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Inventory games

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Abstract

Inventory management studies how a single firm can minimize the average cost per time unit of its inventory. In this paper we extend this analysis to situations where a collective of firms minimizes its joint inventory cost by means of cooperation. Depending on the information revealed by the individual firms, we analyze related cooperative TU games and focus on proportional division mechanisms to share the joint cost.

Keywords: Inventory management; Information; Cooperative games; Proportional division

1. Introduction

Generally speaking, shops or firms trade various types of goods, and to keep their service to their customers at a high level they aim at meeting the demand for all goods on time. To attain this goal, shops may keep inventories in a private warehouse. These inventories bring costs along with them. To keep these costs low, a good management of the inventories is needed. The management of inventory, or inventory management, started at the beginning of this century when manufacturing industries and engineering grew rapidly. A starting paper on mathematical models of inventory management was Harris (1915). Since then, many books on this subject have been published (for example, Hadley and Whitin, 1963; Hax and Candea, 1984; Tersine, 1994). The main objective of inventory management is to minimize the average cost per time unit (in the long run) incurred by the inventory system, while guaranteeing a pre-specified minimal level of service.

In this paper we study an extremely simple model of inventory systems. In this so-called basic inventory model we begin with a single firm that stores a single good. Demand for this good is continuous over time and occurs at a constant rate. The lead-time of the good is deterministic, and without loss of generality assumed to be

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zero. The related inventory cost is assumed to be time-invariant and there are no constraints on the quantity ordered and stored. The inventory cost consists of two parts: the ordering cost and the holding cost. The ordering cost, the cost one has to pay each time an order is placed, is fixed, i.e., it is independent of the quantity ordered. The holding cost reflects the cost per time unit of storage of the good in a private warehouse and is assumed to be linear in the quantity stored. As a decision criterion we use the average inventory cost per time unit, so we have to decide upon an ordering policy that minimizes this cost. This decision criterion is well known in the literature and in text books (see, for example, Winston, 1994).

New aspects and features come in when we consider situations with several firms or shops and a single good. One can think for example of some franchise operators restricted to a single good. Each of these firms has its own private demand and its own private storage possibilities for the good. There is a single supplier where all firms place their orders, concerning the good, at the same ordering cost. By means of placing their orders simultaneously, these firms can reduce their total cost [1] compared to the total cost in the initial situation in which they all order separately, because of the lower total number of orders. An interesting question is what the optimal ordering policy for a group of agents will be. Here, another aspect enters. When coordination leads to joint cost savings, how should these savings be allocated among the firms? This paper provides answers to both questions. In particular, the last question is addressed by means of cooperative game theory arguments.

To provide adequate answers to the two research questions above, we have to specify the exact informational structure we want to consider. Both the constant rates of the demand and the holding cost are assumed to be private information; only the ordering cost, which is the same for all firms, is public information. To coordinate the ordering policy of the cooperating firms, some revelation of information is needed. [2] We will assume first that the only information of a firm that is truthfully revealed to the other firms is its average number of orders per time unit in case this firm would act on its own in an optimal way. In fact, we will show that this is the only information one needs to determine an optimal joint ordering policy. If all information would have been public, we would arrive at the same optimal policy.

In Section 3 we consider this first model and corresponding cooperative inventory cost games. We propose an allocation rule in which the ordering cost is divided proportionally to the square of the individual ordering cost. This cost only depends on the ordering cost and the individual average number of orders per time unit, which is public information. The holding cost component is included implicitly since this cost can only be computed using private information. It turns out that the proportional rule leads to a core allocation of the corresponding game that even can be sustained as a population monotonic allocation scheme (Sprumont, 1990), which is a core allocation supported by a monotonic scheme of core allocations for all subgames. Furthermore, we give an axiomatic characterization of this rule on the class of ordering cost games, i.e., games where we forget about the private holding cost and only consider the ordering cost.

Subsequently, we compare the above results with the results in case all information on demands and holding cost is revealed within a cooperating group of firms. No strengthening can be obtained, so there seems no real need for the disclosure of private information if one only focuses on savings with respect to ordering cost. However, if we have full disclosure of information, no limits to storage capacities, no transport cost and deterministic transport times, one could also consider coordination with regard to holding cost. Stocks will be stored in the warehouse of the firm with lowest

[1] When we write 'cost' we mean average (inventory) cost per time unit.

[2] We keep the amount of revealed information between the firms as low as possible since the firms may be competitors on the consumer market. To establish meaningful cooperation without full disclosure of information some kind of intermediary will be needed.

holding cost. This kind of situations and the corresponding games are considered in Section 4. We show that these games are not necessarily concave but they are permutationally concave.

