the face $F(I)$ for a unique index set $I \in \mathcal{I}$ containing $n-1$ indices.

Lemma 6.3 *Let $x^1 = x^-$ and $\tau = \{x^1\}$. Then there exists a unique subset $I = \{j_1, \cdots, j_{n-1}\}$ of I^- with $|I| = n-1$ such that τ is an I-complete 0-simplex in $F(I)$.*

Proof: Recall that x^- is a unique solution to the problem

$$\min c^\top x, \ \ s.t. \ x \in P.$$

By duality theory there exists a unique solution $\lambda_i > 0$ for all $i \in I^-$ such that $-c = \sum_{i \in I^-} \lambda_i a^i$. In other words, the vectors c and a^i, $i \in I^-$, are affinely independent. Consider the following polyhedron

$$W = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq 1, i \in I^-, c^\top x \leq 1\}.$$

It is easy to see that W is bounded and contains 0^n in its interior and therefore is an n-dimensional polytope. Then the following set

$$Q = \{x \in \mathbb{R}^n \mid {a^i}^\top x \leq 1, i \in I^-, c^\top x = 1\}$$

is an $(n-1)$-dimensional polytope. Let $g = f(x^-)$. Now all the conditions of Theorem 6.2 are satisfied. So there exists a unique subset $I = \{j_1, \cdots, j_{n-1}\}$ of I^- with $|I| = n-1$ such that the following matrix

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -f(x^-) & a^{j_1} & a^{j_2} & \cdots & a^{j_{n-1}} & c \end{bmatrix}^{-1}$$

exists and is semi-lexicopositive. This means that τ is I-complete. Clearly, τ lies in $F(I)$ since $F(I^-)$ is a subset of $F(I)$ and $\tau = F(I^-)$. $\square$

Lemma 6.4 *Let $x^1 = x^+$ and $\tau = \{x^1\}$. Then there exists a unique subset $I = \{j_1, \cdots, j_{n-1}\}$ of I^+ with $|I| = n-1$ such that τ is an I-complete 0-simplex in $F(I)$.*

Proof: Notice that x^+ is the unique solution to the problem

$$\max c^\top x, \ \ s.t. \ x \in \mathrm{P}.$$

By duality theory there exists a unique solution $\lambda_i > 0$ for all $i \in I^+$ such that $c = \sum_{i \in I^+} \lambda_i a^i$. In other words, the vectors $-c$ and a^i, $i \in I^+$ are affinely independent. Consider the following polyhedron

$$W = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq 1, i \in I^+, -c^\top x \leq 1\}.$$

It is easy to see that W is an n-dimensional polytope. Then the following set

$$Q = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq 1, i \in I^+, c^\top x = -1\}$$

is an $(n-1)$-dimensional polytope. Let $g = f(x^+)$. Now all the conditions of Theorem 6.2 are satisfied. So there exists a unique subset $I = \{j_1, \cdots, j_{n-1}\}$ of I^+ with $|I| = n-1$ such that the following matrix

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -f(x^+) & a^{j_1} & a^{j_2} & \cdots & a^{j_{n-1}} & c \end{bmatrix}^{-1}$$

exists and is semi-lexicopositive. This means that τ is I-complete. Clearly, τ lies in $F(I)$ since $F(I^+)$ is a subset of $F(I)$ and τ lies in $F(I^+)$. □

The following lemma is well-known in linear programming theory and can easily be proved. We will invoke it later. Let B be a matrix. We denote its i-th row by $B_{i.}$ and its j-th column by $B_{.j}$.

Lemma 6.5 *Let $B = (B_{.1}, \cdots, B_{.n+1})$ be any non-singular $(n+1) \times (n+1)$ matrix and let x be any vector in $\mathbb{R}^{n+1}$. Let $k \in I_{n+1}$ and $\bar{B} = (B_{.1}, \cdots, B_{.k-1}, x, B_{.k+1}, \cdots, B_{.n+1})$. Then either $(B^{-1}x)_k = 0$ and $\bar{B}$ is singular, or $(B^{-1}x)_k \neq 0$, $\bar{B}$ is non-singular and $\bar{B}^{-1}$ is given by*

$$\bar{B}^{-1} = \begin{bmatrix} (B^{-1})_{1.} - \frac{(B^{-1}x)_1}{(B^{-1}x)_k}(B^{-1})_{k.} \\ \vdots \\ (B^{-1})_{k-1.} - \frac{(B^{-1}x)_{k-1}}{(B^{-1}x)_k}(B^{-1})_{k.} \\ \frac{1}{(B^{-1}x)_k}(B^{-1})_{k.} \\ (B^{-1})_{k+1.} - \frac{(B^{-1}x)_{k+1}}{(B^{-1}x)_k}(B^{-1})_{k.} \\ \vdots \\ (B^{-1})_{n+1.} - \frac{(B^{-1}x)_{n+1}}{(B^{-1}x)_k}(B^{-1})_{k.} \end{bmatrix}.$$

Lemma 6.6 *Let σ be a t-simplex in $F(I)$ where $I \in \mathcal{I}$, $t = n - |I|$ and $I = \{i_{t+1}, \cdots, i_n\}$. If σ has an I-complete facet τ, then exactly one of the following two cases occurs:*

(1) The simplex σ is an $\bar{I}$-complete simplex in $F(\bar{I})$ where $\bar{I} = I \setminus \{i\}$ for precisely one index $i \in I$;

(2) The simplex σ has exactly one other I-complete facet $\bar{\tau}$.

Proof: Let x^{t+1} be the vertex of σ opposite to τ, and let $y = A_{\tau,I}^{-1}(1, -f(x^{t+1})^\top)^\top$. Notice that $y \neq 0$. Let $K = \{i \in I_n \mid y_i > 0\}$. We first prove $|K| > 0$. Since $A_{\tau,I}y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^{t} y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence K is nonempty.

