A note on exponential dispersion models which are invariant under length-biased sampling

Bar-Lev, S.K.; van der Duyn Schouten, F.A.

Publication date:
2003

Citation for published version (APA):
Bar-Lev, S. K., & van der Duyn Schouten, F. A. (2003). *A note on exponential dispersion models which are invariant under length-biased sampling*. (CentER Discussion Paper; Vol. 2003-84). Operations research.

CentER

Discussion Paper

No. 2003–84

A NOTE ON EXPONENTIAL DISPERSION MODELS WHICH ARE INVARIANT UNDER LENGTH-BIASED SAMPLING

By S.K. Bar-Lev, F.A. van der Duyn Schouten

September 2003

ISSN 0924-7815

TILBURG ◆ UNIVERSITY

A Note on Exponential Dispersion Models which are Invariant under Length-Biased Sampling

Shaul K. Bar-Lev*and Frank A. Van der Duyn Schouten†

August 17, 2003

Abstract

Length-biased sampling situations may occur in clinical trials, reliability, queueing models, survival analysis and population studies where a proper sampling frame is absent. In such situations items are sampled at rate proportional to their "length" so that larger values of the quantity being measured are sampled with higher probabilities. More specifically, if $f(x)$ is a p.d.f. presenting a parent population composed of nonnegative valued items then the sample is practically drawn from a distribution with p.d.f. $g(x) = xf(x)/E(X)$ describing the length-biased population. In this case the distribution associated with g is termed a length-biased distribution. In this note we present a unified approach for characterizing exponential dispersion models which are invariant, up to translations, under various types of length-biased sampling. The approach is rather simple as it reduces such invariance problems into differential equations in terms of the derivatives of the associated variance functions.

Key words: Exponential dispersion model; length-biased sampling; variance function.

1 Introduction

Length-biased sampling situations may occur in clinical trials, reliability, survival analysis and population studies where a proper sampling frame is absent. In such situations items are sampled at rate proportional to their

*Department of Statistics, University of Haifa, Haifa 31905, Israel.

†Center for Economic Research, University of Tilburg, 5000 LE Tilburg, The Netherlands.

"length" so that larger values of the quantity being measured are sampled with higher probabilities. This type of sampling sometimes gives rise to unexpected conclusions, as the well-known 'waiting time paradox' in queuing theory shows (see Tijms (1994)). More specifically, if $f(x)$ is a p.d.f. presenting a parent population composed of nonnegative valued items then the sample is practically drawn from a distribution with p.d.f. $g(x) = xf(x)/E(X)$ describing the length-biased population. In this case the distribution associated with g is termed a length-biased distribution. Numerous works on various aspects of length-biased sampling are available in the literature. For recent publications see Oluyede (1999,2000,2003), van Es, Klaassen and Oudshoorn (2000), El Barmi (2002), and the references cited therein. Characterization results on invariant length-biased distributions in the Pearson family can be found in Sankaran and Unnikrishnan (1993). Some other characterizations appear in Sen and Khattree (1996) and Lingappaiah (1988). Such characterizations were obtained by using rather different approaches.

In this note we present a unified approach for characterizing exponential dispersion models (EDM's) which are invariant, up to translations, under various types of length-biased sampling (LBS). This approach has a rather simple nature as it is reduces such invariance problems into differential equations in terms of the derivatives of the associated variance functions.

EDM's have been defined and studied thoroughly by Jorgensen (1987,1998) and others, suggesting them to describe the error component in generalized linear models. An EDM is related to a natural exponential family (NEF) as follows. Let ν be a non-Dirac positive measure on $\mathbb{R}$ with Laplace transform $L_\nu(\theta) = \int_{\mathbb{R}} e^{\theta x}\nu(dx)$ and denote by D_v the effective domain of ν, i.e., $D_v = \{\theta \in \mathbb{R} : L_\nu(\theta) < \infty\}$. Assuming that $\Theta_\nu \doteq int\ D_v \neq \phi$ and letting $\kappa_\nu(\theta) = \ln L_\nu(\theta)$ be the cumulant transform of L_ν, then the NEF F generated by ν is defined by probabilities

$$F = F(\nu) = \{P(\theta, \nu(dx)) = \exp\{\theta x - \kappa_\nu(\theta)\}\,\nu(dx),\ \theta \in \Theta_\nu\}. \qquad (1)$$

The Jorgensen set $\Lambda = \Lambda_F$ associated with F is defined by

$$\Lambda = \left\{\lambda \in \mathbb{R}^+ : \lambda\kappa_\nu(\theta) \text{ is a cumulant transform of some measure } \nu_\lambda \text{ on } \mathbb{R}\right\}. \qquad (2)$$

