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Convex congestion network problems

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Abstract

This paper analyzes convex congestion network problems. It is shown that for network problems with convex congestion costs, an algorithm based on a shortest path algorithm, can be used to find an optimal network for any coalition. Furthermore an easy way of determining if a given network is optimal is provided.

Keywords: Convex congestion network problems, optimal networks.

JEL Classification Number: C63, C71.

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1 Introduction

Economic congestions situations arise if a group of agents uses facilities from a common pool and costs of using a certain facility depends on the number of its users. A congestion problem creates interaction and involves the analysis of a cost allocation problem.

Rather surprisingly, in cooperative game theoretic literature congestion effects have not been considered for a large extend. One branch of cooperative literature especially suited by its very nature to accommodate considerations regarding congestion is the literature on Operations Research Games as surveyed by Borm, Hamers, and Hendrickx (2001). Quant, Borm, and Reijnierse (2003) consider a particular extension of a minimum cost spanning tree problem. In this extension costs of a network depend on the number of users of the various parts. These problems are so called congestion network problems. The problems of cost allocation and network structure are studied for various cost functions with the aid of cooperative game theory. If cost functions are convex, the underlying game turns out to be balanced. In the case of concave cost functions, they show the existence of optimal network structures without cycles and provide an example in which the corresponding game has an empty core. This paper will focus on convex congestion network problems, ie. cost functions of the parts of a network are convex. The aim of this paper is to solve the problem of finding an optimal network for a specific coalition.

The structure of the paper is as follows. In section 2 we formally introduce convex congestion network problems. In Section 3 the issue of finding an optimal network for a coalition is solved. To solve this problem it is first illustrated how it can be determined if a network is optimal for a coalition. Finally the algorithm of finding an optimal network is illustrated by an example.

2 Convex congestion network problems

A convex *congestion network problem* is a triple $T = (N, *, (k_a)_{a \in A_{N^*}})$, where $N = \{1, \dots, n\}$ is a set of agents/players (so $|N| = n$), $*$ is the source and $N^* := N \cup \{*\}$. The set A_{N^*} denotes the set of all arcs between pairs of elements in N^*, i.e. (N, A_{N^*}) denotes the complete digraph on N^*. For each arc $a \in A_{N^*}$ the function $k_a : \{0, \dots, n\} \to \mathbb{R}_+$ is a nonnegative (weakly) increasing convex cost function which depends on the number of users of a.

A cost function k_a, $a \in A_{N^*}$, is convex if for all $m \in \{1, \ldots, n\}$:

$$k_a(m+1) - k(m) \geq k_a(m) - k_a(m-1).$$

We assume that for all $a \in A_{N^*}$ it holds that $k_a(0) = 0$. Elements of A_{N^*} will be denoted by a or by (i, j), where $i, j \in N^*$. The arc (i, j) denotes the connection between i and j in the direction from i to j. Node i is called the tail of (i, j) and node j the head of (i, j). If $a = (i, j)$, a^{-1} denotes the arc in opposite direction, ie. $a^{-1} = (j, i)$. The cost function of arc (i, j) is denoted by k_{ij}. The problem is called symmetric if $k_{ij}(m) = k_{ji}(m)$ for all $m \leq n$ and all $(i, j) \in A_{N^*}$.

A network can be described by $f : A_{N^*} \to \{0, 1, 2, \ldots\}$. Let F be the set consisting of all such networks. A network f assigns to each arc a number of users. The indegree for a network $f \in F$ and $i \in N^*$ is defined by indegree$(i) = \sum_{j \in N^* \setminus \{i\}} f\big((j, i)\big)$. Similarly the outdegree is defined by outdegree$(i) = \sum_{j \in N^* \setminus \{i\}} f\big((i, j)\big)$. The set of arcs used by f, i.e. $\{a \in A_{N^*} \mid f(a) > 0\}$, is denoted by A_f.

For a coalition $S \in 2^N \setminus \{\emptyset\}$, F_S is the set of all networks such that all members of S have a path to the source:

$$\begin{aligned} F_S \quad = \quad \big\{ f \in F \mid & \text{ outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S, \\ & f(a) \in \{0, \ldots, |S|\}, \ \forall a \in A_{N^*} \big\}. \end{aligned}$$

The costs of a network $f \in F_S$ is naturally defined by:

$$k(f) = \sum_{a \in A_{N^*}} k_a\big(f(a)\big).$$

The aim of S is to construct a feasible network such that total costs are minimized.