Consider the ratio vectors $(1/y_j)(A_{\tau,I}^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,I}^{-1}$ is regular, k is uniquely determined. Now, we consider the following two cases (1) and (2).

(1) If $k \in I_n \setminus I_t$, then let $l = i_k$ and $\bar{I} = I \setminus \{l\}$. Clearly, $\bar{I} \in \mathcal{I}$ and σ is in $F(\bar{I})$. Let B be the matrix obtained from $A_{\tau,I}$ by replacing its k-th column by $(1, -f(x^{t+1})^\top)^\top$. It follows from Lemma 6.5 that B^{-1} exists and is semi-lexicopositive. By reordering the columns of B we get $A_{\sigma,\bar{I}}$ whose inverse exists and is semi-lexicopositive. So σ is $\bar{I}$-complete.

(2) If $k \in I_t$, then let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k. Using Lemma 6.5, it follows from the choice of k that $A_{\bar{\tau},I}^{-1}$ exists and is semi-lexicopositive. Hence $\bar{\tau}$ is an I-complete $(t-1)$-simplex in $F(I)$.

It follows immediately from Lemma 6.5 that if any column other than the k-th column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive. □

In Lemma 4.8 we consider the analogous case of making a lexicographic pivot step with a column $(1, a^{i\top})^\top$. First we need the next lemma.

Lemma 6.7 *For any set $I \in \mathcal{I}$ with $I \neq I^-$ and $I \neq I^+$, there exist no solutions λ_0, λ_i, $i \in I$, to $\sum_{i \in I} \lambda_i a^i = \lambda_0 c$ such that $\lambda_i \leq 0$ for all $i \in I$ and $\sum_{i \in I} \lambda_i < 0$.*

Proof: We need to consider the following three cases:
Case (1). If $\lambda_0 = 0$, then $\sum_{i \in I} \lambda_i a^i = \lambda_0 c = 0^n$ contradicts the fact that all vectors a^i, $i \in I$, are linearly independent.
Case (2). If $\lambda_0 < 0$, then by duality theory $\sum_{i \in I} \lambda_i a^i = \lambda_0 c$ and $I \neq I^+$ contradicts the fact that

$$c^\top x^+ = \max_{x \in P} c^\top x.$$

Case (3). If $\lambda_0 > 0$, then by duality theory $\sum_{i \in I} \lambda_i a^i = \lambda_0 c$ and $I \neq I^-$ contradicts the fact that

$$c^\top x^- = \min_{x \in P} c^\top x.$$

□

Lemma 6.8 *Let σ be an I-complete $(t-1)$-simplex in $F(I)$ where $I \in \mathcal{I}$, $t = n - |I|$ and $I = \{i_{t+1}, \cdots, i_n\}$. If σ is in $F(\bar{I})$ and $\bar{I} \neq I^-$ or $\bar{I} \neq I^+$, where $\bar{I} = I \cup \{l\} \in \mathcal{I}$ for some $l \in I_m \setminus I$, then exactly one of the following two cases occurs:*

(1) There exists a unique set $J \in \mathcal{I}$ with $|J| = |I|$ and $J \neq I$ so that σ is in $F(J)$ and is J-complete.

(2) There exists exactly one facet τ of σ which is in $F(\bar{I})$ and is $\bar{I}$-complete.

Proof: Let $x = (0, a^{l\top})^\top$ and $y = A_{\sigma,I}^{-1}x$. Note that $y \neq 0$. Let $K = \{i \in I_n \mid y_i > 0\}$. Note that $A_{\sigma,I}y = (0, a^{l\top})^\top$. We need to consider the following two cases (i) and (ii):
Case (i). If there exists an index $j \in I_t$ such that $y_j < 0$, then there must exist an index $i \in I_t$ such that $y_i > 0$ since $\sum_{k=1}^{t} y_i = 0$. Hence K is nonempty.
Case (ii). Suppose that $y_i = 0$ for all $i \in I_t$. If $y_i \leq 0$ for all $i = t+1, t+2, \cdots, n$, then we have that $a^l = \sum_{i=t+1}^{n} y_i a^{j_{i-t}} + y_{n+1}c$. By Lemma 6.7 it is impossible since $\bar{I}$ is neither equal to I^- nor equal to I^+. Hence there exists at least one index $i \in I_n \setminus I_t$ such that $y_i > 0$. Again K is nonempty.

Consider the ratio vectors $(1/y_j)(A_{\sigma,I}^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,I}^{-1}$ is regular, k is uniquely determined. Now, we consider the following two cases (1) and (2).

(1) If $k \in I_n \setminus I_t$, then let $p = i_k$ and $J = I \cup \{l\} \setminus \{p\}$. Clearly, $J \in \mathcal{I}$, $J \neq I$, $|J| = |I|$ and σ is in $F(J)$. Let B be the matrix obtained from $A_{\sigma,I}$ by replacing its k-th column by x. It follows from Lemma 6.5 that B^{-1} exists and is semi-lexicopositive. It is clear that $A_{\sigma,J} = B$. So σ is a J-complete $t-1$-simplex in $F(J)$.

(2) If $k \in I_t$, then let τ be the facet of σ opposite to the vertex x^k. Clearly, τ is a $(t-2)$-simplex in $F(\bar{I})$. Let B be the matrix obtained from $A_{\sigma,I}$ by replacing its k-th column by x. It follows from Lemma 6.5 that B^{-1} exists and is semi-lexicopositive. By reordering the columns of B we get $A_{\tau,\bar{I}}$ whose inverse also exists and is semi-lexicopositive. So τ is an $\bar{I}$-complete $(t-2)$-simplex in $F(\bar{I})$.