The Jorgensen set is not empty since by convolution it contains $\mathbb{N}$. If ν is infinitely divisible then $\Lambda = \mathbb{R}^+$. For $\lambda \in \Lambda$, the NEF generated by ν_λ is

$$F_\lambda = F_\lambda(\nu_\lambda) = \{P(\theta, \lambda, \nu_\lambda(dx)) = \exp\{\theta x - \lambda\kappa_\nu(\theta)\}\,\nu_\lambda(dx),\ \theta \in \Theta_\nu\}, \qquad (3)$$

where the support and the convex-hull of the support of ν_λ may depend on λ. The set of probabilities

$$\mathbb{G} = \{P(\theta, \lambda, \nu_\lambda) : \theta \in \Theta_\nu, \lambda \in \Lambda\}\,, \tag{4}$$

is called the EDM associated with F. Note that the class of EDM's is abundant since any distribution with Laplace transform generates an EDM. An EDM is parameterized by the two parameters (θ, λ) belonging to the Cartesian product $\Lambda \times \Theta_\nu$. The parameter $\sigma^2 = 1/\lambda$ is termed the dispersion parameter. The original EDM has been defined by Jorgensen by transforming x to $y = x/\lambda$ in (3). However, the representation of F_λ in (3) suffices our characterization purposes.

Alternatively, we shall consider in this note the EDM $\mathbb{G}$ as a set of NEF's defined by

$$\mathbb{G} = \{F_\lambda : \lambda \in \Lambda\}\,, \tag{5}$$

and say that $\mathbb{G}$ is generated by the basis $F = F_1(= F(\nu))$ in the sense described above. Although any other member F_λ of $\mathbb{G}$ can serve as a basis of $\mathbb{G}$, we prefer to use F_1 for simplicity.

A main tool which is utilized to obtain characterization results for NEF's in general and in this note in particular is the variance function (VF) associated with an NEF and its associated EDM. A VF determines an NEF uniquely within the class of NEF's. For a rigorous description of the latter property as well as other fundamental properties of VF's of NEF's the reader is referred Letac and Mora (1990). We shall employ the VF and show that the problem of classifying EDM's which are invariant under LBS simply reduces to solving a differential equation in terms of the VF. For this we gather in Section 2 some required preliminaries on VF's, present a formal description of the problem and provide a characterization of invariant EDM's under first moment LBS. This characterization shows that the invariant EDM's necessarily admit quadratic VF's, implying there are only four possibilities: binomial, negative binomial, Poisson and gamma EDM's. In Section 3, we discuss the problem of EDM's invariant under k-th moment or factorial moment LBS and present, for $k = 2$, characterization results expressed in terms of appropriate differential equations.

2 Preliminaries, a problem formulation and a characterization

Consider the NEF $F = F_1 = F(\nu)$ defined in (1). The cumulant transform κ_ν of ν is strictly convex and real analytic on Θ_ν and

$$\kappa'_\nu(\theta) = \int_{\mathbb{R}} x \exp\{\theta x - \kappa_\nu(\theta)\}\, \nu(dx)$$

is the mean function of F. The open interval $\Omega_F = \kappa'_\nu(\Theta_\nu)$ is called the mean domain of F. Since the map $\theta \longmapsto \kappa'_\nu(\theta)$ is one-to-one, its inverse function $\psi_\nu : \Omega_F \longrightarrow \Theta_\nu$ is well defined. Hence, the map $m \longmapsto P(m, F) = P(\psi_\nu(m), \nu)$ is one-to-one from Ω_F onto F and is called the mean domain parameterization of F. The variance of the probability $P(m, F)$ is $V_F(m) = 1/\psi'_\nu(m) = \kappa''_\nu(\theta)$. The map $m \longmapsto V_F(m)$ from Ω_F into $\mathbb{R}^+$ is called the VF of F. In fact a VF of an NEF F is a pair (V_F, Ω_F). It uniquely determines an NEF within the class of NEF's. Morris (1982) characterized all NEF's having quadratic VF's and Letac and Mora (1990) all NEF's with cubic VF's. Bar-Lev (1987) showed that any k-th degree polynomial $\sum_{i=1}^{k} a_i m^i$ with non-zero coefficients a_i's constitutes an infinitely divisible VF with mean domain $\mathbb{R}^+$.

In the sequel, when no confusion is caused, we shall suppress the dependence of $\kappa_\nu, \kappa'_\nu, \kappa''_\nu, \Theta_\nu, V_F$ and Ω_F, related to the basis $F = F_1$ of $\mathbb{G}$, on ν or F_1 and write $\kappa, \kappa', \kappa'', \Theta, V$ and Ω.