Section 2 starts with an analysis of the optimal ordering policy in a multi-firm situation with a single good and a single supplier of the good. We already described Sections 3 and 4. In Section 5 we provide an example that illustrates all the games and allocation rules of Sections 3 and 4. Section 6 concludes.

2. The basic inventory model

In the basic inventory model, a single firm has to meet the demand for a single good on time. To attain this, the firm keeps stock on hand. We assume that the firm owns or rents a warehouse, which has an unlimited capacity, and there is a single supplier who delivers all orders. The demand for the good is assumed to be known, constant and equals d units per time unit, $d \geqslant 0$. The firm is not allowed to run out of stock. The lead-time, the time between placement of an order and delivery of that order, is assumed to be deterministic and constant, and without loss of generality equal to zero. [3]

There are two types of cost involved. First, there is the ordering cost. We assume that this cost does not depend on the quantity ordered. It includes, for example, telephone charges, delivery costs and the labour cost incurred in processing the order. Each time the firm places an order to replenish its stock, it pays a fixed ordering cost $a > 0$. Second, there is the holding cost; the cost of storing goods. This cost includes insurance, warehouse rental if the warehouse is not owned by the firm, depreciation if the warehouse is owned by the firm, light, maintenance and so on. The cost of carrying one good in stock for one time unit is assumed to be constant and is denoted by the constant $h > 0$.

[3] Since the lead-time of an order only determines the actual time of delivery of an order and does not influence the optimal amount of the good to order, this is not a restrictive assumption.

Since the demand is deterministic and constant and the lead-time equals zero time units, the firm that wants to minimize its average cost per time unit, will order the same quantity each time an order is placed. Also, the size of the on hand inventory when an order is issued, will always be zero, to minimize the average holding cost, since the order is delivered immediately. The firm wants to determine how many orders it should place per time unit and how much to order such that its goal, to minimize the average cost per time unit, is attained. The following analysis follows the lines set out by Hadley and Whitin (1963).

Denote by Q the quantity ordered each time the firm places an order to replenish the stock. The time between two successive placements of orders is thus Q/d time units. A cycle will be defined as an interval of time of length Q/d starting at that point in time when an order is placed. During each cycle, the behaviour of the inventory system is exactly the same. By m we denote the number of orders placed per time unit, that is, $m = d/Q$.

Let us take a look at a single time period of unit length. In this period, the demand for the good equals d units. The firm wants to meet all demand on time, so if the quantity ordered equals Q, then the number of orders placed per time unit is d/Q on the average and the average ordering cost per time unit equals ad/Q. Since an order is placed when the size of the stock equals zero, the average size of the inventory will be $\frac{1}{2}(Q+0) = Q/2$. Then the average holding cost per time unit will be $hQ/2$. The average cost of the firm per time unit, $AC(Q)$, equals the sum of the average ordering and holding cost per time unit:

$$AC(Q) = a\frac{d}{Q} + h\frac{Q}{2}.$$

The minimal cost is obtained in Q^* with $AC' \times (Q^*) = 0$ and $AC''(Q^*) > 0$. It follows that $Q^* = \sqrt{2ad/h}$. The optimal cycle length is $Q^*/d = \sqrt{2a/(dh)}$, the optimal number of orders placed per time unit, m^*, equals $m^* = d/Q^* = \sqrt{dh/(2a)}$ and the minimal average cost per time unit is $AC(Q^*) = \sqrt{2adh} = 2am^*$. Note that in the optimum both the holding and the ordering cost per time unit equal am^*.

In an n-firm inventory situation, there is a set $N = \{1, 2, \dots, n\}$ of firms. We denote the demand, holding cost and order size of firm $i \in N$ by $d_i \geqslant 0$, $h_i > 0$ and $Q_i \geqslant 0$, respectively. There is a single good and each firm has its own private storehouse. When these firms cooperate, they minimize their total cost by placing their orders together as one big order. So, in the optimum, cycle lengths are equal for all firms. Why? Suppose that we have a situation with two firms and unequal cycle lengths, as in Fig. 1. We consider the time interval from t_1 up to and including t_4. Firm Long is the firm with the largest cycle length. Its cycle length equals $t_3 - t_1$. Firm Short has the smallest cycle length, namely $t_2 - t_1$. If both firms decide to cooperate then we see from the figure that they place a joint order at t_1 and separate orders at t_2, t_3 and t_4. This makes a total of four orders. Firm Long can reduce the total cost of the cooperating firms by reducing its cycle length to $t_2 - t_1$, the cycle length of firm Short. If we compare Fig. 1 and Fig. 2 we see that the reduction of the cycle length reduces

