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On the Role of Chance Moves and Information in Two-Person Games

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Abstract

The value of information has been the subject of many studies in a strategic context. The central question in these studies is how valuable the information hidden in the chance moves of a game is for one or more of the players. Generally speaking, only the extra possibilities that are beneficial for the players have been considered so far. In this note we study the value of information for a special class of two-person games. For these games we also investigate how "badly" the players can do, both with and without knowing the result of the chance move. In this way one can determine to what extent the players are restricted in their possibilities by the fact that some information is hidden in the chance moves of the games. This allows for a comparison of the influence of the chance move to the control that the players have over the game result.

Keywords: games with almost perfect information; value of information; player control.
JEL code: C72, D82.

1 Introduction

Picture yourself sitting at a table, playing poker against one opponent. You play for money and your objective is to make as much money as possible in this game. Of course, your opportunities to make money depend on a few things: the dealing of the cards, the strategy of your opponent and your own strategy. The first two factors are outside of your control; you can only influence the third aspect, your own strategy. This strategy tells you, for each possible poker hand that you can be dealt and for all possible actions taken by the opponent, what actions you will take. The prescribed action depends on your hand, but it cannot depend on the hand of your opponent, because you simply don't know his cards. But what if there was a possibility to learn your opponent's hand, for example by paying someone to hold up a mirror behind his back? To what extent would this increase your possibilities? Can you use this information to improve your expected profits in the game? And if that is the case, with what amount does your expected profit change? Or, in other words, how much are you willing to pay this person who holds up the mirror?

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A one-sided cheating option such as this mirror leads to interesting questions already, but things become even more interesting when your opponent can be active behind your back too. What if he also knows your cards? Does this change the value that “your” man with the mirror has for you?

In this note we will provide an answer to the questions above. We do this for the class of two-person zero-sum games in which a chance move at the start of the game determines which game exactly is to be played. We will discuss what will be the consequences when the outcome of this chance move is revealed to one or both of the players before the players have to choose their actions. Loosely formulated, the difference between what a player can do with and without the information, is called the *value of information*. We take into account that this value may depend on the kind of opponent this player faces. For example, it might be very useful to have the information if the opponent does not, while it is less valuable to know the outcome of the chance move if the opponent knows it too. Besides that, one may wonder if it is always valuable to know the random outcome if the opponent has the irrational objective of trying to lose as much as possible. We will formalize the various kinds of opponents later in this note.

The value of information has been a notion of interest for a long time. It has been studied both in a non-strategic and a strategic setting. For the strategic setting, game theoretic analysis has been applied to many classes of games. Ponssard (1975) called the class of games that we will study, games with an initial chance move, *games with “almost” perfect information*. These games were also subject of study in the papers of Ponssard and Sorin (1980, 1982). Value of information in two-person zero-sum games has been studied by Ponssard and Zamir (1973) in the context of sequential games. Ponssard (1976) considers the constant-sum case, while general bimatrix games are the object of study in the articles of Levine and Ponssard (1977), Borm (1988) and Kamien, Tauman and Zamir (1990).

In these papers, most of the definitions concerning the value of information in a strategic conflict are based on the difference between two numbers. However, we think that more numbers may be important if one wants to quantify the worth of information in a game to one or both of the players. In computing the value of information, we will use the idea of an information buying pre-game that Sakaguchi (1993) introduced. In such a pre-game, both players get the opportunity to buy information about the outcome of the initial chance move before the start of the game. The value of information is then determined by setting the “price of information” in this pre-game at a reasonable level. The values of information that we compute, will be used to determine how restrictive the chance move in the game is for the players. In fact, these values will be used to quantify the extent to which the players have influence on the game result by defining the (derived) notion of *relative control*. This notion will also be referred to as *relative influence*.

An interesting aspect of our way of analysing information in two-person zero-sum games is that it makes use of various game-theoretic concepts within a larger framework. Apart from the zero-sum games themselves, coordination games and amalgamations of games play an important role. Coordination games form a nice subclass of the (exact) potential games. For an extensive overview of potential games we refer to Voorneveld (1999). Amalgamations of games were introduced by Borm, García-Jurado, Potters and Tijs (1996).

The analysis of relative player influence is closely related to the analysis of the skill level of a game. The skill level of a game plays a role in the consistent classification of specific (casino) games as game of chance or as game of skill. In many countries, this classification is important for the legally approved exploitation possibilities of the game. A complete overview of all aspects that play a role in the analysis of skill can be found in Dreef, Borm and Van der Genugten (2004). The main goal of that analysis is rather similar to our goal: with both methods one can draw conclusions about the role of the chance moves in a game. Central in the skill analysis are three types of players who can play the game: beginners, optimal players and fictive players. The second and third category will also appear in our analysis of relative player influence. The category of the beginners, however, which is, generally speaking, the most difficult to describe, will not play a role here.

The paper is organized as follows. In the next section, we will give some preliminaries and introduce the most important basic notions that are used in the text. Then, the sections 3 and 4 describe the way in which the value of information and the role of the chance moves are studied. Section 5 illustrates the analysis with an example that is based on a simple poker game. To conclude, section 6 contains a few remarks about our model.

2 Notation and definitions

In this section we will introduce the notation that we use throughout this paper. A *(strategic) two-person game* is a tuple $G = \langle X_1, X_2, u_1, u_2 \rangle$, where

- X_i denotes the finite, nonempty set of pure strategies of player i,
- each player i has a payoff function $u_i : X_1 \times X_2 \to \mathbb{R}$ specifying for each strategy profile $x = (x_1, x_2) \in X_1 \times X_2$ player i's payoff $u_i(x) \in \mathbb{R}$.

The set of pure strategy profiles will be denoted by $X = X_1 \times X_2$. A two-person game is called *zero-sum* if $u_1(x) = -u_2(x)$ for each strategy profile $x \in X$. The set of probability distributions over a finite set S is denoted $\Delta(S)$:

$$\Delta(S) = \{p : S \to [0,1] \mid \sum_{s \in S} p(s) = 1\}.$$

The mixed extension of the finite game $G = \langle X_1, X_2, u_1, u_2 \rangle$ allows each player i to choose a mixed strategy from $\Delta(X_i)$; a mixed strategy for player 1 (2) is denoted by p (q). Payoffs are extended to mixed strategies as follows:

$$u_i(p, q) = \sum_{x \in X} p(x_1) q(x_2) u_i(x),$$

i.e., the payoff to a mixed strategy profile is simply the expected payoff. A pure strategy $x_i \in X_i$ can be identified with the mixed strategy that assigns probability one to x_i.