Using the above definitions and properties of VF's, the mean function, mean domain and VF of the NEF $F_\lambda \in \mathbb{G}$, $\lambda \in \Lambda$, can be expressed in terms of the corresponding elements of F as

$$m_{F_\lambda} = \lambda\kappa'(\theta) = \lambda m, \ \Omega_{F_\lambda} = \lambda\Omega \tag{6}$$

and

$$V_{F_\lambda}(m) = \lambda\kappa''(\theta) = \lambda V(m), m \in \Omega \ \text{ or } \ V_{F_\lambda}(m_{F_\lambda}) = \lambda V_{F_\lambda}(m_{F_\lambda}/\lambda), m_{F_\lambda} \in \Omega_{F_\lambda}. \tag{7}$$

Let $S = S_\nu$ and $C = C_\nu$ denote, respectively, the support and the convex-support of the measure ν. Then, S and C are also the common support and convex-support of F. Moreover, for $\lambda \in \Lambda$, λS and λC are the common support and convex-support of the NEF $F_\lambda \in \mathbb{G}$. If C is contained in a half line then by using an appropriate affine transformation on ν, C can be assumed without any loss of generality to have the form $C = [0, a)$, for some $a \leq \infty$. Such an assumption is made throughout Section 2 as it is required for defining the set of lengh-biased distributions. It entails that the common convex-support of $F_\lambda \in \mathbb{G}$ is $\lambda C = [0, \lambda a)$.

The set $\tilde{\mathbb{G}}$ of length-biased probabilities based on the the EDM $\mathbb{G}$ in (4) is constructed as follows. For fixed $\lambda \in \Lambda$, define the measure $\mu_\lambda(dx) = x\nu_\lambda(dx)$. Its cumulant transform is clearly $\lambda\kappa(\theta) + \ln\kappa'(\theta) + \ln\lambda$. Hence, by absorbing the term $\ln\lambda$ into μ_λ, the NEF generated by μ_λ is

$$\tilde{F}_\lambda = \tilde{F}_\lambda(\mu_\lambda) = \left\{P(\theta, \lambda, \mu_\lambda(dx)) = \exp\left\{\theta x - \lambda\kappa(\theta) - \ln\kappa'(\theta)\right\}\mu_\lambda(dx),\ \theta \in \Theta\right\}. \tag{8}$$

Therefore, the set of length-biased probabilities associated with $\mathbb{G}$ can be represented as the set of NEF's

$$\tilde{\mathbb{G}} = \left\{\tilde{F}_\lambda : \lambda \in \Lambda\right\}. \tag{9}$$

The mean function and VF of the NEF $\tilde{F}_\lambda \in \tilde{\mathbb{G}}$ are

$$m_{\tilde{F}_\lambda} = \lambda\kappa'(\theta) + \frac{\kappa''(\theta)}{\kappa'(\theta)} = \lambda m + \frac{V(m)}{m}, m \in \Omega, \tag{10}$$

and

$$\begin{aligned} V_{\tilde{F}_\lambda}(m) &= \lambda\kappa''(\theta) + \frac{\kappa'(\theta)\kappa'''(\theta) - \left[\kappa''(\theta)\right]^2}{\left[\kappa'(\theta)\right]^2} \\ &= \lambda V(m) + \frac{mV'(m)V(m) - V^2(m)}{m^2}, m \in \Omega. \end{aligned} \tag{11}$$

Before posing a formal description of the invariance problem, we need one more definition. For $\alpha \in \mathbb{R}$ denote by $f_\alpha : x \mapsto x + \alpha$ a translation mapping. If P is a measure then $f_\alpha(P)$ denotes the image of P by the translation f_α. If $\tilde{\mathbb{G}}$ is a length-biased model as defined in (9), then $f_\alpha(\tilde{\mathbb{G}})$ denotes the set of all images of the members of $\tilde{\mathbb{G}}$ by f_α; i.e.,

$$f_\alpha(\tilde{\mathbb{G}}) = \left\{f_\alpha\left(P(\theta, \lambda, \mu_\lambda)\right) = P(\theta, \lambda, f_\alpha(\mu_\lambda)) : P(\theta, \lambda, \mu_\lambda) \in \tilde{\mathbb{G}}\right\}, \tag{12}$$

or, equivalently, it represents the set of NEF's

$$f_\alpha(\tilde{\mathbb{G}}) = \left\{f_\alpha\left(\tilde{F}_\lambda\right) : \lambda \in \Lambda\right\}. \tag{13}$$