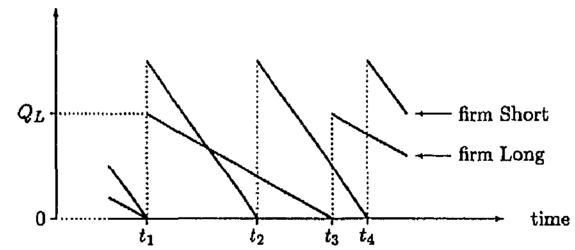

Fig. 1. Unequal cycle lengths.

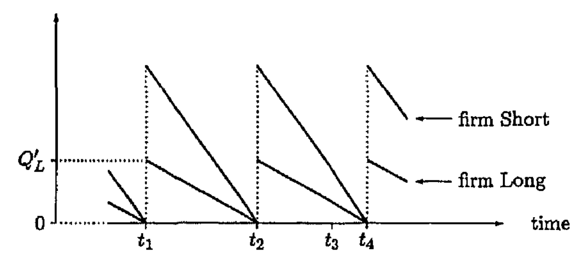

Fig. 2. Equal cycle lengths.

the order-size of firm Long from Q_L to Q'_L since it is optimal to issue an order when the inventory level equals zero. Consequently, the average inventory level goes down from $Q_L/2$ to $Q'_L/2$ and the holding cost of firm Long decreases. The reduction of the cycle length also implies that the firms place joint orders at times t_1, t_2 and t_4 and no order is placed at time t_3. The total number of orders has fallen from four to three, so, the ordering cost will decrease.

From the explanation above, it follows that if the total cost is minimized then the cycle lengths of all firms are equal. The cycle length of firm $i \in N$ equals Q_i/d_i, so it should hold that $Q_i/d_i = Q_j/d_j$ for all $i, j \in N$. If we take $j = 1$ then we can express Q_i as a function of Q_1:

$$Q_i = \frac{d_i}{d_1} Q_1. \tag{1}$$

The average cost per time unit for the firms in N consists of ordering and holding cost. One order is placed per cycle, so the average ordering cost per time unit equals ad_1/Q_1. Since each firm stores its goods in its own storehouse, the holding cost will be the sum of the individual holding cost. Thus, the average cost per time unit for the firms in N equals

$$a\frac{d_1}{Q_1} + \sum_{i \in N} h_i \frac{Q_i}{2}.$$

Compare this to the individual average cost per time unit $ad_i/Q_i + h_i Q_i/2$. To express this cost as a function of Q_1 only, we substitute (1) and get

$$a\frac{d_1}{Q_1} + \frac{Q_1}{2d_1} \sum_{i \in N} h_i d_i.$$

Minimizing this with respect to Q_1 gives the following results. The optimal order size $\widehat{Q}_i$ for firm i is

$$\widehat{Q}_i = \sqrt{\frac{2ad_i^2}{\sum_{j \in N} d_j h_j}}.$$

The optimal cycle length equals

$$\frac{\widehat{Q}_i}{d_i} = \sqrt{\frac{2a}{\sum_{j \in N} d_j h_j}}$$

for all $i \in N$. The optimal number of orders the firms should place, m_N, is

$$m_N = \frac{d_i}{\widehat{Q}_i} = \sqrt{\frac{\sum_{j\in N} d_j h_j}{2a}} = \sqrt{\sum_{j\in N} m_j^2}.$$

Here, $m_i = d_i/Q_i^* = \sqrt{d_i h_i/(2a)}$ denotes the number of orders that minimizes the cost of firm i. The minimal average cost equals $2am_N$. As in the one-firm situation, the ordering and holding cost both equal am_N in the optimum. Also notice that the minimal cost only depends on a, which is assumed to be public information, and m_N, which only depends on all m_i. So, to calculate the minimal cost, it suffices for each firm to reveal its number m_i, the optimal number of orders if the firm would operate on its own. The firms do not have to reveal their private demand or holding cost; we do not need full disclosure of information. However, the amount of information disclosed may influence the possible allocation mechanism. In Section 3, where each firm only reveals its individual optimal number of orders m_i, we propose an allocation mechanism that allocates the total cost proportionally to the square of the individual cost. In Section 4, where we have full disclosure of information, we could use the same allocation mechanism as in Section 3. But now we have more information available. Each firm reveals its demand and holding cost, so we might as well design an allocation mechanism that depends on this information. We will propose a mechanism that allocates the total cost proportionally to the demands.