A mixed-strategy profile $(p, q) \in \Delta(X_1) \times \Delta(X_2)$ is a *(mixed-strategy) Nash equilibrium* of the game G if

$$\forall x_1 \in X_1 \quad : \quad u_1(p,q) \geq u_1(x_1, q) \quad \text{and} \tag{1}$$

$$\forall x_2 \in X_2 \quad : \quad u_2(p,q) \geq u_2(p, x_2). \tag{2}$$

Let A and B be two matrices of equal size. With slight abuse of notation, we define a *bimatrix game* $\langle A, B\rangle$ as a strategic two-person game with payoff functions $u_1(p,q) = p^\top Aq$ and $u_2(p,q) = p^\top Bq$. Here, p is a column vector of which the i-th element gives the probability with which player 1 plays his i-th pure strategy; q is defined analogously. The column vector corresponding to the i-th pure strategy of a player is written as e_i. A *matrix game* is a bimatrix game with $B = -A$, written as $\langle A\rangle$. We will write matrix games and bimatrix games without the brackets if they are used as the argument of a function.

A *bimatrix game with almost perfect information* is a bimatrix game in which the payoff matrices A and B are formed in a special way. A chance move determines which of k possible bimatrix games will be played. The game $\langle A_i, B_i\rangle$ is played with probability μ_i $(1 \leq i \leq k)$. Of course, the μ_i are such that each element is selected with positive probability ($\mu_i > 0$) and the sum of the probabilities equals one ($\sum_{i=1}^k \mu_i = 1$). The payoff matrix A is formed by taking the weighted sum of the k underlying payoff matrices: $A = \sum_{i=1}^k \mu_i A_i$. Similarly, $B = \sum_{i=1}^k \mu_i B_i$. All matrices A_i and B_i must have the same size. In the naming of this type of games we follow Ponssard (1975). A *matrix game with almost perfect information* is a matrix game $\langle A\rangle$ that is based on the matrix games $\langle A_i\rangle$ in the sense that is described above.

3 The strategic possibilities of the players

In the remainder of this paper, the basic object of study will be a matrix game with almost perfect information $\langle A\rangle$, based on the matrix games $\langle A_1\rangle, \ldots, \langle A_k\rangle$. $\langle A_i\rangle$ will be played with probability μ_i $(1 \leq i \leq k)$.

3.1 Player types and related games

For the investigation of the possibilities of the players and the role of the chance move in this game, we distinguish four types of players. On the one side players can be either egoistic or altruistic, whereas on the other side players can be clairvoyant or not. *Egoistic* players want to maximize their own payoffs, while the aim of an *altruistic* player is to minimize his own payoff. The naming stems from the fact that the latter type of player helps his opponent when playing a zero-sum game. A similar distinction of behaviour patterns in non-cooperative games is given by Szép and Forgó (1985); for zero-sum games our altruistic players coincide with both their *mazochist* and *philantropic* types. Also in evolutionary settings the distinction between altruistic and selfish attitudes is made. A discussion on the context dependence of these types of preferences can be found in Bester and Güth (1998). The *clairvoyance* relates to the outcome of the chance move: clairvoyant players know beforehand which matrix game will be played. However, they cannot influence the randomization procedure. Table 1 summarizes the terms we will use when we refer to the resulting four possible player types as well as the corresponding abbreviations.

	Not clairvoyant	Clairvoyant
Altruistic	worst player (W)	fictive worst player (FW)
Egoistic	optimal player (O)	fictive optimal player (FO)

Table 1: Four types of players.

For a given matrix game of the type that we discussed at the beginning of this section, we want to know to what extent the players are in control. More precisely, we determine the range of payoffs that can be reached by the players, given the rules of the game. For each player we want to know how well he can do, but we are also interested in how badly he can do. Moreover, we want to know if the uncertainty that is caused by the chance move really restricts the possibilities of the players. To investigate these questions, we let each of the four player types take both player roles in the matrix game and we let all combinations of player types play the game. If we assume that players always know what type of opponent they are facing, this idea gives rise to the 16 games that are given in Table 2.

Player 1 type \ Player 2 type	FW	W	O	FO
FW	$\langle -A_i \rangle$ $(1 \le i \le k)$	$\langle -A^1 \rangle$	$\langle -A^1, -A^1 \rangle$	$\langle -A_i, -A_i \rangle$ $(1 \le i \le k)$
W	$\langle -A^2 \rangle$	$\langle -A \rangle$	$\langle -A, -A \rangle$	$\langle -A^2, -A^2 \rangle$
O	$\langle A^2, A^2 \rangle$	$\langle A, A \rangle$	$\langle A \rangle$	$\langle A^2 \rangle$
FO	$\langle A_i, A_i \rangle$ $(1 \le i \le k)$	$\langle A^1, A^1 \rangle$	$\langle A^1 \rangle$	$\langle A_i \rangle$ $(1 \le i \le k)$

Table 2: All combinations of player types and the resulting games.

Let us explain the contents of table 2 in more detail. The basic situation is the case where two optimal players face each other. These players both try to maximize their payoffs in the matrix game $\langle A \rangle$. If player 2 acts as a worst player and thus tries to obtain the lowest possible payoff in $\langle A \rangle$, the resulting strategic situation can be modelled by the coordination game $\langle A, A \rangle$. In this game the payoff for player 1 is the same as in the original game, whereas player 2 now acts as if he maximizes his payoff, pretending that his payoff matrix is A instead of $-A$. However, after we have found an equilibrium in this game, we have to reverse the sign of player 2's payoff again. This reasoning explains the four cells in the middle, the situations where two non-clairvoyant players meet.