Based on (10) and (11), the mean function, mean domain and VF of $f_\alpha(F_\lambda) \in f_\alpha(\mathbb{G})$ are

$$m_{f_\alpha(\tilde{F}_\lambda)} = m_{\tilde{F}_\lambda} + \alpha,\ \Omega_{f_\alpha(\tilde{F}_\lambda)} = \Omega_{\tilde{F}_\lambda} + \alpha, \tag{14}$$

$$V_{f_\alpha(\tilde{F}_\lambda)}(m) = V_{\tilde{F}_\lambda}(m), m \in \Omega \tag{15}$$

or

$$V_{f_\alpha(\tilde{F}_\lambda)}(m_{f_\alpha(\tilde{F}_\lambda)}) = V_{\tilde{F}_\lambda}(m_{f_\alpha(\tilde{F}_\lambda)} - \alpha), m_{f_\alpha(\tilde{F}_\lambda)} \in \Omega_{f_\alpha(\tilde{F}_\lambda)}. \tag{16}$$

Our characterization problem is concerned with the search of all EDM's $\mathbb{G}$ defined in (5) which coincide with the corresponding sets of NEF's $f_\alpha(\tilde{\mathbb{G}})$ defined in (13). Proposition 1 provides the solution for this search. Its proof is carried out by deriving a differential equation in terms of the VF (V, Ω) of the basis F_1 of $\mathbb{G}$.

We now present four known examples of EDM's invariant under length-biased sampling. While demonstrating the above notation and relations, these examples are also required for the proof of Proposition 1. We shall use VF's terminology and compute for each of the four examples the appropriate functionals related to the basis NEF $F = F_1$ and the three EDM's: $\mathbb{G}$ (see (6) and (7)), $\tilde{\mathbb{G}}$ (see (10) and (11)) and $f_\alpha(\tilde{\mathbb{G}})$ (see (14) and (15)). The binomial and negative binomial NEF's are displayed in Table 1, the Poisson and gamma in Table 2.

NEF Type	Binomial	Negative Binomial
F_1	$B(1,p)$	$NB(1,p)$
θ	$\ln\left(pq^{-1}\right)$	$\ln q$
Θ	$\mathbb{R}$	$\mathbb{R}^-$
$\kappa(\theta)$	$\ln(1+e^\theta)$	$-\ln(1-e^\theta)$
Ω	$(0,1)$	$\mathbb{R}^+$
V	$m(1-m)$	$m(m+1)$
F_λ	$B(\lambda,p)$	$NB(\lambda,p)$
Λ	$\mathbb{N}$	$\mathbb{R}^+$
m_{F_λ}	λm	λm
Ω_{F_λ}	$(0,\lambda)$	$\mathbb{R}^+$
V_{F_λ}	$m(1-m/\lambda)$	$\lambda^{-1}m(m+\lambda)$
$\tilde{F}_\lambda$	$LB\ B(\lambda,p)$	$LB\ NB(\lambda,p)$
$m_{\tilde{F}_\lambda}$	$(\lambda-1)m+1$	$(\lambda+1)m+1$
$\Omega_{\tilde{F}_\lambda}$	$(1,\lambda)$	$(1,\infty)$
$V_{\tilde{F}_\lambda}$	$(m-1)\ (1-(m-1)/(\lambda-1))$	$(\lambda+1)^{-1}(m-1)(m+\lambda)$
α	-1	-1
$f_\alpha\left(\tilde{F}_\lambda\right)$	$B(\lambda-1,p)$	$NB(\lambda+1,p)$
$m_{f_\alpha(\tilde{F}_\lambda)}$	$m_{\tilde{F}_\lambda}-1$	$m_{\tilde{F}_\lambda}-1$
$\Omega_{f_\alpha(\tilde{F}_\lambda)}$	$(0,\lambda-1)$	$\mathbb{R}^+$
$V_{f_\alpha(\tilde{F}_\lambda)}$	$m(1-m/(\lambda-1))$	$(\lambda+1)^{-1}m(m+\lambda+1)$

Table 1. The NEF's F_1, $F_\lambda, \tilde{F}_\lambda$ and $f_\alpha\left(\tilde{F}_\lambda\right)$ for binomial and negative binomial