3. Ordering cost

In this section we consider situations in which each firm only reveals m_i, its individual optimal number of orders per unit of time. Its private information thus consists of d_i, h_i and Q_i^*.

We have seen that when all firms work together, the optimal amount to order equals $\widehat{Q}_i = d_i/m_N$ for firm $i \in N$. This amount is smaller than the individual optimal amount to order, $Q_i^* = d_i/m_i$, since $m_N = \sqrt{\sum_{j\in N} m_j^2} \geqslant m_i$ for all $i \in N$. So, the average inventory level will be lower for each firm: $\widehat{Q}_i/2 \leqslant Q_i^*/2$. Each firm saves on holding cost. Since the holding cost of each firm is private information, we cannot consider how to divide total holding cost among the firms. Therefore, we assume that each firm pays its own holding cost.

The optimal order size $\widehat{Q}_i = d_i/m_N$ of firm i is private information because of d_i. To be able to place a joint order without revealing any private information, there is an intermediary who will place all orders. Each firm $i \in N$ tells this intermediary its optimal order size $\widehat{Q}_i$ and the intermediary will place an order of size $\sum_{i\in N} \widehat{Q}_i$. The numbers m_i are known by the intermediary but not by the supplier. Thus the supplier only knows $\sum_{i\in N} \widehat{Q}_i$. Furthermore, the intermediary will not pass information about one firm to another firm thus ensuring that all private information remains private.

We are only interested in allocations of the optimal ordering cost am_N. In short, an ordering cost situation is described by the 3-tuple $\langle N, a, \{m_i\}_{i\in N}\rangle$. If a coalition S of firms cooperates then their optimal ordering cost equals

$$a\sqrt{\sum_{i\in S} m_i^2}. \tag{2}$$

Consequently, one can define the corresponding *ordering cost game* (N, c_o) as follows. For all coalitions $S \subset N$, the cost $c_o(S)$ equals the cost in (2) and $c_o(\emptyset) = 0$. We will consider some properties of ordering cost games. A cost game (N, c) is *concave* if for all $i \in N$ and for all $S \subset T \subset N \setminus \{i\}$ we have that $c(S \cup \{i\}) - c(S) \geqslant c(T \cup \{i\}) - c(T)$ and it is *monotone* if for all $S \subset T \subset N$ it holds that $c(S) \leqslant c(T)$.

Proposition 1. *Let $\langle N, a, \{m_i\}_{i\in N}\rangle$ be an ordering cost situation and let (N, c_o) be the corresponding ordering cost game. Then the game (N, c_o) is concave and monotone.*

Proof. Let (N, c_o) be the corresponding ordering cost game. Since $\sum_{i\in S} m_i^2$ is increasing in the number of elements in S and since $\sqrt{x}$ is a monotonically increasing and concave function, it follows immediately that (N, c_o) is monotone and concave. □

One of the main issues treated in cooperative game theory is how to divide the benefits from cooperation if coalition N has formed. One way to share these benefits is according to an allocation in the core. The *core* of a cost game (N, c) is the set

$$C(c) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leqslant c(S) \text{ for all } S \subset N, S \neq \emptyset \right\}.$$

When an element of the core $x \in C(c)$ is proposed as a distribution of the total cost $c(N)$ where firm i has to pay x_i, then a coalition S of firms has to pay at most its own cost since $\sum_{i \in S} x_i \leqslant c(S)$. So, no coalition has an incentive to leave the grand coalition N. A game is *balanced* if it has a nonempty core (see Bondareva, 1963; Shapley, 1967), and it is called *totally balanced* if each subgame $(S, c_{|S})$ is balanced, where $c_{|S}(T) := c(T)$ for all $T \subset S$. Since ordering cost games are concave, it follows from Shapley (1971) that these games are totally balanced.