Again it follows from Lemma 6.5 that if any other column is replaced, then the new matrix is no longer semi-lexicopositive. □

We construct a graph $G = (\mathcal{V}, \mathcal{A})$ where $\mathcal{V}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of edges. Each I-complete $(n-|I|-1)$-simplex is a *node* in $\mathcal{V}$. An I-complete $(n-|I|-1)$-simplex τ^1 in $F(I)$ and a J-complete $(n-|J|-1)$-simplex τ^2 in $F(J)$ are said to be *adjacent* complete simplices if $I = J = L$ and τ^1 and τ^2 are both facets of an

$(n-|L|)$-simplex σ in $F(L)$, or τ^1 is a facet of τ^2 and τ^2 is an $(n-|I|)$-simplex in $F(I)$, or τ^2 is a facet of τ^1 and τ^1 is an $(n-|J|)$-simplex in $F(J)$. Two adjacent complete simplices τ^1 and τ^2 are connected by an edge $e = \{\tau^1, \tau^2\} \in \mathcal{A}$. The degree of a node τ in G is defined to be the number of nodes connected with it, denoted by $deg(\tau)$. A path in G from node $\tau^0 = \{x^-\}$ to node τ^l is defined as a sequence of the form $(\tau^0, e_1, \tau^1, \cdots, e_l, \tau^l)$ where τ^0, τ^1, $\cdots$, τ^l are nodes and e_1, $\cdots$, e_l are edges, such that $e_i = \{\tau^{i-1}, \tau^i\}$ for $i \in I_l$. A path is simple if all its nodes and edges are different.

Theorem 6.9 *Let $\mathcal{T}$ be a triangulation of P. Starting at the vertex x^-, the algorithm generates a finite sequence of adjacent J-complete simplices for varying $J \in \mathcal{I}$ which leads to the vertex x^+.*

Proof: By Lemma 6.6, $\{x^-\}$ is an I-complete 0-simplex in $F(I)$ for some unique set $I \in \mathcal{I}$ with $|I| = n-1$. Since $\{x^-\}$ lies in the boundary of $F(I)$, there exists a unique 1-simplex σ in $F(I)$ having $\{x^-\}$ as its facet. By Lemma 6.6, either σ is an $\bar{I}$-complete simplex in $F(\bar{I})$ where $\bar{I} = I \setminus \{i\}$ for some unique $i \in I$, or σ has exactly one other I-complete facet $\bar{\tau}$. Hence there exists a unique adjacent complete simplex to $\{x^-\}$. That is, $deg(\{x^-\}) = 1$. Similarly, by using Lemmas 6.6 and 6.8, we can prove $deg(\{x^+\}) = 1$.

In all other cases, we prove that if τ is an I-complete $(n-|I|-1)$-simplex in $F(I)$ for some $I \in \mathcal{I}$, τ has exactly two adjacent complete simplices. There are two possibilities: either τ lies in the interior of $F(I)$ or τ lies in the boundary of $F(I)$. If τ lies in the interior of $F(I)$, then τ is a facet of exactly two $(n-|I|)$-simplices in $F(I)$. It follows from Lemma 6.8 that τ is adjacent to exactly two complete simplices. If τ lies in the boundary of $F(I)$, then there exists exactly one $(n-|I|)$-simplex σ in $F(I)$ having τ as its facet. By Lemma 6.8 either σ is an $\bar{I}$-complete $(n-|\bar{I}|-1)$-simplex in $F(\bar{I})$ for some unique $\bar{I} \in \mathcal{I}$ with $|\bar{I}| = |I|-1$ and has no other I-complete facets, or σ has exactly one other I-complete facet. This yields one adjacent complete simplex to τ. On the other hand, since τ lies in the boundary of $F(I)$, τ lies in $F(\tilde{I})$ for some unique set $\tilde{I} \in \mathcal{I}$ with $|\tilde{I}| = |I|+1$. By Lemma 6.8 either τ is J-complete for some unique set $J \in \mathcal{I}$ with $|J| = |I|$ and $J \neq I$, or τ has exactly one $\tilde{I}$-complete facet. In the former case, τ lies in $F(\tilde{I})$ and hence there exists exactly one simplex $\bar{\sigma}$ in $F(\tilde{I})$ having τ as its facet. It follows again from Lemma 6.8 that there exists exactly one other complete simplex adjacent to τ. This concludes that τ has exactly two adjacent complete simplices. In other words, we have $deg(\tau) = 2$.

As shown above, the degree of each node in the graph $G = (\mathcal{V}, \mathcal{A})$ is at most two. Exactly two nodes have degree equal to one. Since the number of simplices in P is finite, the number of nodes in G must be finite, too. Since $deg(\{x^-\}) = 1$, it is easy to see that there exists a simple finite path starting from $\{x^-\}$. The end node of this path must be a node τ of degree one and different from x^-. The only possibility is that τ is equal to $\{x^+\}$. □

From the above theorem we see that every simplex on the path between $\{x^-\}$ and $\{x^+\}$ contains a parametrized stationary point of the piecewice linear approximation f of φ with respect to $\mathcal{T}$. By taking the straight line segments between the parametrized stationary points of any two adjacent simplices, we obtain a piecewise linear path of parametrized stationary points of f connecting the points x^- and x^+.