A transferable utility cost game consists of a pair (N, c), in which $N = \{1, \ldots, n\}$ is a set of players and $c : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a cost of $c(S)$. By definition $c(\emptyset) = 0$. With each congestion network problem $T = (N, *, (k_a)_{a \in A_{N^*}})$ one can associate a congestion network game (N, c^T), such that $c^T(S)$ denotes the minimum costs of a network connecting all players of S to the source:

$$c^T(S) = \min_{f \in F_S} k(f).$$

Quant, Borm, and Reijnierse (2003) study congestion network problems with arbitrary cost functions. It is shown that if cost functions are concave, the set of optimal networks contains trees. Furthermore if cost functions are convex, the corresponding cooperative congestion network game has a non-empty core. This paper focusses on convex congestion network problems, in particular the problem of finding an optimal network.

Example 2.1 *Consider a symmetric convex congestion network problem, in which there are three players. For the arcs, the costs of one, two and three users respectively are given by:*

$$\begin{aligned} k_{1*} &= (6, 12, 18), \\ k_{2*} &= (1, 4, 8), \\ k_{3*} &= (3, 8, 13), \\ k_{12} &= (5, 10, 15), \\ k_{13} &= (1, 7, 14), \\ k_{23} &= (1, 5, 9). \end{aligned}$$

The corresponding TU-game c^T is given below.

S	1	2	3	12	13	23	N
$c^T(S)$	6	1	3	7	9	4	9

An optimal network of N is drawn in Figure 1, the costs of this network are 9.

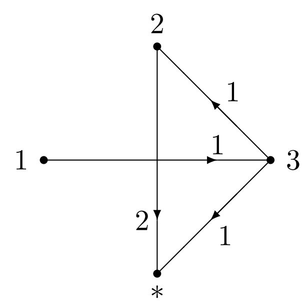

Figure 1: Optimal network of the problem given in Example 2.1.

It is not directly clear how the optimal network of Example 2.1 can be found, nor that this indeed is an optimal network. In the sequel we concentrate on determining if a network is optimal and how one can find such an optimal network. Example 2.1 will be revisited at several instances.

3 Finding an optimal network

In this section we consider convex congestion network problems and solve the problem of finding an optimal network.

First we derive a special condition for a network being optimal for a certain coalition. This condition is based on finding a negative circuit with respect to a length function determined by the network. Let $T = (N, *, (k_a)_{a \in A_{N^*}})$ be a convex congestion network problem and $f \in F_N$. We assume that network f satisfies that $f(a^{-1}) = 0$ whenever $f(a) > 0$ for $a \in A_{N^*}$, since if both $f(a)$ and $f(a^{-1})$ are positive, the network remains feasible if both $f(a)$ and $f(a^{-1})$ are decreased by one. Because this new network is as least as cheap as f, it is reasonable to make the above assumption. Furthermore, we assume that A_f does not contain any circuit, since the network arising from f by decreasing the number of users of the arcs in one circuit of A_f by one yields a network as least as cheap as f.

Given f one can define a length function l_f on the complete digraph (N, A_{N^*}) as follows:

$$l_f(a) \quad := \quad \begin{cases} \infty & \text{if } f(a) \geq n \text{ (so } f(a^{-1}) = 0), \\ k_a(f(a)+1) - k_a(f(a)) & \text{if } f(a^{-1}) = 0 \text{ and } f(a) < n, \\ k_{a^{-1}}(f(a^{-1})-1) - k_{a^{-1}}(f(a^{-1})) & \text{if } f(a^{-1}) > 0. \end{cases}$$

This function can be interpreted as the marginal costs of an extra user of an arc. Note that if the opposite a^{-1} of an arc is used, an extra user of arc a should be interpreted as the reduction of the number of users of a^{-1} by one. The following lemma proves that if f is not optimal for N, then A_{N^*} contains a negative circuit[1] with respect to the length function l_f.

Before stating this lemma, we first show the reverse: an optimal network does not contain a negative circuit with respect to its length function. Consider an optimal network f. Suppose that A_{N^*} contains a negative circuit C with respect to l_f. Then $f(a) < n$ for all a in C. One can change f by increasing the numbers of users of the arcs of C by one. If an arc of C is used in opposite direction, it is meant that the number of users of the opposite arc is decreased by one. This yields a new network, which is feasible as well, since the number of users of arcs are nonnegative and does not exceed n. The costs of this new network equal $k(f) + \sum_{a \in C} \big(l_f(a)\big)$. Since C is a negative circuit, this contradicts the optimality of f.

[1] A set of arcs C is called a circuit if the arcs in C form a sequence $(i_1, i_2), (i_2, i_3), \ldots, (i_p, i_1)$ such that all (intermediate) nodes involved differ. A circuit C is a negative circuit with respect to a length function l if $\sum_{a \in C} l(a) < 0$.