NEF Type	Poisson	Gamma
F_1	$Pois(\gamma)$	$Gamma(1,\gamma)$
θ	$\ln\gamma$	$-\gamma$
Θ	$\mathbb{R}$	$\mathbb{R}^-$
$\kappa(\theta)$	e^θ	$-\ln\theta$
Ω	$\mathbb{R}^+$	$\mathbb{R}$
V	m	m^2
F_λ	$Pois(\lambda\gamma)$	$Gamma(\lambda,\gamma)$
Λ	$\mathbb{R}^+$	$\mathbb{R}^+$
m_{F_λ}	λm	λm
Ω_{F_λ}	$\mathbb{R}^+$	$\mathbb{R}^+$
V_{F_λ}	m	m^2/λ
$\tilde{F}_\lambda$	$LB\ Pois(\lambda\gamma)$	$Gamma(\lambda+1,\gamma)$
$m_{\tilde{F}_\lambda}$	$\lambda m+1$	$(\lambda+1)m$
$\Omega_{\tilde{F}_\lambda}$	$(1,\infty)$	$\mathbb{R}^+$
$V_{\tilde{F}_\lambda}$	$m-1$	$m^2/(\lambda+1)$
α	-1	0
$f_\alpha\left(\tilde{F}_\lambda\right)$	$Pois(\lambda\gamma)$	$Gamma(\lambda+1,\gamma)$
$m_{f_\alpha(\tilde{F}_\lambda)}$	λm	$(\lambda+1)m$
$\Omega_{f_\alpha(\tilde{F}_\lambda)}$	$\mathbb{R}^+$	$\mathbb{R}^+$
$V_{f_\alpha(\tilde{F}_\lambda)}$	m	$m^2/(\lambda+1)$

Table 2. The NEF's F_1, $F_\lambda, \tilde{F}_\lambda$ and $f_\alpha\left(\tilde{F}_\lambda\right)$ for Poisson and gamma

Proposition 1 *Let $\mathbb{G}$ be an EDM as defined in (5)-(7) with F_1 being a basis of $\mathbb{G}$ with VF (V,Ω). Assume that the convex support C of F_1 has the form $[0,a)$, for some $a\leq\infty$. Then $\mathbb{G}$ is invariant, up to a translation, under length-biased sampling if and only if $\mathbb{G}$ is either the binomial, negative binomial, Poisson or gamma EDM's.*

Proof. The 'if' part of the statement follows from Tables 1 and 2. For proving the 'only if' part we use VF's techniques. If $\mathbb{G}$ is invariant up to a translation under length-biased sampling then $\mathbb{G} = f_\alpha(\tilde{\mathbb{G}})$, for some $\alpha\in\mathbb{R}$.

Hence, there exist pairs (λ_1,λ_2), $\lambda_i\in\Lambda$, $i=1,2$, such that

$$F_{\lambda_1} = f_\alpha\left(\tilde{F}_{\lambda_2}\right), \tag{17}$$

implying that the VF's of these two NEF's coincide. Accordingly, relation (17) is equivalent to requiring that $V_{F_{\lambda_1}}(m) = V_{f_\alpha\left(\tilde{F}_{\lambda_2}\right)}(m)$ for all $m\in\Omega$, or, by (7) and employing (11) in (15), that

$$\lambda_1 - \lambda_2 = \frac{mV'(m) - V(m)}{m^2} = \left(\frac{V(m)}{m}\right)', \; m \in \Omega. \tag{18}$$

Letting $a = \lambda_1 - \lambda_2$, the general solution of the differential equation (18) is $V(m) = am^2 + bm$, a quadratic form in m. Morris (1982) showed that there are only six types of NEF's with quadratic VF's, of which two, the normal and hyperbolic cosine, are supported on $\mathbb{R}$ and therefore are excluded as solutions. The other four are the binomial, negative binomial, Poisson and gamma NEF's (or, EDM's in their general settings). In our formulation, the VF's of the bases F_1 corresponding to the latter EDM's are given, respectively, by: $(V, \Omega) = (m(1-m), (0,1))$ with $b = 1, a = -1$ (i.e., $\lambda_2 = \lambda_1 - 1$); $(V, \Omega) = (m(1+m), \mathbb{R}^+)$ with $b = 1, a = 1$ ($\lambda_2 = \lambda_1 + 1$); $(V, \Omega) = (m, \mathbb{R}^+)$ with $b = 1, a = 0$ ($\lambda_2 = \lambda_1$); and $(V, \Omega) = (m^2, \mathbb{R}^+)$ with $b = 1, a = 1$ ($\lambda_2 = \lambda_1 + 1$). This completes the proof. ■