The notations A^1 and A^2 require some explanation too. We will give the interpretation for A^1; the story for A^2 goes analogously. The payoff matrix A^1 is used in the cells where player 1 is fictive and player 2 is not. Such a situation can be modelled as an *amalgamation of games*, following the definition of Borm, García-Jurado, Potters and Tijs (1996). In an amalgamation of games the player set is partitioned into two parties and the game is an aggregation of the conflicts between two players, one in each party. The payoff of a player is just the sum of the payoffs in the separate conflicts. In the situation we study we have k "instances" of player 1 (all members of the first party) playing against one opponent (the only member of the second party). Taking into account the fact that instance i of player 1 is called to play with probability μ_i, the matrix $\mu_i A_i$ is the payoff matrix that is used in the game between instance i of player j and his opponent ($1 \leq i \leq k$). Borm *et al.* (1996, p. 574, Proposition 1) showed for the $k+1$-person game that models this situation that each equilibrium corresponds to an equilibrium in the related two-person *correlation game*. In this correlation game the k instances of player 1 are considered as one player and are allowed to pick correlated strategies. Together they choose a strategy from $\Delta(\prod_{i=1}^{k} X_1^i)$, where X_1^i is the set of pure strategies of player 1 in the matrix game $\langle A_i \rangle$. The payoff matrix of this correlation game, with player 1 as a fictive player, is represented as A^1. Together with the reasoning about the reversed sign as before, this explains the contents of the eight cells where one player is clairvoyant and his opponent is not.

The remaining four cells, corresponding to the situations with two clairvoyant players, speak more or less for themselves. Both players know which of the k matrix games they play, so they can optimize their strategic behaviour for each of these games separately.

The following example will illustrate the types of games described above.

Example 3.1 Let $\langle A \rangle$ be a matrix game with almost complete information, based on the matrix games $\langle A_1 \rangle$ and $\langle A_2 \rangle$ that are played with equal probability ($\mu_1 = \mu_2 = \frac{1}{2}$). The payoff matrices are given in Figure 1. Figure 2 gives the payoff matrices as well as the available strategies both

$$\begin{array}{c|c|c|}
 & f_1 & f_2 \\ \hline
e_1 & 5 & 0 \\ \hline
e_2 & 0 & 2 \\ \hline
\end{array}
\qquad\qquad
\begin{array}{c|c|c|}
 & f_1 & f_2 \\ \hline
e_1 & 2 & 5 \\ \hline
e_2 & 4 & 0 \\ \hline
\end{array}$$

$\langle A_1 \rangle$ $\qquad$ $\langle A_2 \rangle$

Figure 1: The matrix games underlying $\langle A \rangle$.

for the situation where player 1 is a fictive player and the situation where he is a normal player.

With the strategy $e_a e_b$ we denote the choice of player 1 to play e_a if payoff matrix A_1 is used and play e_b if A_2 is used.

	f_1	f_2
e_1	$\frac{7}{2}$	$\frac{5}{2}$
e_2	2	1

$\langle A \rangle$

	f_1	f_2
e_1e_1	$\frac{7}{2}$	$\frac{5}{2}$
e_1e_2	$\frac{9}{2}$	0
e_2e_1	1	$\frac{7}{2}$
e_2e_2	2	1

$\langle A^1 \rangle$

Figure 2: The resulting games with player 2 as a normal player and player 1 as a normal player ($\langle A \rangle$) and as fictive player ($\langle A^1 \rangle$) player.

$\triangleleft$

3.2 Expected payoffs

We want to compare the equilibrium payoffs of the games that are played in each of the 16 cells of Table 2. Since the central game in this analysis is a zero-sum game, we can restrict our attention to player 1's payoff. Player 2's payoff is the same number with opposite sign. Note that it is possible to interchange the roles of the players. With player 2 as the row player the basic game would be $\langle -A^\top \rangle$ and we could construct Table 2 in a similar way as we did for $\langle A \rangle$.

In half of the cases equilibrium payoffs are unambiguous. If the players are both egoistic, the games we have to solve are matrix games and therefore they have a uniquely defined value. The same holds if both players are altruistic. In the other eight cells, with one player being altruistic while his opponent is egoistic, we have to make a selection out of the possibly many Nash equilibria of bimatrix games. In fact, all bimatrix games we have to solve are coordination games. This type of games forms a nice subclass of the (exact) potential games. For an extensive overview of potential games we refer to Voorneveld (1999). An obvious Nash equilibrium refinement choice for these games is the potential maximizer. For coordination games, the logical choice for a potential function is $u_1(x)$. We will denote the payoff for the players for any x that maximizes $u_1(x)$ in the game G as $u_{PM}(G)$.

We give three reasons justifying the choice of the potential maximizer. First of all, we are exploring the range of possibilities the players have in the game. Potential-maximizing strategies certainly form an extremity of the strategic options of the players. It gives a theoretic bound of the game. Secondly, the potential maximizer forms an attractive focal point for the players, being a pure-strategy equilibrium with high payoffs. Finally, Reijnierse, Voorneveld and Borm (2003) showed that the potential maximizer is in the set of informationally robust equilibria of the game. This equilibrium refinement concept is closely related to our idea of leaking of information to one (or both) of the players. Table 3 presents the expected payoffs for player 1 in each of the 16 situations.

Player 1 type \ Player 2 type	FW	W	O	FO
FW	$-\sum_{i=1}^{k}\mu_i v(-A_i)$	$-v(-A^1)$	$-u_{PM}(-A^1,-A^1)$	$-\sum_{i=1}^{k}\mu_i u_{PM}(-A_i,-A_i)$
W	$-v(-A^2)$	$-v(-A)$	$-u_{PM}(-A,-A)$	$-u_{PM}(-A^2,-A^2)$
O	$u_{PM}(A^2,A^2)$	$u_{PM}(A,A)$	$v(A)$	$v(A^2)$
FO	$\sum_{i=1}^{k}\mu_i u_{PM}(A_i,A_i)$	$u_{PM}(A^1,A^1)$	$v(A^1)$	$\sum_{i=1}^{k} v(A_i)$

Table 3: The expected payoffs of player 1 for all combinations of player types.

Example 3.2 We computed the numbers that are given in Table 3 for the game we introduced in Example 3.1. The results are in Table 4. $\triangleleft$

Player 1 type \ Player 2 type	FW	W	O	FO
FW	$2\frac{1}{7}$	$1\frac{5}{7}$	0	0
W	$2\frac{3}{7}$	2	1	0
O	5	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{6}{7}$
FO	5	$4\frac{1}{2}$	$2\frac{11}{14}$	$2\frac{1}{7}$

Table 4: The expected payoffs of player 1 for each combination of player types.