Lemma 3.1 *Let* $T = (N, *, (k_a)_{a\in A_{N^*}})$ *be a convex congestion network problem. Let* $f \in F_N$*. If* f *is not optimal, then* A_{N^*} *contains a negative circuit with respect to the length function* l_f*.*

Proof: Let $f \in F_N$ and suppose that f is not optimal for N. Let $\bar{f}$ be an optimal network for N. By comparing f and $\bar{f}$ we will find a negative circuit with respect to l_f.

Define the network $\bar{f} \ominus f$ that measures the difference between $\bar{f}$ and f as follows:

$$\bar{f} \ominus f(a) \quad = \quad \max\{\bar{f}(a) - f(a) + f(a^{-1}) - \bar{f}(a^{-1}), 0\}.$$

It assigns a positive number of users to an arc $a \in A_{N^*}$, if the arc is used more in $\bar{f}$ than in f or/and if the arc in opposite direction is used more in f than in $\bar{f}$. Since both $\bar{f}$ and f are feasible for N, and $\bar{f} \ominus f$ measures the difference between $\bar{f}$ and f, it holds for all $i \in N$ that:

$$\text{indegree}^{\bar{f}\ominus f}(i) - \text{outdegree}^{\bar{f}\ominus f}(i) = 0.$$

This implies that $A_{\bar{f}\ominus f}$ contains a circuit C. We will show that it contains a negative circuit with respect to l_f.

Let a be an element of C. In the following table the five possibilities of the presence of a and a^{-1} in $A_{\bar{f}}$ and A_f are illustrated. An arrow to the right indicates the presence of a, an arrow to the left the presence of a^{-1}, whereas x indicates that neither a nor a^{-1} are present.

	C_1	C_2	C_3	C_4	C_5
$A_{\bar{f}}$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	x	$\longleftarrow$
A_f	$\longrightarrow$	x	$\longleftarrow$	$\longleftarrow$	$\longleftarrow$

For example, the arrows in the last column indicate that $0 < \bar{f}(a^{-1}) < f(a^{-1})$, since $a \in A_{\bar{f}\ominus f}$ and in both networks a is used in opposite direction. Note that there is no column with a left arrow in the row of $A_{\bar{f}}$ and a right arrow in the row of A_f, since then a^{-1} and not a were present in C. For the same reason, the column with a left arrow and a x and there is no column with a x and a right arrow. The set C can be partitioned into five sets $C_1, \ldots, C_5$, each corresponding to a column of the table above.

Let C^{-1} be the circuit C in opposite direction, then:

$$\begin{aligned}\sum_{a^{-1}\in C^{-1}} l_{\bar{f}}(a^{-1}) \quad &= \quad \sum_{a\in C_1\cup C_2\cup C_3} \big(k_a(\bar{f}(a)-1) - k_a(\bar{f}(a))\big) + \\ &\quad + \sum_{a\in C_4\cup C_5} \big(k_{a^{-1}}(\bar{f}(a^{-1})+1) - k_{a^{-1}}(\bar{f}(a^{-1}))\big) \\ &\geq \quad 0. \end{aligned} \tag{1}$$

Inequality (1) is true, because $\bar{f}$ is optimal. Assume for the time being that inequality (1) is strict. The length of C with respect to l_f equals:

$$\begin{aligned}\sum_{a\in C} l_f(a) &= \sum_{a\in C_1\cup C_2}\big(k_a(f(a)+1)-k_a(f(a))\big)+\\ &\quad+\sum_{a\in C_3\cup C_4\cup C_5}\big(k_{a^{-1}}(f(a^{-1})-1)-k_{a^{-1}}(f(a^{-1}))\big)\\ &\le \sum_{a\in C_1\cup C_2}\big(k_a(\bar{f}(a))-k_a(\bar{f}(a)-1)\big)+\\ &\quad+\sum_{a\in C_4\cup C_5}\big(k_{a^{-1}}(\bar{f}(a^{-1}))-k_{a^{-1}}(\bar{f}(a^{-1})+1)\big)+\\ &\quad+\sum_{a\in C_3}\big(k_{a^{-1}}(f(a^{-1})-1)-k_{a^{-1}}(f(a^{-1}))\big)\\ &\le -\sum_{a^{-1}\in C^{-1}} l_{\bar{f}}(a^{-1})\\ &< 0.\end{aligned}$$

Here the first inequality follows from the convexity of the functions k_a and the fact that $\bar{f}(a)\ge f(a)+1$ if $a\in C_1\cup C_2$ and $f(a^{-1})\ge \bar{f}(a^{-1})+1$ if $a\in C_4\cup C_5$. Consequently C is a negative circuit with respect to the length function l_f.