3 Some remarks, extensions and the like

By the definition of LBS used in previous sections we have been unable to expose those invariant EDM's, if such exist, supported on $\mathbb{R}$. For delineating these, we have to use different types of LBS. For instance, consider a continuous random variable X with a p.d.f. $f(x)$. Then $xf(x)/E(X)$ is not a p.d.f. when S_f, the support of f, contains negative values. As opposed to this, $x^2 f(x)/E(X^2)$ is always a p.d.f. regardless of the form S_f. A similar situation occurs for higher moments too. Accordingly, for $k \in \mathbb{N}$, we shall call $x^k f(x)/E(X^k)$,when is well defined, a k-th moment LBS distribution. Here, x^k is a special case of what is called a weight function $w(x)$ used in general framework of LBS. Obviously, if a continuous EDM supported on $[0, a)$, for some $a \leq \infty$, is invariant under first order moment LBS, then it is invariant under any other k-th order moment LBS. Consequently, the set of continuous EDM's which are invariant under second order moment LBS contains that of continuous EDM's invariant under first moment LBS. The latter situation is not entirely parallel for the discrete case. Indeed, if a discrete EDM is supported, say, on $\{0, 1, ..., a\}$, for some $a \in \mathbb{N}$, then its invariance under first moment LBS does not imply its invariance under second moment LBS. For the latter discrete case we have to use $w(x) = x(x-1)$ as a weight function rather than $w(x) = x^2$; or more generally, to use $w(x) = \prod_{i=1}^{k}(x - (i-1))$ instead of x^k. Accordingly, if such a discrete EDM is invariant under a first (factorial) moment LBS then it is invariant, up to a translation under any k-th factorial moment LBS. In a manner analogous to that used in Section 2, we shall derive necessary and sufficient conditions for an EDM to be invariant under both, second moment and second factorial moment LBS. These conditions are expressed in terms of differential equations in V, the VF of the basis F_1.

Proposition 2 *Let $\mathbb{G}$ be an EDM as defined in (5)-(7) and F_1 be a basis of $\mathbb{G}$ with VF (V, Ω). Then*

i) $\mathbb{G}$ is invariant under second moment LBS if and only if the VF (V, Ω) of its basis F_1 is a solution of the differential equation

$$V\left(\frac{V' + 2m\lambda_2}{V + \lambda_2 m^2}\right) = (\lambda_1 - \lambda_2)m + b, \quad m \in \Omega, \tag{19}$$

where $\lambda_1, \lambda_2 \in \Lambda$ and b is some constant.

ii) $\mathbb{G}$ is invariant, up to a translation, under second factorial moment LBS if and only if the VF (V, Ω) of its basis F_1 solves the differential equation

$$V\left(\frac{V' + 2\lambda_2 m - 1}{V + \lambda_2 m^2 - m}\right) = (\lambda_1 - \lambda_2)m + b, \quad m \in \Omega. \tag{20}$$

Proof. *i*) We follow similar lines as in the proof of Proposition 1 and construct the set $\mathbb{G}_2$ of second moment LBS NEF's related to the EDM $\mathbb{G}$. Fix $\lambda \in \Lambda$ and define the measure $\pi_\lambda(dx) = x^2 \nu_\lambda(dx)$. Following (6) and (7) and absorbing $\ln \lambda$ into π_λ, the cumulant transform of π_λ and the NEF generated by π_λ are given, respectively, by

$$\kappa_{\pi_\lambda}(\theta) = \lambda\kappa(\theta) + \ln\left[\kappa''(\theta) + \lambda\left(\kappa'(\theta)\right)^2\right] \tag{21}$$

and

$$F_{2,\lambda} = F_{2,\lambda}(\pi_\lambda) = \left\{P(\theta, \pi_\lambda(dx)) = \exp\left\{\theta x - \kappa_{\pi_\lambda}(\theta)\right\}\pi_\lambda(dx),\ \theta \in \Theta\right\}. \tag{22}$$

The set $\mathbb{G}_2$, the mean function and VF of $F_{2,\lambda}$ expressed in terms of V and $m \in \Omega$ are

$$\mathbb{G}_2 = \left\{F_{2,\lambda} : \lambda \in \Lambda\right\},$$

$$m_{F_{2,\lambda}} = \lambda m + \frac{V'(m)V(m) + 2\lambda m V(m)}{V(m) + \lambda m^2}, m \in \Omega, \tag{23}$$

and

$$V_{F_{2,\lambda}} = V\left\{\lambda + \left[V\left(\frac{V' + 2\lambda m}{V + \lambda m^2}\right)\right]'\right\}, m \in \Omega. \tag{24}$$