The following lemma states two simple observations regarding the interchange of player roles in a matrix game $\langle A\rangle$ and a coordination game $\langle A, A\rangle$.

Lemma 3.3

1. $v(-A^\top) = -v(A)$;
2. $u_{PM}(A,A) = u_{PM}(A^\top, A^\top)$.

The first part of the lemma states that equilibrium payoffs in a matrix game do not depend on who is the row player and who is the column player. The second part formalizes the fact that interchanging the player roles in a coordination game does not influence the potential maximizing payoffs. We use the observations in Lemma 3.3 to explain some symmetry arguments in the proof of Theorem 3.5.

Lemma 3.4 *Let $\langle A, B\rangle$ be a bimatrix game and let $\langle A\rangle$ be the (corresponding) matrix game. Then*

$$(\hat{p}, \hat{q}) \in NE(A, B) \Rightarrow \hat{p}^\top A\hat{q} \geq v(A).$$

Proof. Let $(\hat{p}, \hat{q}) \in NE(A, B)$. Then $\hat{p}^\top A\hat{q} = \max_p p^\top A\hat{q} \geq \min_q \max_p p^\top Aq = v(A)$. □

Let us introduce some notation for the payoffs that are given in Table 3. We write the collection of player types as $\mathcal{T} = \{FW, W, O, FO\}$ and we will use the notation u_{τ_1,τ_2} for the expected payoff of player 1 in the game between player 1 of type τ_1 and player 2 of type τ_2 for all $\tau_1, \tau_2 \in \mathcal{T}$. Using these definitions, we can formulate Theorem 3.5. This theorem states that the payoffs in Table 3 are higher for egoistic players than for altruistic players against any opponent type and that clairvoyancy helps a player, independent of the type of opponent he faces.

Theorem 3.5

$$u_{\tau_1,FW} \geq u_{\tau_1,W} \geq u_{\tau_1,O} \geq u_{\tau_1,FO} \quad \textit{for all } \tau_1 \in \mathcal{T}, \tag{3}$$

$$u_{FW,\tau_2} \leq u_{W,\tau_2} \leq u_{O,\tau_2} \leq u_{FO,\tau_2} \quad \textit{for all } \tau_2 \in \mathcal{T}. \tag{4}$$

Proof. Since the potential maximizing strategies together form a Nash equilibrium, it follows directly from Lemma 3.4 that $u_{\tau_1,W} \geq u_{\tau_1,O}$ for all $\tau_1 \in \mathcal{T}$. By writing down the expression for $u_{FW,FW}$ we see that

$$\begin{aligned}
u_{FW,FW} &= -\sum_{i=1}^{k} \mu_i v(-A_i) = -\sum_{i=1}^{k} \mu_i \min_{q_i\in\Delta(X_2)} \max_{p_i\in\Delta(X_1)} p_i^\top(-A_i)q_i \\
&\geq -\min_{q\in\Delta(X_2)} \sum_{i=1}^{k} \max_{p_i\in\Delta(X_1)} p_i^\top(-\mu_i A_i)q = -\min_{q\in\Delta(X_2)} \max_{p\in\prod_{i=1}^{k}(\Delta X_1)} p^\top(-A^1)q \\
&\overset{(*)}{=} -\min_{q\in\Delta(X_2)} \max_{p\in\Delta(\prod_{i=1}^{k} X_1)} p^\top(-A^1)q = -v(-A^1) = u_{FW,W},
\end{aligned}$$

where $(*)$ was shown by Borm *et al.* (1996, p. 574, Proposition 1). Analogously, one can show that $u_{W,FW} \geq u_{W,W}$, $u_{O,O} \geq u_{O,FO}$ and $u_{FO,O} \geq u_{FO,FO}$. In a similar way, one can show that the inequality between $u_{O,FW}$ and $u_{O,W}$:

$$\begin{aligned}
u_{O,FW} &= u_{PM}(A^2, A^2) = \max_{(p,q)\in\Delta(X_1)\times\Delta(\prod_{i=1}^{k} X_2)} p^\top A^2 q \\
&= \max_{(p,q_1,\ldots,q_k)\in\Delta(X_1)\times\prod_{i=1}^{k}\Delta(X_2)} \sum_{i=1}^{k} p^\top(\mu_i A_i)q_i \geq \max_{(p,q)\in\Delta(X_1)\times\Delta(X_2)} \sum_{i=1}^{k} p^\top(\mu_i A_i)q \\
&= \max_{(p,q)\in\Delta(X_1)\times\Delta(X_2)} p^\top Aq = u_{PM}(A, A) = u_{O,W},
\end{aligned}$$

where the third equality again is an application of the result of Borm *et al.* (1996). The validity of the remaining three inequalities in (3) can be shown analogously.

Finally, the inequalities in (4) can be derived from (3) by writing down a table like Table 3 with player 2 as the row player and applying Lemma 3.3 to the inequalities that (3) gives for the payoffs in this table. □

Theorem 3.5 states that the payoffs for player 1, as they were defined in Table 3, are (weakly) decreasing in value if we go through the table from bottom-left to top-right. Therefore, we know that differences like, for example, $u_{FO,O} - u_{O,O}$ and $u_{FW,W} - u_{FW,O}$ are nonnegative.

4 The role of information

In section 4.2 we will use the definitions and results of section 3.2 to present a well-defined way to quantify the restrictive role of the chance moves in matrix games with almost perfect information. But first, in section 4.1, we will see which different types of value of information that are distinguished in the literature can be derived from Table 3.

4.1 The value of information

In this section we will see how the numbers from Table 3 are related to the various definitions one finds in the literature on the value of information in a strategic context. For an overview of various types of information and a discussion on relations between them, we refer to Borm (1988).

The games between two egoistic players, one of the two being clairvoyant, are games with *private information* for the clairvoyant player. The difference $u_{FO,O} - u_{O,O}$ is often referred to as the *value of private information* for player 1. According to Theorem 3.5, this value is positive. This is a confirmation of a result of Ponssard (1976). Two egoistic players, both being clairvoyant, play a game with *public information*. Therefore, the difference $u_{FO,FO} - u_{O,O}$ is called the *value of public information* for player 1. It is not possible to say anything about the sign of the value of public information; all we can say is that the value of public information for player 1 will be the opposite of the value of public information for player 2.