In the case inequality (1) is tight, so $\sum_{a^{-1}\in C^{-1}} l_{\bar{f}(a^{-1})}=0$, one can change the network $\bar{f}$ as follows: increase the numbers of users of the arcs of C^{-1} by one. If an arc of C^{-1} is used in opposite direction in $\bar{f}$ (so $a^{-1}\in C^{-1}$ and $\bar{f}(a)>0$), it is meant that the number of users of the opposite arc is decreased by one. The resulting network $\bar{f}_1$ is feasible and costs $k(\bar{f})+\sum_{a^{-1}\in C^{-1}} l_{\bar{f}}(a^{-1})=k(\bar{f})$ and is also optimal, since $\bar{f}$ is optimal. One can measure the difference between f and $\bar{f}_1$ in a similar way as the difference between f and $\bar{f}$. In comparison to $\bar{f}\ominus f$ it holds that for all arcs a in A_{N^*}, $\bar{f}_1\ominus f(a)=\bar{f}\ominus f(a)$ if a is not in C and $\bar{f}_1\ominus f(a)=\bar{f}\ominus f(a)-1$ if a is in C. The set of edges $A_{\bar{f}_1\ominus f}$ also contains a circuit and one can follow the lines of the proof above. If inequality (1) is again tight, one must define networks $\bar{f}_2$, $\bar{f}_3$ and $\bar{f}_2\ominus f$, $\bar{f}_3\ominus f$ and so on, until a strict inequality arises. Eventually, this will be the case, since the values of $\bar{f}_k\ominus f(a)$ are decreasing for all $a\in A_{N^*}$. Note that $\bar{f}_k\ominus f$ is the zero network if and only if $f(a)=\bar{f}_k(a)$ for all $a\in A_{N^*}$. Because f is not optimal, $\bar{f}_k\ominus f$ cannot be the zero-network; there is a k for which inequality (1) is strict. □

Combining Lemma 3.1 with the remark that if a network f is optimal, then A_{N^*} does not contain a negative circuit with respect to l_f, yields the following theorem.

Theorem 3.1 *Let $T = (N, *, (k_a)_{a \in A_{N^*}})$ be a convex congestion network problem. Then $f \in F_N$ is optimal if and only if A_{N^*} contains no negative circuit with respect to the length function l_f.*

The existence of negative circuits can be detected by a shortest path[2] algorithm, as e.g. is described in *Floyd-Warshall* algorithm (Papadimitriou and Steiglitz (1982), page 132). If the algorithm finds that the 'shortest path' (cheapest way) to go from some node to itself has negative costs, there must be a negative circuit containing this node.

Example 3.1 *In this example it is shown that the network f depicted in Figure 1 is indeed optimal for N. According to Theorem 3.1 it is sufficient to show that there is no negative circuit with respect to l_f. In figure 2 the complete digraph with length function l_f is drawn. The Floyd-Warshall algorithm finds the following matrix of shortest paths:*

$$\begin{bmatrix} 0 & -6 & -3 & -5 \\ 6 & 0 & 3 & 1 \\ 4 & -2 & 0 & -1 \\ 5 & -1 & 2 & 0 \end{bmatrix}.$$

There is no negative circuit, since there are only zeroes on the diagonal.

Theorem 3.1 provides the tools to introduce an algorithm which determines an optimal network for each convex congestion network problem. The algorithm described below is based on the following idea. First an order is chosen on N. An order of N is a bijective function $\sigma : \{1, \ldots, n\} \to N$. The player at position i in the order σ is denoted by $\sigma(i)$. The set of all orders of N is denoted by $\Pi(N)$. Players are connected to the source in this order in an optimal way. It is allowed to use all arcs to establish a connection to the source. The first player, $\sigma(1)$, chooses a shortest path to the source, this results in a network. The second player faces this network and chooses a shortest path to the source. The costs of this path depend on the path chosen by the first player. If player two uses an arc that is also used by the first player, it costs the marginal costs from one to two users. If player two uses an arc that is used by the first player in opposite direction, this cancels

[2] A set of arcs P is called a path from i to j if the arcs in P form a sequence $(i, i_2), (i_2, i_3), \ldots, (i_p, j)$, such that all nodes involved differ, except possibly i and j themselves. In the latter case, the path is a circuit.

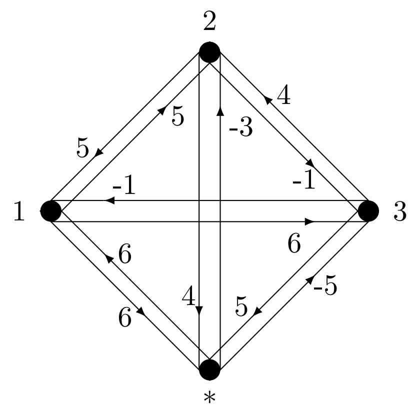

Figure 2: Digraph (N, A_{N^*}) with length function l_f.

out. In general: using an arc in the same direction yields the marginal costs of a player extra and using an arc in opposite direction yields a benefit of the marginal costs of one user less. Hence if a player chooses a shortest path to the root he faces the length function of the current network. One can continue until all players are connected. We first give a small example to illustrate the idea of the algorithm.