Corollary 6.10 *Let $\mathcal{T}$ be any triangulation of P. Then with respect to the vector c there is a piecewise linear path of parametrized stationary points of the piecewise linear approximation f of φ with respect to $\mathcal{T}$ and this path connects x^- and x^+.*

7 Proofs for the Existence Theorems

In this section we still assume that x^- and x^+ are unique. First it will be argued in Theorem 7.1 and Theorem 7.2 that the points lying on the path given in Corollary 6.10 indeed all correspond to approximate parametrized stationary or zero points of the mapping φ. To show this, a sequence of triangulations $\mathcal{T}^r$ with mesh size converging to zero is taken. This yields according to Corollary 6.10, for every $r \in \mathbb{N}$, a continuous piecewise linear function $\pi^r : [0,1] \to P$ with image set $\pi^r([0,1])$ connecting x^- and x^+. It will be shown that if q^r is an arbitrary point in $\pi^r([0,1])$ and the sequence $(q^r)_{r\in\mathbb{N}}$ converges to q, then q is a parametrized stationary point of $\varphi(q)$ with respect to c. Under the conditions of Theorems 5.2, 5.3 and 5.4, the piecewise linear approximation can be chosen in such a way that q is a zero point of φ. Furthermore, it will be shown in Theorem 7.4 by a limiting argument that there exists a connected set of zero points of φ, containing both x^- and x^+, that is being approximated.

Theorem 7.1 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be a correspondence satisfying Assumption 4.2 and let $c \in \mathbb{R}^n \setminus \{0\}$. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $f^r : P \to \mathbb{R}^n$ be a piecewise linear approximation of φ with respect to $\mathcal{T}^r$ and let $\pi^r : [0,1] \to P$ be the corresponding continuous function with image set connecting x^- and x^+. Let $(q^r)_{r\in\mathbb{N}}$ be an arbitrary convergent sequence of points in P with limit q^* where $q^r \in \pi^r([0,1])$. Then q^* is a parametrized stationary point of φ with respect to c.*

Proof: Let $(\lambda_1^r, \ldots, \lambda_{n+1}^r, x^{1^r}, \ldots, x^{n+1^r}, s^{1^r}, \ldots, s^{n+1^r})_{r\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}_+^{n+1} \times \prod_{i=1}^{n+1} P \times \prod_{i=1}^{n+1} \mathbb{R}^n$ satisfying $\sum_{j=1}^{n+1} \lambda_j^r = 1$, $\sigma^r(x^{1^r}, \ldots, x^{n+1^r})$ is a simplex of $\mathcal{T}^r$, $q^r = \sum_{j=1}^{n+1} \lambda_j^r x^{j^r} \in \pi^r([0,1])$, and $s^{j^r} = f^r(x^{j^r})$. Notice that it may happen that $\lambda_j^r = 0$ for some $j \in I_{n+1}$. By definition, $f^r(q^r) = \sum_{j=1}^{n+1} \lambda_j^r s^{j^r}$. Define $s^r = f^r(q^r)$, then $s^r = \beta^r c + \sum_{j\in I^r} \mu_j^r a^j$, for some $\beta^r \in \mathbb{R}$, for some $\mu_j^r \geq 0$, $\forall j \in I^r$, and for some $I^r \in \mathcal{I}$ satisfying that q^r lies in $F(I^r)$. Since $\cup_{q\in P}\varphi(q)$ is bounded, the sequence given above remains in a compact set, and without loss of generality it can be assumed to converge to an element $(\lambda_1^*, \ldots, \lambda_{n+1}^*, x^{*1}, \ldots, x^{*n+1}, s^{*1}, \ldots, s^{*n+1})$. Define $s^* = \sum_{j=1}^{n+1} \lambda_j^* s^{*j}$. Clearly, it holds that $s^r \to s^*$. Since for every $r \in \mathbb{N}$ the mesh size of $\mathcal{T}^r$ is smaller than $\frac{1}{r}$, it holds for every $j \in I_{n+1}$ that $x^{*j} = q^*$. Using that φ is upper semi-continuous this implies that for every $j \in I_{n+1}$, $s^{*j} \in \varphi(q^*)$. Moreover, $\beta^r \to \beta^*$ for some number β^*, without loss of generality $I^r = I^*$ for all r for some $I^* \in \mathcal{I}$, and $\mu_j^r \to \mu_j^*$ for all $j \in I^*$ for some nonnegative μ_j^*. Since φ is convex valued, $\sum_{j=1}^{n+1} \lambda_j^* = 1$, and $\lambda_j^* \geq 0$, $\forall j \in I_{n+1}$, it holds that $s^* \in \varphi(q^*)$. Moreover, $q^* \in F(I^*)$ and $s^* = \beta^* c + \sum_{j \in I^*} \mu_j^* a^j \in A(I^*)$. Hence, according to Lemma 4.3, q^* is a parametrized stationary point of φ with respect to c. □

In order to give a constructive proof of Theorems 5.2, 5.3 and 5.4 the piecewice linear approximation f of φ with respect to a triangulation $\mathcal{T}$ should be chosen as follows. We call such a piecewise linear approximation a proper one. In case of Theorem 5.2 any piecewise linear approximation of φ with respect to $\mathcal{T}$ can be chosen. Next consider Theorem 5.3. If a point x in the (relative) interior of a face $F(I)$ is a vertex of a simplex of the triangulation, this implies that at least one element in $\varphi(x)$ lies in the set $A_0^*(I)$, and this element is assigned to the piecewice linear approximation at x. In case of Theorem 5.4 an element of the set $A^*(I)$ in $\varphi(x)$ is assigned to a vertex x of a simplex if x lies in (the interior of) $F(I)$.