With each matrix game with almost perfect information, we want to associate eight values of information, one for each player in each possible combination of attitudes (egoistic or altruistic). We will do this using the payoffs in Table 3. As the two definitions in the previous paragraph illustrate, most definitions concerning the value of information in a strategic conflict are based on the difference between two numbers. However, we think that at least four numbers are important if one wants to quantify the worth of information in a game to one or both of the players. If both players are egoistic, then the four numbers in the lower right part of Table 3 should be taken into account: $u_{O,O}$, $u_{FO,O}$, $u_{O,FO}$ and $u_{FO,FO}$. To do this, we follow the approach of

Sakaguchi (1993), who defined an information buying pre-game. In this game both players have to decide whether to buy information or not. Player i has to pay an amount c_i to his opponent if he wants to be informed about the outcome of the chance move. These prices are set by an external person, someone like the "maven" from Kamien, Tauman and Zamir (1990). Figure 3 shows the payoffs of the information buying pre-game. The prices are set in such a way that for

		Player 2: Don't buy	Player 2: Buy
Player 1	Don't buy	$u_{O,O}$	$u_{O,FO}+c_2$
	Buy	$u_{FO,O}-c_1$	$u_{FO,FO}-c_1+c_2$

Figure 3: The information buying pre-game when both players are egoistic.

both players the pre-game is interesting in the sense that information is neither too cheap nor too expensive. They both really have to think about buying or not buying it. More formally, c_1 and c_2 are set such that neither player has a strongly dominant strategy in the pre-game.

Buy is strongly dominant for player 1 if $c_1 < \underline{c}_1 = \min\{u_{FO,O} - u_{O,O}, u_{FO,FO} - u_{O,FO}\}$. In other words, it should be profitable to buy the information against an optimal player, but also against a fictive optimal player. Similarly, *Buy* is strongly dominant for player 2 if $c_2 < \underline{c}_2 = \min\{u_{O,O} - u_{O,FO}, u_{FO,O} - u_{FO,FO}\}$. Analogously, *Don't buy* is a strongly dominant strategy for player 1 if $c_1 > \overline{c}_1 = \max\{u_{FO,O} - u_{O,O}, u_{FO,FO} - u_{O,FO}\}$. Buying is neither profitable against an optimal player nor against a fictive optimal player. For player 2 *Don't buy* is a strongly dominant strategy if $c_2 > \overline{c}_2 = \max\{u_{O,O} - u_{O,FO}, u_{FO,O} - u_{FO,FO}\}$.

Only between the boundaries $\underline{c}_i$ and $\overline{c}_i$ it is possible for player i to be made indifferent between buying and not buying by his opponent. Therefore, the price of information for player i, c_i, should definitely lie between these boundaries. We follow the approach of Sakaguchi (1993) and set the prices at a level that ensures that buying the information with probability $\frac{1}{2}$ is an equilibrium strategy for both players. The prices for which the pre-game has this nice characteristic, are $c_i = \frac{1}{2}(\underline{c}_i + \overline{c}_i)(i \in \{1,2\})$. To make clear that these prices are the values of information for the players in the game when they both act egoistically, we will write them as $c_1^{O,O}$ and $c_2^{O,O}$.

For the games in which altruistic players are involved, we can define the prices of information in a similar way. Consider the situation where player 1 is egoistic and player 2 is altruistic. The four payoffs of interest can then be found in the bottom-left part of Table 3: $u_{O,FW}$, $u_{O,W}$, $u_{FO-,FW}$ and $u_{FO,W}$. Figure 4 shows the payoffs of the information buying pre-game for this situation. It is clear that the "fair" price of information for player 1 can be set in the way that we described for the game between two optimal players. Does this method also work for player 2, who has altruistic motives? Yes, it does. We know from Theorem 3.5 that $u_{O,W} \leq u_{O,FW}$ and $u_{FO,W} \leq u_{FO,FW}$. The altruistic player 2 is better off if more money in the game is transferred to his opponent. Since the game is zero-sum, we already implicitly

		Player 2	
		Don't buy	Buy
Player 1	Don't buy	$u_{O,W}$	$u_{O,FW} + c_2$
	Buy	$u_{FO,W} - c_1$	$u_{FO,FW} - c_1 + c_2$

Figure 4: The information buying pre-game when player 1 is egoistic and player 2 is altruistic.

assumed that an altruistic player was willing to pay one unit for each unit that the opponent gets extra. Therefore, consistent reasoning leads to the conclusion that player 2 is willing to pay at least $\underline{c}_2 = \min\{u_{O,FW} - u_{O,W}, u_{FO,FW} - u_{FO,W}\}$. Using similar reasoning, we can also define the upperbound $\overline{c}_2$. In order to give the game the property that that buying the information with probability $\frac{1}{2}$ is an equilibrium strategy for both players, we have to define $c_i = \frac{1}{2}(\underline{c}_i + \overline{c}_i)(i \in \{1, 2\})$ here too.

For the other two combinations of attitudes, with only player one or both players being altruistic, we can do similar computations. In this way we can associate with each matrix game *eight* values of information, four for each player. We will give the numbers for our example.

Example 4.1 For the game that was discussed in examples 3.1 and 3.2, the value of information for player 1, in the situation where he is egoistic and his opponent is altruistic, should lie between $\underline{c}_1^{O,W} = \min\{5 - 5, 4\frac{1}{2} - 3\frac{1}{2}\} = 0$ and $\overline{c}_1^{O,W} = \max\{0, 1\} = 1$. Taking the average, we get $c_1^{O,W} = \frac{1}{2}$. This number is given in the list below, together with the other seven relevant values of information.

$$
\begin{array}{rclcrcl}
c_1^{O,O} & = & \frac{2}{7} & \qquad & c_2^{O,O} & = & \frac{9}{14} \\
c_1^{O,W} & = & \frac{1}{2} & & c_2^{O,W} & = & 1 \\
c_1^{W,O} & = & \frac{1}{2} & & c_2^{W,O} & = & \frac{1}{2} \\
c_1^{W,W} & = & \frac{2}{7} & & c_2^{W,W} & = & \frac{3}{7}
\end{array}
$$

◁

We will use these values of information in the next section to quantify the relative influence of the players on the game result, compared to the influence of the initial chance move.