Example 3.2 *Consider a symmetric convex congestion network problem with two players, $N = \{1, 2\}$. Cost functions are defined by*

$$\begin{aligned} k_{1*} &= (5, 10), \\ k_{2*} &= (1, 4), \\ k_{12} &= (3, 6). \end{aligned}$$

Suppose that player 1 is the first one to be connected to the source. The cheapest way to do this is taking the path $\big((1,2),(2,*)\big)$. The resulting network is the left picture of Figure 3. For player 2 there are two possible paths to the source. First the direct path $\big((2,*)\big)$. This costs 3, since $(2,*)$ is also used by player 1. Second the path via player 1, $\big((2,1),(1,*)\big)$. Since player 1 is using $(2,1)$ in the opposite direction, this cancels the use of $(1,2)$ and yields a benefit of 3. The resulting costs are $-3+5=2$. Hence it is better for player 2 to choose the indirect path. In fact by choosing this path, he changes the path of player 1. The resulting (optimal) network for the two players is depicted in the right sided network of Figure 3.

Before giving a formal definition of the algorithm, we provide some additional notations. Let P be a path, then f_P is the network induced by

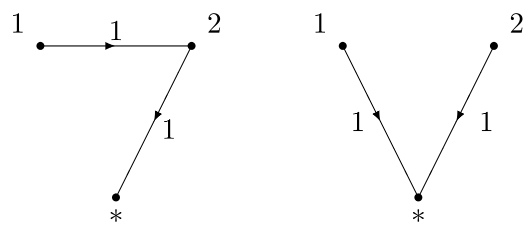

Figure 3: Optimal networks for player 1 (left) and coalition $\{1,2\}$ (right) of the two person convex congestion network problem of Example 3.2.

P:

$$f_P(a) := \begin{cases} 1 & \text{if } a \in P, \\ 0 & \text{if } a \notin P. \end{cases}$$

Let f_1 and f_2 be two networks, then the sum $f_1 \oplus f_2$ is defined by:

$$f_1 \oplus f_2(a) := \max\{f_1(a) + f_2(a) - f_1(a^{-1}) - f_2(a^{-1}), 0\}$$

for all $a \in A_{N^*}$. This operation takes into account that the usage of two oppositely directed arcs cannot be beneficial. If there is two ways traffic between nodes, the numbers of users are subtracted instead of added.

The above described procedure can be described in the following way. At some moment a network is chosen by a group of players. The next player chooses a shortest path with respect to the length function of the network present at that moment. The combination of the network and the chosen path leads to a new network. This new network can be found by adding the network which corresponds to the path and the original network using the operation $\oplus$.

Algorithm 3.1
Input: *a convex congestion network problem $\mathcal{T} = (N, *, (k_a)_{a \in A_{N^*}})$ and an order $\sigma \in \Pi(N)$.*
Output: an optimal network $f^n \in F_N$.

1. *Initialize $f^0(a) = 0$ for all $a \in A_{N^*}$ and $t = 1$.*
2. *Find a shortest path P^t in (N, A_{N^*}) from $\sigma(t)$ to $*$ with respect to the length function $l_{f^{t-1}}$.*
3. *Set $f^t(a) = f^{t-1} \oplus f_{P^t}$.*

4. If $t \neq n$, set $t := t + 1$ and return to step 2.

Finding shortest paths can be done again by the Floyd-Warshall algorithm. In its purest form, it finds only the values of all shortest paths, but finding the paths themselves is just a matter of keeping track which arcs optimal paths use. It has a complexity of order $\mathcal{O}(n^3)$. Since we have to perform it n times, the complete algorithm has a complexity of order $\mathcal{O}(n^4)$.

The following theorem states that this algorithm yields an optimal network for N.

Theorem 3.2 *Let $\mathcal{T} = (N, *, (k_a)_{a \in A_{N^*}})$ be a convex congestion network problem and $\sigma \in \Pi(N)$ an order. The output f^n of Algorithm 3.1 is an optimal network for coalition N.*

Before giving the proof of Theorem 3.2, we first take a closer look at the specific characteristics of the algorithm. In each step an extra node is connected to the source and for this node it is allowed to use all arcs. This differs from our approach that a coalition can only use arcs for which both head and tail correspond to members of that coalition. In fact this gives rise to a relaxation in which all arcs are fully public. The set of feasible networks for a coalition $S \subseteq N$ becomes:

$$\begin{aligned}
\hat{F}_S \quad = \quad \big\{ f \in F \mid & \text{ outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\
& \text{ outdegree}(i) - \text{indegree}(i) = 0 \text{ for all } j \in N \backslash S, \\
& \ f(a) \in \{0, \ldots, |S|\}, \ \forall a \in A_{N^*} \big\}.
\end{aligned}$$

The aim of a coalition S is to connect all its members to the source, such that total costs are minimized. Denote the corresponding coalitional value by $\hat{c}^T(S)$. Note that $\hat{c}^T(N) = c^T(N)$. We then have the following generalization of Theorem 3.1.