Theorem 7.2 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be a correspondence satisfying the conditions in one of Theorems 5.2, 5.3 and 5.4. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $f^r : P \to \mathbb{R}^n$ be a proper piecewise linear approximation of φ with respect to $\mathcal{T}^r$. Let $(q^r)_{r \in \mathbb{N}}$ be an arbitrary convergent sequence of points in P with limit q^* such that for any $r \in \mathbb{N}$ it holds that $q^r \in \pi^r([0,1])$. Then q^* is a zero point of φ.*

Proof: First consider Theorem 5.2. Following the proof of Theorem 7.1 the limit point $s^* \in \varphi(q^*)$ is an element of $A(I^*)$ whereas q^* is an element of $F(I^*)$. The latter property implies that s^* is not an element of $A(I^*)$, unless $s^* = 0^n$. Hence, $s^* = 0^n$. Next consider Theorem 5.3. Consider again the convergent sequence of simplices σ^r in $F(I^*)$ mentioned in the proof of Theorem 7.1. Then the vertex x^{j^r} of σ^r lies in some face $F(I^{j^r})$ of P with $I^* \subset I^{j^r}$. Hence, we have that $A_0^*(I^{j^r}) \subset A_0^*(I^*)$, and so $s^r \in A_0^*(I^*)$ for all r because of the properness of f^r. Consequently, also $s^* \in A_0^*(I^*)$ and therefore $s^* \in A_0^*(I^*) \cap A(I^*) \subset \{0^n\}$, i.e. $s^* = 0^n$. In case of Theorem 5.4 following a similar argument as for the previous case we obtain that $s^* \in A^*(I^*) \cap A(I^*)$. Since the latter intersection consists of the zero-vector only, we obtain again that $s^* = 0^n$. □

From Theorem 7.2 the next result follows immediately.

Corollary 7.3 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be a correspondence satisfying the conditions in one of Theorems 5.2, 5.3 and 5.4. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $f^r : P \to \mathbb{R}^n$ be a proper piecewise linear approximation of φ with respect to $\mathcal{T}^r$. Then for every $\varepsilon > 0$ there exists an $R \in \mathbb{N}$ such that for every $r \geq R$ it holds that $q^r \in \pi^r([0,1])$ implies $\|f^r(q^r)\|_\infty < \varepsilon$.*

Proof: Suppose a sequence $(q^r, f^r(q^r))_{r\in\mathbb{N}}$ exists with $q^r \in \pi^r([0,1])$ and $\|f^r(q^r)\|_\infty \geq \varepsilon$ for every $r \in \mathbb{N}$. Since P and $\cup_{q\in P}\varphi(q)$ are compact, there exists a converging subsequence $(q^{r^s}, f^{r^s}(q^{r^s}))_{s\in\mathbb{N}}$, with limit say (q^*, s^*), where $\|s^*\|_\infty \geq \varepsilon > 0$. As in the proof of Theorem 7.2 it can be shown that $s^* = 0^n$, yielding a contradiction. □

Using Theorem 6.9 and Theorem 7.2 it will be shown that there exists a connected set C in P such that $x^- \in C$, $x^+ \in C$, and $0^n \in \varphi(q)$, $\forall q \in C$. Hence, there is a continuum of zero points of φ being approximated by the algorithm of Section 4. For a non-empty, compact set $S \subset \mathbb{R}^n$, define the continuous function $d_S : \mathbb{R}^n \to \mathbb{R}$ by $d_S(x) = \min\{\|x - y\|_\infty \mid y \in S\}$.

Theorem 7.4 *Let $\varphi : P \Longrightarrow \mathbb{R}^n$ be a correspondence satisfying the conditions in one of Theorems 5.2, 5.3 and 5.4. In case of Theorems 5.2, 5.3 and 5.4 there exists a connected set C of points in P such that $x^- \in C$, $x^+ \in C$, and $0^n \in \varphi(q)$, $\forall q \in C$.*

Proof: Define $Q = \{q \in P \mid 0^n \in \varphi(q)\}$. From the conditions of the theorems it immediately follows that $x^- \in Q$, $x^+ \in Q$, and Q is compact. Suppose the theorem is false. Then x^+ is not an element of the component of Q containing x^-. By Munkres (1975, p. 235) it holds for every compact set X in some Euclidean space and for every element $x \in X$ that the component of X containing x equals the intersection of all sets containing x which are both open and closed in X. Hence, there exists a set Q^0, which is open and closed in Q, such that $x^- \in Q^0$ and $x^+ \notin Q^0$. Define $Q^1 = Q \setminus Q^0$. Then Q^1 is open and closed in Q, $x^- \notin Q^1$, and $x^+ \in Q^1$. Since Q is compact, it follows that Q^0 and Q^1 are disjoint, compact sets. Hence, there exists $\varepsilon > 0$ such that $\min\{\|q^0 - q^1\|_\infty \mid q^0 \in Q^0,\ q^1 \in Q^1\} \geq \varepsilon$. For every $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$, let $f^r : P \to \mathbb{R}^n$ be a proper piecewise linear approximation of φ with respect to $\mathcal{T}^r$, and let $\pi^r : [0,1] \to P$ be the corresponding continuous function with image set connecting x^- and x^+. Define $g^r : [0,1] \to \mathbb{R}$ by

$$g^r(t) = d_{Q^0}(\pi^r(t)) - d_{Q^1}(\pi^r(t)),\ \forall t \in [0,1].$$

Since g^r is continuous, $g^r(0) \leq -\varepsilon$, and $g^r(1) \geq \varepsilon$, there exists a point $t^r \in [0,1]$ such that $g^r(t^r) = 0$. Hence, $d_{Q^0}(\pi^r(t^r)) = d_{Q^1}(\pi^r(t^r)) = d_Q(\pi^r(t^r)) \geq \frac{1}{2}\varepsilon$. Without loss of generality it can be assumed that $(\pi^r(t^r))_{r\in\mathbb{N}}$ converges to a point $q^* \in P$. Hence,

$$d_Q(q^*) = d_Q(\lim_{r\to\infty} \pi^r(t^r)) = \lim_{r\to\infty} d_Q(\pi^r(t^r)) \geq \frac{1}{2}\varepsilon > 0.$$

However, by Theorem 7.2, $d_Q(q^*) = 0$, yielding a contradiction. $\square$

Similarly, one can easily show that in case of Theorem 5.1 there exists a connected set C of parametrized stationary points in P with respect to c such that $x^- \in C$ and $x^+ \in C$.