4.2 Player control and influence of the chance move

In this section we will use the payoffs from Table 3, together with the corresponding values of information, to quantify the restrictive role of the chance moves. By Theorem 3.5 we know that the highest possible equilibrium payoff for player 1 occurs in the situation with two clairvoyant players, with player 1 being egoistic and player 2 being altruistic. In terms of Theorem 3.5, this payoff is written as $u_{FO,FW}$. Similarly, the minimal payoff for player 1 is $u_{FW,FO}$. These numbers represent the maximum and minimum payoff for player 1, given that the information on the chance moves can be used by the players. The difference between these numbers, $u_{FO,FW} - u_{FW,FO}$, indicates the size of the *fictive range* of the game's payoffs.

We want to compare numbers within this fictive range with the payoffs that can be attained by non-clairvoyant players, as displayed in the four central cells in Table 3. The payoff information in these four cells can be summarized in a logical way by considering four differences: for the two players, we compute the difference between the maximum and minimum payoff, both against a worst opponent and against an optimal opponent. These differences give the payoff variation that can be caused by the players themselves. We are interested in the relative size of these numbers, compared to the restrictions caused by the chance move. The restriction can be quantified by two numbers: the value of information in the case a player tries to minimize his payoff and the value that this information has to him when he tries to maximize his payoff.

We define γ_i^O (γ_i^W) to be the *relative control* level of player i against an optimal (worst) opponent. Formally,

$$\begin{aligned}
\gamma_1^O &= \frac{(u_{O,O} - u_{W,O})}{c_1^{O,O} + (u_{O,O} - u_{W,O}) + c_1^{W,O}},\\
\gamma_1^W &= \frac{(u_{O,W} - u_{W,W})}{c_1^{O,W} + (u_{O,W} - u_{W,W}) + c_1^{W,W}},\\
\gamma_2^O &= \frac{(u_{O,W} - u_{O,O})}{c_2^{O,W} + (u_{O,W} - u_{O,O}) + c_2^{O,O}},\\
\gamma_2^W &= \frac{(u_{W,W} - u_{W,O})}{c_2^{W,W} + (u_{W,W} - u_{W,O}) + c_2^{W,O}}.
\end{aligned}$$

From these definitions and the result of Theorem 3.5, it is clear that the numbers $0 \leq \gamma_i^O \leq 1$ and $0 \leq \gamma_i^W \leq 1$ for $i \in \{1, 2\}$. If $\gamma_i^O = 1$, then the chance move is not restrictive at all for player i against an optimally playing opponent.

It is interesting to note that the opponents against whom the relative controls γ_2^O and γ_1^W are computed have the same objective. These opponents both try to maximize the expected payoff of player 1, but they have to operate from different role perspectives. In the first case, the opponent has to play from the "row position", whereas the opponent against whom γ_1^W is computed uses the "column position". If $\gamma_1^W < \gamma_2^O$, then player 2's control against a rational opponent, who is maximizing his own payoff, is higher if he can operate from the row position than if he can act as the column player. Similar comparisons can be made between γ_1^O and γ_2^W to say something about the relative control of player 1 against a payoff-maximizing opponent.

5 An example: minipoker

In this section we will illustrate the analysis with a more lively example than the ones we used so far to explain our definitions. We will study a two-person game called *minipoker*. This is a game of cards played by two players, player 1 and player 2, and with three cards, namely Q(ueen), K(ing) and A(ce). As usual, A is higher than K and Q is the lowest card of these three. Before play starts, both players donate 1 unit to the stakes. After (re)shuffling the deck of cards each player is dealt one card. Each player knows his own card, but not the card of his opponent. Thus, the card which remains in the deck is not shown to either of the players. Player 1 starts the play and has to decide between P(assing) and B(etting). If he decides to pass, a showdown follows immediately. In the showdown both cards are compared and the player with the highest card gets the stakes. If player 1 decides to bet, he has to add one extra unit to the stakes. Subsequently, player 2 has to decide between F(olding) and C(alling). If he decides to fold, player 1 gets the stakes. If player 2 decides to call, he also has to add one extra unit to the stakes and a showdown follows.

We can model this game as a matrix game with almost perfect information in the way that is shown in Figure 5: after the initial chance move, the players play one of the six 2×2 matrix games (all with equal probability). However, if we model minipoker this way, there is a difference between the normal players in this game and the normal players we have studied so far. So far, the normal players were not able to make any distinction between the k matrix games they could possibly face. In minipoker, both players can exclude outcomes of the chance move by looking at their own card. In fact, we can say that the outcome space of the chance move is the set $\{(Q,K),(Q,A),(K,Q),(K,A),(A,Q),(A,K)\}$ and that player 1 faces can distinguish between elements of the partition $\{\{(Q,K),(Q,A)\},\{(K,Q),(K,A)\},\{(A,Q),(A,K)\}\}$. Similarly, player 2 faces the partition $\{\{(Q,K),(A,K)\},\{(Q,A),(K,A)\},\{(K,Q),(A,Q)\}\}$. The fact that the non-fictive players, together with this formulation of the game as a matrix game with almost perfect information, do not completely fit into the framework of this paper is not a problem. It is not difficult to see that the proof of Theorem 3.5 only uses the fact that the information partition of the fictive players is a refinement of the partition of the normal players. The payoffs in all 16 games that are relevant for determining the relative influence of the players

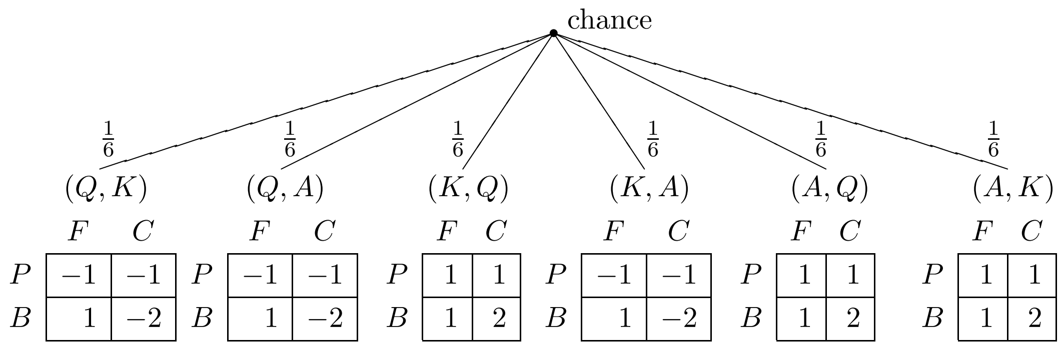


Figure 5: Minipoker as a matrix game with “almost” perfect information.