Theorem 3.3 *Let $T = (N, *, (k_a)_{a \in A_{N^*}})$ be a convex congestion network problem. Then $f \in \hat{F}_S$ is optimal (i.e. $\hat{c}^T(S) = k(f)$) if and only if (N, A_{N^*}) does not contain a negative circuit with respect to l_f.*

The proof of this theorem follows exactly the lines of the proof of Lemma 3.1 and is therefore omitted. We can now give the proof of Theorem 3.2.

Proof of Theorem 3.2: Assume without loss of generality that σ is the identity. It is easy to see that f^n is feasible for N, since at step t the net degree (which equals the difference between outdegree and indegree) of $*$ decreases from $t-1$ to t, the net degree of the nodes $1,\ldots,t-1$ remains 1, the net degree of node t increases from 0 to 1 and the net degree of the nodes $t+1,\ldots,n$ remains 0. In the end, all nodes but the root have net degree one, hence f^n is feasible. To prove the optimality of f^n an induction argument is used. It is sufficient to prove that for all $t \in \{1,\ldots,n\}$, the network f^t is optimal for the coalition $\{1,\ldots,t\}$ in the relaxed game, i.e. $\hat{c}(\{1,\ldots,t\}) = k(f^t)$.

If $t = 1$, then $l_{f^{t-1}}(a) = k_a(1) \geq 0$ for all $a \in A_{N^*}$. P^1 is the shortest path from 1 to $*$ and trivially $\hat{c}(\{1\}) = k(f^1)$.

Let $t \in \{1,\ldots,n-1\}$. Assume that the network f^t is optimal for coalition $\{1,\ldots,t\}$. There exists a shortest path $P := P^{t+1}$ from player $t+1$ to $*$ with respect to l_{f^t}. We have to prove that f^{t+1} is an optimal network for coalition $\{1,\ldots,t+1\}$. According to Theorem 3.1 this can be done by showing that there is no negative circuit with respect to $l_{f^{t+1}}$. This is proved is by contradiction.

Suppose that C is a negative circuit with respect to $l_{f^{t+1}}$. The following argument shows that with the help of C and P a network feasible for coalition $\{1,\ldots,t\}$ can be found, that costs less than f^t, which yields a contradiction.

Note that $f^{t+1} = f^t \oplus f_{P^{t+1}}$. Let $\hat{f}$ be the network arising from f^{t+1} if one user walks along the circuit C. Then $\hat{f} = f^{t+1} \oplus f_C$ and can also be written as $f^t \oplus h$, with $h = f_P \oplus f_C$. Hence, for all $a \in A_{N^*}$:

$$h(a) := \begin{cases} 2 & \text{if } a \text{ is used by both } P \text{ and } C, \\ 1 & \text{if } a^{-1} \text{ is neither used by } P \text{ nor } C \text{ and } a \text{ is either used by} \\ & P \text{ or } C, \\ 0 & \text{otherwise.} \end{cases}$$

The net degree outdegree$^h(i)-$indegree$^h(i)$ of node i is 1 if i equals $t+1$, -1 if i is the root and 0 otherwise. It follows that there is a path $\bar{P} \subset A_h$ from $t+1$ to $*$. Notice that if arc a is used by the path and its opposite a^{-1} by the circuit (or conversely), then $h(a) = h(a^{-1}) = 0$. Therefore $\bar{P}$ is not necessarily the same path as P. Decompose network h into three 0,1-networks $f_{\bar{P}}$, h_1 and h_2 such that $h(a) = f_{\bar{P}}(a) + h_1(a) + h_2(a)$ (note that here a regular operation $+$ is used) for all $a \in A_{N^*}$, $h_1 + h_2$ is a circulation network.[3] After path $\bar{P}$ has been chosen, h_1 and h_2 are defined

[3] A circulation network is a network in which all nodes have net degree zero.