8 The General Case

The algorithm proposed in the previous section can be adapted for computing a continuum of parametrized stationary points or zero points of φ on P in case the faces F^- and F^+ are not vertices of P. First we take any point v in the interior of F^- and a triangulation of P such that the face F^- itself is being triangulated according to the V-triangulation of Talman and Yamamoto (1989). For $J \in \mathcal{I}$ such that $I^- \subset J$, let $VF(J) = \{x \in F^- | x = \lambda v + (1-\lambda)y, 0 \leq \lambda \leq 1, y \in F(J)\}$. The V-triangulation subdivides any such set $VF(J)$ into $(n - |J| + 1)$-simplices.

Then we apply the algorithm of Talman and Yamamoto (1989) to find a parametrized stationary point x^- in F^- of the piecewise linear approximation f of the correspondence φ on P with respect to c. To initiate the algorithm, we solve $\max x^\top f(v)$ subject to $x \in F^-$, yielding by using Theorem 2.6 of Fujishige and Yang (1998) for its dual a uniquely determined vertex $F(I_0)$ of F^- as solution, with $|I_0| = n$ and $I^- \subset I_0$. Then starting with the 0-simplex $\{v\}$ and I equal to I_0 the algorithm generates a sequence of adjacent simplices in $VF(I)$ for varying $I \in \mathcal{I}$ such that $I^- \subset I$ and the common facets $\tau(x^1, \ldots, x^t)$ satisfy that the following system of equations

$$\sum_{j=1}^{t} \lambda_j \begin{pmatrix} 1 \\ -f(x^j) \end{pmatrix} + \sum_{i \in I} \mu_i \begin{pmatrix} 0 \\ a^i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has a solution (λ, μ) satisfying $\lambda_j \geq 0, j \in I_t, \mu_i \geq 0$ for $i \in I \setminus I^-$. Notice that $I^- \subset I$ and that we allow I to be equal to I^-.

The algorithm of Talman and Yamamoto stops as soon as a $(t-1)$-simplex $\tau^-(x^1, \ldots, x^t)$ in $F(I)$ for some I containing I^- is generated for which the system has a feasible solution (λ, μ). Then $x = \sum_{j=1}^{t} \lambda_j x^j$ is a parametrized stationary point in $F(I^-)$ with respect to c of the piecewise linear approximation f of φ. Next the vector $(c^\top, 0)$ is pivoted semi-lexicographically into the system, making any $\mu_i, i \in I$, nonnegative. Since $-c = \sum_{i \in I^-} \lambda_i a^i$ for unique $\lambda_i > 0$, one of the μ_i's, say, μ_{i_0}, for some $i_0 \in I^-$, will leave the basis. Now the algorithm continues in $F(I \setminus \{i_0\})$ with the unique t-simplex σ in $F(I \setminus \{i_0\})$ having τ^- as facet and a semi-lexicographic pivot step is made with $(1, -f^\top(x^{t+1}))^\top$ where x^{t+1} is the vertex of σ opposite to τ, and so on.

In this way the algorithm generates for varying $I \in \mathcal{I}$ by semi-lexicographic pivoting a unique sequence of adjacent simplices in $F(I)$ with common I-complete facets until a facet τ being a simplex in $F(H)$ for some H containing I^- or a complete facet τ^+ being a simplex in $F(J)$ for some J containing I^+ is generated. In the former case the algorithm continues in the subset $VF(H)$ of $F(I^-)$ as above until again an I-complete simplex in $F(I)$ for some I containing I^- is found and so on. In the latter case the point $x^+ = \sum_{j=1}^{t} \lambda_j x^j$ at the solution (λ, μ, β) lies in τ^+ and is a parametrized stationary point in F^+ of f with respect to c.

Letting x^- be the last point being generated in F^-, the algorithm generates a piecewise linear path of parametrized stationary points of f with respect to the vector c. This path connects the point x^- in the face F^- with a point x^+ in the face F^+. Taking a sequence of triangulations of P with mesh tending to zero, in the limit a connected set of parametrized stationary points of φ is obtained with respect to c connecting the faces F^- and F^+. In case the correspondence φ satisfies the conditions of Theorems 5.2, 5.3, or 5.4 and the piecewise linear approximations are chosen in an appropriate way, there exists a connected set of zero points of φ connecting F^- and F^+.

Notice that if a sequence of triangulations with mesh tending to zero is taken, for any triangulation in this sequence the points x^- in F^- and x^+ in F^+ being connected through the piecewise linear path generated by the algorithm may differ. In the limit these points converge on a subsequence to two different zero points of φ, one lying in F^- and the other one lying in F^+.

9 Special Cases

In this section we will derive two existing existence theorems from Theorem 5.3 and Theorem 5.4. The following result is obtained by Herings, Talman and Yang (1996) (Theorem 4.3, p. 690). We will show that this theorem is a special case of Theorem 5.3. Let $U^n = \{x \in \mathbb{R}^n \mid 0 \le x_i \le 1, \forall\, i \in I_n\}$ be the n-dimensional unit cube. Let 1^n denote the n-vector of ones and for $i \in I_n$ let $e(i)$ denote the ith unit vector in $\mathbb{R}^n$.

Theorem 9.1 *Let $\varphi : U^n \Longrightarrow \mathbb{R}^n$ be any correspondence satisfying Assumption 4.2. Moreover it holds that*

(A) for every $x \in U^n$, there exists $f \in \varphi(x)$ such that, for every $j \in I_n$, $x_j = 0$ implies $f_j \ge 0$, and $x_j = 1$ implies $f_j \le 0$;

(B) for every $x \in U^n$, for every $f \in \varphi(x)$, there exists some $p \in \mathbb{R}^n_{++}$ such that $f^\top p = 0$.