on their game result, are given in Figure 6. We learn from the table that the value of the game is $\frac{1}{18}$. And, as we can see, the numbers in this table satisfy the order that we need to make all definitions regarding information and control sensible. Using the numbers in Figure 6, we can determine the values of information and relative control levels for both players in minipoker. We find that three of the eight prices of information are unequal to zero: $c_1^{O,O} = \frac{1}{36}$, $c_2^{O,O} = \frac{1}{12}$

player 1 type \ player 2 type	FW	W	O	FO
FW	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
W	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
O	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{1}{18}$	0
FO	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{1}{9}$	0

Figure 6: Expected payoffs for player 1 in minipoker, all combinations of player types.

and $c_2^{O,W} = \frac{1}{6}$. Using these numbers, we find $\gamma_1^O = \frac{20}{21}$, $\gamma_1^W = 1$, $\gamma_2^O = \frac{46}{55}$ and $\gamma_2^W = 1$. For both players, it turns out that playing against a player who minimizes his own payoff is better from a control point of view.

What's the explanation for this extremely high level of relative control against a worst opponent? Well, let us consider the case of player 1 facing an opponent who tries to minimize his own payoff. If neither player is fictive and player 1 acts as an optimal player, then in the unique equilibrium player 1 always bets, while player 2 calls with a queen and folds with a king or with an ace. The expected payoff for player 1 is then $\frac{4}{3}$. If player 1 acts as a worst player against his worst opponent, the best thing he can do with any card is passing (or he can bet with a queen, but this will not be useful since player 2 will not risk winning by calling with a king or an ace). His expected payoff will then be 0. Since $c_1^{O,W} = c_1^{W,W} = 0$, revealing player 2's card to player 1 does not give player 1 additional strategic possibilities. The fact that $u_{O,W} > u_{W,W}$ is therefore sufficient to obtain a relative control level equal to 1.

From the numbers γ_i^O and γ_i^W $(i = 1, 2)$, we want to draw two conclusions. In the first place, since both γ_1^O and γ_2^O are smaller than 1, we know that the dealing of the cards really leads to restrictions in the possibilities of rational players. In the second place, since $\gamma_1^O < \gamma_2^W$ and $\gamma_2^O < \gamma_1^W$, we can conclude that both players in minipoker would prefer the position of the opponent from a control point of view. In the "column position" of player 2 one has more control about the expected payoff of player 1 when one is trying to maximize this payoff. Similarly, the "row position" of player 1 gives more control over the maximization of player 2's payoff.

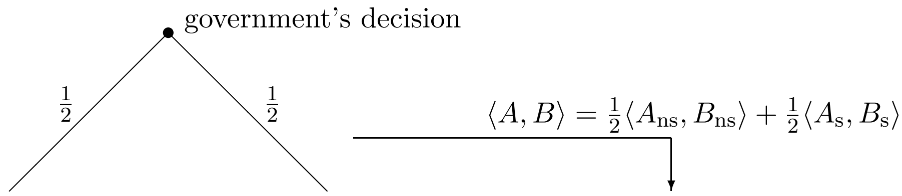
6 Concluding remarks

In this paper we have presented a way to analyse the role of chance moves for a specific class games. We have given a method that enables us to determine the value of information about those chance moves for the players. Using this valuation of the information, we quantified the restrictive role of the chance moves with respect to the influence of the players on their own payoffs. To conclude, we wish to make a few remarks about our analysis.

6.1 Extension to other classes of games

The starting point in this paper was a matrix game with almost perfect information. We used the zero-sum property in the proof of Theorem 3.5. Of course, it is interesting to check whether our analysis can be carried out for a broader class of bimatrix games. Example 6.1 supports the conjecture that this is possible. Before we give the example, we have to think for a moment about the definition of the altruistic players. Although we prefer this positive terminology over the term masochistic in the zero-sum context, we want to stress here that the original idea was that these players try to minimize their own payoffs. Thus, the interpretation that we give to the worst player in a bimatrix game, really is a masochistic player.

Example 6.1 Consider the following duopoly situation. Two firms produce some good and they can choose to use *grey* or *green* energy for the production process. The second type is better for the environment, but it is also more expensive. Therefore, the selling price of the product should be higher. This leads to a decrease in demand. However, the government is discussing the possibility of giving subsidy to consumers who buy products that have been produced in an ecologically sound way. This subsidy is expected to stimulate the consumers so much that the demand for the product will increase, even if the price is higher. The probability that the government will decide to give this subsidy, is estimated by the firms to be 50%. We model this situation as a bimatrix game with almost perfect information. This game is shown in Figure 7. Figure 8 contains the payoffs corresponding to the 16 games from which we wish to



(no subsidy)

	grey	green
grey	$4, 4$	$4, 3$
green	$3, 4$	$3, 3$

$\langle A_{\text{ns}}, B_{\text{ns}}\rangle$

(subsidy)

	grey	green
grey	$4, 4$	$3, 7$
green	$7, 3$	$5, 5$

$\langle A_{\text{s}}, B_{\text{s}}\rangle$

(compound game)

	grey	green
grey	$4, 4$	$3\frac{1}{2}, 5$
green	$5, 3\frac{1}{2}$	$4, 4$

$\langle A, B\rangle$

Figure 7: Duopoly with possible consumer subsidy for ecologically sound products.