Now, if $\mathbb{G}$ is invariant under second moment LBS then $\mathbb{G} = \mathbb{G}_2$, in which case there exist $\lambda_i \in \Lambda, i = 1, 2$, such that $F_{\lambda_1} = F_{2,\lambda_2}$ or, equivalently, in terms of the associated VF's, that $V_{F_{\lambda_1}}(m) = V_{F_{2,\lambda_2}}(m), m \in \Omega$. By employing (7) and (24) in the latter relation and rearranging terms, (20) follows. By assuming that (20) holds for $\lambda_1, \lambda_2 \in \Lambda$, the reverse implication follows easily.

ii) This statement can be proved analogously. Indeed, the set $\tilde{\mathbb{G}}_2$ of second factorial moment LBS NEF's stemming from $\mathbb{G}$ is defined as follows. For fixed $\lambda \in \Lambda$, the cumulant transform of the measure $\tilde{\pi}_\lambda(dx) \doteq x(x-1)\nu_\lambda(dx)$ and its related NEF are

$$\kappa_{\tilde{\pi}_\lambda}(\theta) = \lambda\kappa(\theta) + \ln\left[\kappa''(\theta) + \lambda\left(\kappa'(\theta)\right)^2 - \kappa'(\theta)\right]$$

and

$$\tilde{F}_{2,\lambda} = \tilde{F}_{2,\lambda}(\pi_\lambda) = \left\{P(\theta, \tilde{\pi}_\lambda(dx)) = \exp\left\{\theta x - \kappa_{\tilde{\pi}_\lambda}(\theta)\right\}\pi_\lambda(dx),\ \theta \in \Theta\right\},$$

so that the set $\tilde{\mathbb{G}}_2$, the mean function and VF of $F_{2,\lambda}$ are

$$\tilde{\mathbb{G}}_2 = \left\{\tilde{F}_{2,\lambda} : \lambda \in \Lambda\right\},$$

$$m_{\tilde{F}_{2,\lambda}} = \lambda m + V(m)\left(\frac{V'(m) + 2\lambda m - 1}{V(m) + \lambda m^2 - m}\right), m \in \Omega,$$

$$V_{\tilde{F}_{2,\lambda}} = V\left\{\lambda + \left[V\left(\frac{V' + 2\lambda m - 1}{V + \lambda m^2 - m}\right)\right]'\right\}, m \in \Omega.$$

As in (13) of Section 2, the group of translations acting on $\tilde{\mathbb{G}}_2$ generates the following set of NEF's

$$f_\alpha(\tilde{\mathbb{G}}_2) = \left\{f_\alpha(\tilde{F}_{2,\lambda}) : \lambda \in \Lambda\right\},$$

where, in analogy with (14) and (15), the mean function and mean domain of $f_\alpha(\tilde{F}_{2,\lambda})$ are

$$m_{f_\alpha\left(\tilde{F}_{2,\lambda}\right)} = m_{\tilde{F}_{2,\lambda}} + \alpha,\ \Omega_{f_\alpha\left(\tilde{F}_{2,\lambda}\right)} = \Omega_{\tilde{F}_{2,\lambda}} + \alpha,$$

where its VF expressed in terms of $m \in M$ is

$$V_{f_\alpha(\tilde{F}_\lambda)}(m) = V_{\tilde{F}_\lambda}(m) = V\left\{\lambda + \left[V\left(\frac{V' + 2\lambda m - 1}{V + \lambda m^2 - m}\right)\right]'\right\}, m \in \Omega\ .$$

If $\tilde{\mathbb{G}}_2$ is invariant, up to a translation, under second factorial moment LBS, i.e., if $\mathbb{G} = f_\alpha(\tilde{\mathbb{G}}_2)$, then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $F_{\lambda_1} = f_\alpha(F_{2,\lambda_2})$ or that $\lambda_1 V(m) = V_{f_\alpha(\tilde{F}_{\lambda_2})}(m), m \in \Omega$. The rest of the proof can now be continued as part (i). ■

In contrast to Proposition 1 where the differential equation (18) has been entirely solved, yielding the four EDM's in Tables 1 and 2, the situation in Proposition 2 is different, posing difficulties of two kinds. One is related to the general solutions of (19) and (20), and the other to the question whether the resulting solutions are VF's of NEF's. Indeed, both differential equations (19) and (20) have infinitely many two-parametric solutions. In principle, once $V(m)$ and $V'(m)$ are fixed at some point $m = \gamma \neq 0$, one can find series of functions, represented by iterative relations, that converge to the solutions of (19) and (20). Special solutions, however, of the two differential equations do exist. For instance, the gamma EDM presented by a basis VF $V(m) = m^2$ solves (19) with $\lambda_1 = \lambda_2 + 2$ and $b = 0$. The binomial, negative binomial and Poisson EDM's with VF's bases as described in Tables 1 and 2 solve (20) with $\lambda_1 = \lambda_2 - 2, \alpha = -b = -2$; $\lambda_1 = \lambda_2 + 2, \alpha = -b = -2$; and $\lambda_1 = \lambda_2 + 2, \alpha = -b = -2$, respectively.