Then there exists a connected set C of zero points of φ such that $0^n \in C$ and $1^n \in C$.

Proof: We can rewrite the set U^n as

$$U^n = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i, \forall\, i \in I_{2n}\}$$

where $a^i = e(i)$, $b_i = 1$, $a^{n+i} = -e(i)$, and $b_{n+i} = 0$ for all $i \in I_n$. Now let $c = 1^n$. Clearly, $x^- = 0^n$ and $x^+ = 1^n$. Moreover, it is easy to verify that Condition (ii) of Theorem 5.3 is satisfied by Condition (A). We have to check Condition (i). For $I = \emptyset \in \mathcal{I}$, Condition (i) is trivially satisfied. Now take any nonempty set I from $\mathcal{I}$. We can partition I into two disjoint subsets I^1 and I^2 with $I^1 \subset I_n$ and $I^2 \subset I_{2n} \setminus I_n$. Notice that for any $J \in \mathcal{I}$, $i \in J$ implies $i + n \notin J$ if $i \leq n$ and $i - n \notin J$ if $i > n$. Suppose there is some $x \in F(I)$ and some $f \in \varphi(x)$ such that $f \in (A_0^*(I) \cap A(I)) \setminus \{0^n\}$. This implies that there exist $\mu_i \geq 0$, $\forall\, i \in I$, and $\beta \in \mathbb{R}$ such that

$$\begin{aligned} f &= \textstyle\sum_{i \in I} \mu_i a^i + \beta c \\ f^\top a^i &\leq 0, \forall\, i \in I \\ f &\neq 0^n. \end{aligned}$$

Equivalently,

$$\begin{aligned} f &= \textstyle\sum_{i \in I^1} \mu_i e(i) - \sum_{i \in I^2} \mu_i e(i-n) + \beta c \\ f_i &\leq 0, \forall\, i \in I^1 \\ f_i &\geq 0, \forall\, n+i \in I^2 \\ f &\neq 0^n. \end{aligned}$$

This implies that

$$\begin{aligned} \beta &\geq \max_{i \in I^2} \mu_i, \\ \beta &\leq -\max_{i \in I^1} \mu_i. \end{aligned} \tag{9.1}$$

In case $I^2 = \emptyset$, we have $f < 0^n$. This contradicts Condition (B). In case $I^1 = \emptyset$, we have $f > 0^n$, again contradicting Condition (B). In case $I^1 \neq \emptyset$ and $I^2 \neq \emptyset$, without loss of generality there exist $i \in I^1$ and $j \in I^2$ such that $\mu_i > 0$ and $\mu_j > 0$. This would mean both $\beta > 0$ and $\beta < 0$ from (9.1) which is impossible. Hence Condition (i) is satisfied by Condition (B). □

Now we show that the fundamental fixed point theorems of Browder (1960) and Mas-Colell (1974) can be derived from Theorem 5.4. Browder proved the continuous function case and Mas-Colell extended the result to the upper semi-continuous correspondence case.

Theorem 9.2 *Let P be an n-dimensional polytope and let $\varphi : P \times [0,1] \Longrightarrow P$ be any correspondence satisfying Assumption 4.2. Then the set*

$$D = \{(x,t) \in P \times [0,1] \mid x \in \varphi(x,t)\}$$

contains a connected set C such that

$$C \cap (\mathrm{P} \times \{0\}) \neq \emptyset \quad \textit{and} \quad C \cap (\mathrm{P} \times \{1\}) \neq \emptyset.$$

Proof: We can rewrite the set $P \times [0,1]$ as

$$W = \{(x,t) \in \mathbb{R}^{n+1} \mid (a^{i\top}, 0)(x^\top, t)^\top \leq b_i, \forall i \in I_m, \\ (0, \cdots, 0, -1)(x^\top, t)^\top \leq 0, \\ (0, \cdots, 0, 1)(x^\top, t)^\top \leq 1\}.$$

Let $c = (0, \cdots, 0, 1)^\top \in \mathbb{R}^{n+1}$. Obviously, W is simple and no constraint is redundant. Moreover, $F^+ = \{(x,t) \in W \mid t = 1\}$ and $F^- = \{(x,t) \in W \mid t = 0\}$.

Construct the correspondence $\psi : W \Longrightarrow \mathbb{R}^{n+1}$ as

$$\psi(x,t) = (\varphi(x,t) - \{x\}) \times \{0\}.$$

We will show that for any $I \in \mathcal{I}$ and any $(x,t) \in F(I)$, we have $\psi(x,t) \subset A^*(I)$. For $I = \emptyset \in \mathcal{I}$, clearly $\psi(x,t) \subset A^*(\emptyset)$ since $c^\top z = 0$ for any $z \in \psi(x,t)$. Now take any nonempty set I from $\mathcal{I}$. We have to consider the following two cases:
(1) In case $I \subset I_m$, take any (x,t) in the face $F(I)$ of W and any z in $\psi(x,t)$. We have

$$a^{i\top} x = b_i, \ \forall i \in I; \ z = ((k-x)^\top, 0)^\top$$

for some $k \in P$. Take any $y \in A(I)$. That is,

$$y = \sum_{i \in I} \lambda_i (a^{i\top}, 0)^\top + \beta c$$

for some $\lambda_i \geq 0$ and $\beta \in \mathbb{R}$. Thus, we have

$$s^\top y = \sum_{i \in I} \lambda_i (k-x)^\top a^i.$$

Since $k \in P$, we have $a^{i\top} k \leq b_i$ for all $i \in I_m$. Since $a^{i\top} x = b_i$ for all $i \in I$, we have $s^\top y \leq 0$. This means that $\psi(x,t)$ is a subset of $A^*(I)$.
(2) If $I \not\subset I_m$, either $m+1$ or $m+2$ is contained in I. For example, suppose that $m+2$ is in I, i.e. $t = 1$. The case $m+1 \in I$ follows the same argument. Take any (x,t) in the face $F(I)$ of W and any z in $\psi(x,t)$. We have