by ($a \in A_{N^*}$):

$$h_1(a) \;:=\; \begin{cases} 1 & \text{if } h(a) - f_{\bar{P}}(a) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2(a) \;:=\; \begin{cases} 1 & \text{if } h(a) - f_{\bar{P}}(a) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The costs to obtain network $\hat{f}$ from network f^t can be computed in two ways:

$$k(\hat{f}) - k(f^t) \;=\; \sum_{a \in P} l_{f^t}(a) + \sum_{a \in C} l_{f^{t+1}}(a), \tag{2}$$

and

$$k(\hat{f}) - k(f^t) \;=\; \sum_{a \in \bar{P}} l_{f^t}(a) + \sum_{a \in A_{h_1}} l_{f^t \oplus f_{\bar{P}}}(a) + \sum_{a \in A_{h_2}} l_{f^t \oplus f_{\bar{P}} \oplus h_1}(a). \tag{3}$$

Since P is the shortest path from $t+1$ to $*$ it holds that $\sum_{a \in P} l_{f^t}(a) \leq \sum_{a \in \bar{P}} l_{f^t}(a)$. From the fact that C is a negative circuit with respect to $l_{f^{t+1}}$ and equations (2) and (3) we can conclude that:

$$\sum_{a \in A_{h_1}} l_{f^t \oplus f_{\bar{P}}}(a) + \sum_{a \in A_{h_2}} l_{f^t \oplus f_{\bar{P}} \oplus h_1}(a) < 0. \tag{4}$$

It is shown below that for each arc $a \in A_{h_1}$ it holds that $l_{f^t}(a) \leq l_{f^t \oplus f_{\bar{P}}}(a)$. Let a be an arc in A_{h_1} (so $h(a) > f_{\bar{P}}(a)$). Then a^{-1} is not used by $\bar{P}$, since $h(a) > 0$ implies that $h(a^{-1}) = 0$. If a is neither in $\bar{P}$, then $l_{f^t}(a) = l_{f^t \oplus f_{\bar{P}}}(a)$. If a is used by $\bar{P}$, we show that $l_{f^t \oplus f_{\bar{P}}}(a)$ exceeds $l_{f^t}(a)$. We distinguish between three cases. First if $f^t(a) > 0$ (this implies that $f^t(a^{-1}) = 0$), then:

$$\begin{aligned} l_{f^t \oplus f_{\bar{P}}}(a) &= k_a(f^t(a) + 2) - k_a(f^t(a) + 1) \\ &\geq k_a(f^t(a) + 1) - k_a(f^t(a)) \\ &= l_{f^t}(a). \end{aligned}$$

The inequality follows from the convexity of the function k_a.

Secondly if $f^t(a) = 0$ and $f^t(a^{-1}) = 1$, it is true that:

$$\begin{aligned} l_{f^t \oplus f_{\bar{P}}}(a) &= k_a(1) - k_a(0) \\ &\geq 0 \\ &\geq k_{a^{-1}}(0) - k_{a^{-1}}(1) \\ &= l_{f^t}(a). \end{aligned}$$

Finally if $f^t(a) = 0$ and $f^t(a^{-1}) > 1$, it holds that:

$$\begin{aligned} l_{f^t \oplus f_{\bar{P}}}(a) &= k_{a^{-1}}(f^t(a^{-1}) - 2) - k_{a^{-1}}(f^t(a^{-1}) - 1) \\ &\geq k_{a^{-1}}(f^t(a^{-1}) - 1) - k_{a^{-1}}(f^t(a^{-1})) \\ &= l_{f^t}(a). \end{aligned}$$

The inequality follows from the convexity of the function $k_{a^{-1}}$.

Because $\bar{P} \cap A_{h_2} = \emptyset$, we have for all a in A_{h_2}:

$$l_{f^t \oplus f_{\bar{P}} \oplus h_1}(a) = l_{f^t \oplus h_1}(a).$$

We conclude that:

$$\begin{aligned} k(f^t \oplus (h_1 + h_2)) &= k(f^t) + \sum_{a \in A_{h_1}} l_{f^t}(a) + \sum_{a \in A_{h_2}} l_{f^t \oplus h_1}(a) \\ &\leq k(f^t) + \sum_{a \in A_{h_1}} l_{f^t \oplus f_{\bar{P}}}(a) + \sum_{a \in A_{h_2}} l_{f^t \oplus f_{\bar{P}} \oplus h_1}(a) \\ &< k(f^t). \end{aligned} \tag{5}$$

On the other hand, because $h_1 + h_2$ is a circulation network, the network $f^t \oplus (h_1 + h_2)$ is feasible for the coalition $\{1, \ldots, t\}$. Inequality (5) contradicts the assumption that f^t is an optimal network. □

In the following example Algorithm 3.1 is used to determine an optimal structure of the convex congestion network problem of Example 2.1.