draw conclusions about relative control. The bimatrix game corresponding to the situation where firm 1 knows in advance if the government will give subsidy or not, is given in this figure too, since it helps in quickly checking the 16 payoffs. The presence of a strongly dominant strategy in each bimatrix game that we have to consider simplifies the computations. Using the information in Figure 8, we find that the price of information is $\frac{1}{2}$ for each combination of attitudes. These prices are computed in the same way as for the zero-sum case: we construct the information buying

	grey	green
(grey, grey)	$4, 4$	$3\frac{1}{2}, 5$
(grey, green)	$5\frac{1}{2}, 3\frac{1}{2}$	$4\frac{1}{2}, 4$
(green, grey)	$3\frac{1}{2}, 4$	$3, 5$
(green, green)	$5, 3\frac{1}{2}$	$4, 4$

$\langle A^1, B^1 \rangle$

firm 1 type \ firm 2 type	FW	W	O	FO
FW	$3\frac{1}{2}, 3\frac{1}{2}$	$3\frac{1}{2}, 4$	$3, 5$	$3, 5\frac{1}{2}$
W	$4, 3\frac{1}{2}$	$4, 4$	$3\frac{1}{2}, 5$	$3\frac{1}{2}, 5\frac{1}{2}$
O	$5, 3$	$5, 3\frac{1}{2}$	$4, 4$	$4, 4\frac{1}{2}$
FO	$5\frac{1}{2}, 3$	$5\frac{1}{2}, 3\frac{1}{2}$	$4\frac{1}{2}, 4$	$4\frac{1}{2}, 4\frac{1}{2}$

Figure 8: Information about the duopoly needed for analysing the relative influence of the firms on their profits.

pre-game and we set the prices such that the situation in which both players buy information with probability $\frac{1}{2}$ is an equilibrium. These numbers lead to $\gamma_1^O = (4-3\frac{1}{2})/(\frac{1}{2}+(4-3\frac{1}{2})+\frac{1}{2}) = \frac{1}{3}$. Similarly, we find $\gamma_1^W = \frac{1}{2}$ and, by the symmetry of the game, $\gamma_2^O = \frac{1}{3}$ and $\gamma_2^W = \frac{1}{2}$. For both firms, relative control is smaller against an optimal opponent than against a worst opponent. Phrased differently, for both firms the uncertainty about the subsidy being given or not has more influence on the firm's profits when its competitor acts as a profit maximizer.

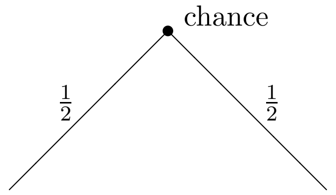
In non-zero sum games, an egoistic player and his masochistic opponent do not necessarily have the same objective. Minimizing one's own payoff is not the same as maximizing the payoff of the opponent anymore. As a result, a comparison between the relative control numbers γ_2^O and γ_1^W does not make sense anymore. $\triangleleft$

In the example above we have a wisely structured bimatrix game (with an underlying story) with almost perfect information for which we can carry out our analysis. The following example shows that our analysis in general does not work for bimatrix games, not even if the game is such that each of the underlying bimatrix games, as well as the compound bimatrix game itself, has an equilibrium in strongly dominating strategies.

Example 6.2 This example is based on an example given by Bassan, Scarsini and Zamir (1997). It is a bimatrix game with almost perfect information in which each of the underlying bimatrix games has a unique Nash equilibrium, consisting of strongly dominant strategies. The numbers that we need for determining relative influence of the players in this game, are given in Figure 10. We see that player 1's payoffs in Figure 10 are not decreasing from bottom-left to top-right, which is a necessary condition to enable computation of the value of information in a way that is analogous to the approach presented in this paper. $\triangleleft$

6.2 Relation with the analysis of skill

As stated in the introduction, the analysis of relative player control is related to the skill analysis of a game. The goal of that analysis, of which Dreef, Borm and Van der Genugten (2004) give a detailed description, is similar: with both methods one can draw conclusions about the role of the chance moves in a game. The methods even share two "building blocks": the expected



$$\langle A, B \rangle = \tfrac{1}{2}\langle A_1, B_1 \rangle + \tfrac{1}{2}\langle A_2, B_2 \rangle$$

	L	R
T	$0, 0$	$6, -3$
B	$-3, 6$	$5, 5$

$\langle A_1, B_1 \rangle$

	L	R
T	$-20, -20$	$-7, -16$
B	$-16, -7$	$-5, -5$

$\langle A_2, B_2 \rangle$

	L	R
T	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$
B	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$

$\langle A, B \rangle$

Figure 9: The bimatrix game with almost perfect information from Bassan, Scarsini and Zamir (1997).

	L	R
(T, T)	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$
(T, B)	$-8, -3\frac{1}{2}$	$\frac{1}{2}, -4$
(B, T)	$-11\frac{1}{2}, -7$	$-1, -5\frac{1}{2}$
(B, B)	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$

$\langle A^1, B^1 \rangle$

player 1 type \ player 2 type	FW	W	O	FO
FW	$-7\frac{1}{2}, -7\frac{1}{2}$	$-11\frac{1}{2}, -7$	$-1, -5\frac{1}{2}$	$-5, -5$
W	$-7, -11\frac{1}{2}$	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$	$-4, \frac{1}{2}$
O	$-5\frac{1}{2}, -1$	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$	$-3\frac{1}{2}, -8$
FO	$-5, -5$	$\frac{1}{2}, -4$	$-8, -3\frac{1}{2}$	$-2\frac{1}{2}, -2\frac{1}{2}$

Figure 10: Information needed for analysing the relative player control in the game of Bassan, Scarsini and Zamir (1997).

payoffs of the optimal player and the fictive optimal player. The way the skill analysis defines an optimal player is exactly the same as the definition we used, but for the fictive optimal player two different definitions have been used in the skill analysis. In the analysis of Borm and Van der Genugten (2001) the fictive optimal player is assumed to have the same kind of information about the chance moves of the game as our clairvoyant, egoistic player. In the alternative approach, that was described in Dreef, Borm and Van der Genugten (2003), the clairvoyancy of the player is assumed to be even stronger: the fictive optimal player also knows in advance the outcome of possible randomization by his opponent. This type of uncertainty, caused by the players themselves, is sometimes called an *internal* chance move. To indicate the contrast, the chance moves of the game itself are called *external* chance moves. As stated, our fictive optimal player, whose clairvoyance only helps him as far as the external chance moves are concerned, is like the fictive player in Borm and Van der Genugten (2001).

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