One other solution of (19), for example, is $V(m) = \left(m^4 + C\right)^{1/2}$ with the specific values $(\lambda_1, \lambda_2) = (3, 1)$. If $C = 0$, then this reduces to the gamma case. Otherwise, since Ω is the largest open interval on which V is positive real analytic, it necessarily follows that $C > 0$ and $\Omega = \mathbb{R}$, in which case the convex-support of the corresponding measure, if such exists, is the whole real line. The question then whether $(\left(m^4 + C\right)^{1/2}, \mathbb{R})$ is a VF of some NEF is not a simple one. Theoretically, one can invert such a pair, obtain the corresponding pair $(\kappa(\theta), \Theta)$ and then determine whether $\kappa(\theta)$ is a cumulant transform of some positive measure on $\mathbb{R}$. Practically, however, this procedure turns out to be in most cases rather cumbersome. There are though some easier ways for such a determination (see, for example, Kokonendji and Seshadri (1994,1996), Bar-Lev, Bshouty and Enis (1992) or Letac (1992), Letac and Mora (1990) and the references cited therein). Whether there are additional VF's solutions of (19), other than the gamma case, is left as an open problem. An availability of a rich and accessible 'dictionary' of VF's items may help solving this problem, as well as other characterization problems related to EDM's.

In this note, however, we have used a unified approach for classifying EDM's invariant under various types of LBS. This approach, while demonstrating the applicability of VF's techniques, has the appealing property of allowing one to reduce the problem of invariance into a problem of solving a differential equation in terms of the corresponding VF's. Invariance of EDM's under any other k-th moment or factorial moment LBS can be derived accordingly and expressed in terms of appropriate differential equations. The solutions of these equations are obviously involved with an increasing complexity as k increases.

References

[1] Bar Lev, S.K., Bshouty, D. and Enis, P. (1992). "On polynomial variance functions." *Probability Theory and Related Fields*, 94, 69-82.

[2] El Barmi, H. (2002). "A note on estimating a non-increasing density in the presence of selection bias." *Journal of Statistical Planning and Inference,* 107, 353-364.

[3] Jorgensen, B. (1987). "Exponential dispersion models" (with discussion), *Journal of the Royal Statistical Society,* Ser. B, 49, 127-162.

[4] Jorgensen, B. (1997). *The Theory of Exponential dispersion models*, Monographs on Statistics and probability, Vol. 76, Chapman and Hall.

[5] Kokonendji, C.C. and Seshadri, V. (1994). "The Lindsay transform of natural exponential families." *Canadian Journal of Statistic*s, 22, 259-272.

[6] Kokonendji, C.C. and Seshadri, V. (1996). "On the determinant of the second derivative of Laplace transform." *Annals of Statistics*, 24, 1813-1827.

[7] Letac, G. and Mora, M. (1990). "Natural real exponential families with cubic variance functions", *The Annals of Statistics*, 18, 1-37.

[8] Lingappaiah, G.S. (1988). "Generalized length biased distributions." *Appl. Mat.*, 33, 354-361.

[9] Morris, C.N. (1982). "Natural exponential families with quadratic variance functions", *The Annals of Statistics*, 10, 65-80.

[10] Oluyede, B.O. (1999). "On inequalities and selection of experiments for length biased distributions." *Probability in the Engineering and Informational Sciences*, 13, 169-185.

[11] Oluyede, B.O. (2000). "On some length biased inequalities for reliability measure." *J. Inequal. App.*, 5, 447-466.

[12] Oluyede, B.O. (2003). "Inequalities and bounds for kernel length-biased density estimation." *App. Math. Comput.*, 135, 541-551.

[13] Sankaran, P.G. and Unnikrishnan, N.N. (1993). "On form-invariant length biased models from Pearson family." *Journal of the Indian Statistical Association*, 31, 85-89.

[14] Sen, A. and Khattree, R. (1996). "Length biased distributions, equilibrium distribution and characterizations of probability laws."

[15] Tijms, H.C. (1994). *Stochastic models; an algorithmic approach*, John Wiley, New York.

[16] van Es, B., Klaassen, C.A.J., Oudshoorn, K. (2000). "Survival analysis under cross-sectional sampling: length bias and multiplicative censoring." *Journal of Statistical Planning and Inference*, 91, 295-312.