Example 3.3 *Consider the congestion network problem of Example 2.1. A number near an arc denotes its length. Choose σ as the identity. Then first player 1 has to find a shortest path in the first digraph in Figure 4. The path chosen is $\big((1,3),(3,2),(2,*)\big)$. The resulting network f^1 is depicted next to the digraph.*

Player 2 faces the length function l_{f^1} as illustrated in Figure 5. His shortest path is $\big((2,3),(3,*)\big)$. Since player 2 uses arc $(2,3)$ in the opposite direction of player 1 this cancels out. Hence in f^2 the arcs $(2,3)$ and $(3,2)$ are both not used.

Finally player 3 searches for a shortest path with respect to l_{f^2}, this is shown in Figure 6. His shortest path is given by $\big((3,2),(2,*)\big)$. The final network f^3 is also depicted in Figure 6. According to Theorem 3.2 this network is optimal, which also has been shown in Example 3.1.

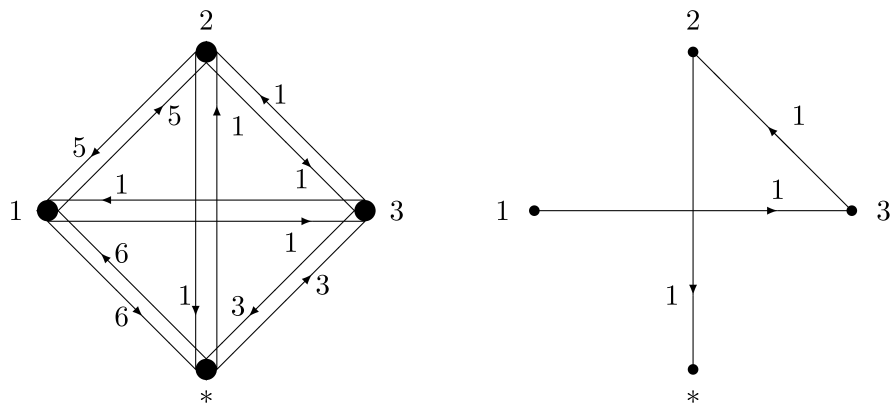

Figure 4: Left: digraph (N, A_{N^*}) with length function l_{f^0} and right: network f^1.

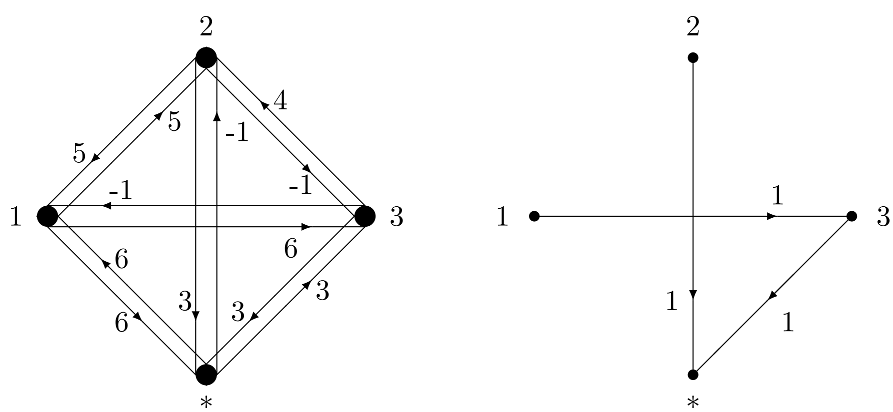

Figure 5: Left: digraph (N, A_{N^*}) with length function l_{f^1} and right: network f^2.

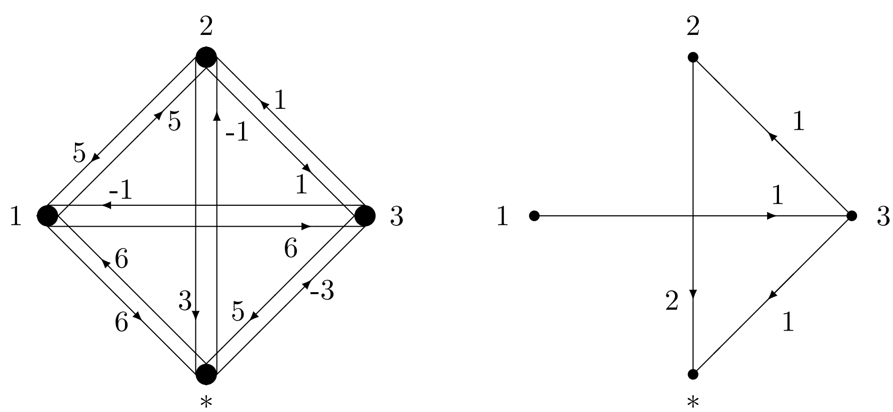

Figure 6: Left: digraph (N, A_{N^*}) with length function l_{f^2} and right: network f^3.

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