$$a^{i\top} x = b_i, \ \forall i \in I \setminus \{m+2\}; \ t = 1; \ z = ((k-x)^\top, 0)^\top$$

for some $k \in P$. Take any $y \in A(I)$. That is,

$$y = \sum_{i \in I \setminus \{m+2\}} \lambda_i (a^{i\top}, 0)^\top + \beta c$$

for some $\lambda_i \geq 0$ and $\beta \in \mathbb{R}$. Thus, we have

$$s^\top y = \sum_{i \in I \setminus \{m+2\}} \lambda_i (k-x)^\top a^i.$$

Since $k \in P$, we have $a^{i\top} k \leq b_i$ for all $i \in I_m$. Since $a^{i\top} x = b_i$ for all $i \in I \setminus \{m+2\}$, we have $s^\top y \leq 0$. This means that $\psi(x,t)$ is again a subset of $A^*(I)$. By Theorem 5.4 there exists a connected set C in W such that

$$0^{n+1} \in \psi(x,t),\ \forall (x,t) \in C;\ \ F^+ \cap C \neq \emptyset;\ \ F^- \cap C \neq \emptyset.$$

Clearly, $x \in \varphi(x,t)$ for each $(x,t) \in C$. □

We remark that the above theorem can also directly be derived from Theorem 5.2 and Theorem 5.3.

Finally we use Theorem 5.4 to show that there is a continuum of constrained equilibria in a pure exchange economy with general price rigidities; see e.g., Schalk and Talman (1999) for details. Price vectors in such an economy with n commodities are restricted to an n-dimensional simple polytope P without redundant constraints in the interior of $\mathbb{R}^n_+$. Each vector $a^i, i \in I_m$, determining a facet of P is normalized to have length one. Let c be any element of $\mathbb{R}^n_{++}$ with length equal to one. Define $Q = \{q \mid a^{i\top} q \leq b_i + 1,\ \forall i \in I_m\}$. Clearly, Q is a polytope containing P in its interior. For any q in Q, let $p(q)$ be the orthogonal projection of q on P and let $I(q)$ be such that $p(q)$ is in the interior of the face $F(I(q))$ of P. Then there exist unique nonnegative numbers $\mu^i, i \in I(q)$ such that $q = p(q) + \sum_{i \in I(q)} \mu_i a^i$.

At $q \in Q$ define the price vector by $p(q)$ and a continuous rationing scheme $(r^i, d_i(q)), i \in I_m$, such that $r^i = a^i - (c^\top a^i)c$, $d_i(q) = 0$ if $\lambda_i(q) = 1$ and $d_i(q) = \infty$ if $\mu_i(q) = 0$ or i not in $I(q)$. This rationing scheme determines the constraints on the net-supply of the consumers. Given a utility function u^h and initial endowment w^h consumer $h \in H$ maximizes his utility $u^h(x)$ over his budget constraint given by $p(q)^\top x \leq p(q)^\top w^h$ and rationing constraints ${r^i}^\top (x - w^h) \leq d_i(q), i \in I_m$. The solution set $x^h(q)$ yields the constraint excess demand set $z^h(q) = x^h(q) - \{w^h\}$ of consumer $h \in H$ at q. Adding up these sets over all consumers in H gives the constraint excess demand $z(q)$ induced by q. Under certain standard economic conditions, the mapping z satisfies Assumption 4.2 and $p(q)^\top z = 0$ for any $z \in z(q), q \in Q$.

A constrained equilibrium is obtained if $q \in P$ is such that $0^n \in z(q)$. At such an equilibrium for any $i \in I_m$ the ith rationing scheme is active, i.e. $d_i(q)$ is finite, only if the ith constraint on the prices is binding, i.e. $p(q)^\top a^i = b_i$.

Let F^- be the face of Q on which $c^\top x$ is minimized on Q and let F^+ be the face of Q on which $c^\top x$ is maximized on Q. These two faces induce trivial constrained equilibria where either all demand or all supply is completely rationed away.

To see that there is a connected set of constrained equilibria linking the F^- and F^+, define the mapping φ on Q by $\varphi(q) = \{y | y = z - (c^\top z)c\,, z \in z(p)\}, q \in Q$. We will show that $\varphi(q) \subset A^*(I)$ when q lies in the interior of the face $F(I)$ of Q. Clearly, $c^\top y = 0$ for any $y \in \varphi(q)$. Moreover, for any $y \in \varphi(q)$ and $i \in I(q)$ it holds that ${a^i}^\top y = {a^i}^\top z - ({a^i}^\top c)c^\top z =$

${r^i}^\top z \leq 0$ for some $z \in z(q)$. Therefore, any $y \in \varphi(q)$ is an element of $A^*(I)$, and hence $\varphi(q) \subset A^*(I)$ if $q \in F(I)$. According to Theorem 5.4 there exists a connected set C in Q containing both F^- and F^+ such that every point $q \in C$ is a zero point of φ, i.e. $0^n \in \varphi(q)$. For such a q it holds that there is an $z \in z(q)$ satisfying $z - (c^\top z)c = 0^n$. Because $p(q)^\top z = 0$, we obtain that $(c^\top z)p(q)^\top c = 0$. Hence, $c^\top z = 0$, since $p(q)^\top c > 0$. This implies that $z = 0^n$ and therefore $0^n \in z(q)$, inducing a constrained equilibrium. Consequently there exists a connected set of constrained equilibria linking the two faces F^- and F^+.

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