Another property of ordering cost games is that a nonnegative multiple of such a game is another ordering cost game. Take a nonnegative number λ, then for all coalitions S of firms in N it holds that

$$\lambda c_o(S) = a \sqrt{\sum_{i \in S} (\lambda m_i)^2}$$

and this describes the value of coalition S in an ordering cost game corresponding to the ordering cost situation $\langle N, a, \{\lambda m_i\}_{i \in N} \rangle$. Such a situation arises for example when all individual demands and holding costs increase by the factor λ. Hence, $(N, \lambda c_o)$ is an ordering cost game. Nevertheless, the sum of two ordering cost games (N, c_o) and (N, c'_o), $(N, c_o + c'_o)$, does not have to be another ordering cost game. For example, take $N = \{1, 2\}$, $a = 2$, $m_1 = 1$, $m_2 = 2$, $a' = 5$, $m'_1 = 2$, $m'_2 = 3$. Let the game (N, c_o) correspond to the ordering cost situation $\langle N, a, \{m_i\}_{i \in N} \rangle$ and (N, c'_o) to $\langle N, a', \{m'_i\}_{i \in N} \rangle$. Then $c_o(\{1\}) = 2$, $c_o(\{2\}) = 4$, $c_o(N) = 2\sqrt{5}$, $c'_o(\{1\}) = 10$, $c'_o(\{2\}) = 15$ and $c'_o(N) = 5\sqrt{13}$. If we sum these games we get $(c_o + c'_o) \times (\{1\}) = 12$, $(c_o + c'_o)(\{2\}) = 19$ and $(c_o + c'_o)(N) = 2\sqrt{5} + 5\sqrt{13}$. Suppose that we can find values for a'', m''_1 and m''_2 such that $\langle N, a'', \{m''_i\}_{i \in N} \rangle$ is the ordering cost situation corresponding to the game $(N, c_o + c'_o)$. Then the following equations should hold:

$$a'' m''_1 = (c_o + c'_o)(\{1\}) = 12, \tag{3}$$

$$a'' m''_2 = (c_o + c'_o)(\{2\}) = 19, \tag{4}$$

$$a'' \sqrt{(m''_1)^2 + (m''_2)^2} = (c_o + c'_o)(N) = 2\sqrt{5} + 5\sqrt{13}. \tag{5}$$

From (3) it follows that $m''_1 = 12/a''$ and from (4) $m''_2 = 19/a''$. When we substitute this in (5) we get

$$a'' \sqrt{(m''_1)^2 + (m''_2)^2} = a'' \sqrt{\frac{144}{(a'')^2} + \frac{361}{(a'')^2}} = a'' \frac{1}{a''} \sqrt{144 + 361} = \sqrt{505}$$

which is not equal to $2\sqrt{5} + 5\sqrt{13}$ (though very close). We conclude that $(N, c_o + c'_o)$ is not an ordering cost game.

Ordering cost games are a special kind of production games, as introduced by Shapley and Shubik (1967). A production game is a cooperative game with player set N and the value of a coalition S of players equals $g(b(S))$ with g a (concave) production function and $b(S) = \sum_{i \in S} b(\{i\})$ the resources owned by coalition S. To specify an ordering cost game we set $c_o(S) = g(b(S))$ with $g(x) = a\sqrt{x}$ and $b(S) = \sum_{i \in S} m_i^2$. If each unit of production costs \$1 then $g(b(S))$ not only denotes how much is produced by coalition S but it also denotes the cost of these produced goods. The amount of resources held by firm i equals $b(\{i\}) = m_i^2$. An interesting solution concept for these games is the *proportional rule*. We will define the proportional rule $\pi(c_o)$ as the rule that divides the total cost $c_o(N)$ proportionally to the individual resources. This implies that firm $i \in N$ has to pay

$$\pi_i(c_o) = \frac{b(\{i\})}{\sum_{j \in N} b(\{j\})} c_o(N) = \frac{m_i^2}{\sum_{j \in N} m_j^2} c_o(N) = \frac{a m_i^2}{\sqrt{\sum_{j \in N} m_j^2}} \tag{6}$$

where the last equality follows from (2) for $S = N$. Another interpretation of this proportional rule follows from the fact that $c_o(\{i\}) = am_i$ for all firms i. If we decide to divide the total cost $c_o(N)$ proportionally to the square of the individual cost then firm i has to pay

$$\frac{c_o^2(\{i\})}{\sum_{j\in N} c_o^2(\{j\})} c_o(N) = \frac{a^2 m_i^2}{\sum_{j\in N} a^2 m_j^2} c_o(N) = \frac{m_i^2}{\sum_{j\in N} m_j^2} c_o(N)$$

so we end up with the same proportional rule. [4] This rule has some nice properties.

First, for all ordering cost games (N, c_o) it holds that $\pi(c_o)$ is an element of the core $C(c_o)$. This is easy to see. From (6) it follows that $\sum_{i\in N} \pi_i \times (c_o) = c_o(N)$ and for all nonempty coalitions S of N it holds that

$$\sum_{i\in S} \pi_i(c_o) = \sum_{i\in S} \frac{am_i^2}{\sqrt{\sum_{j\in N} m_j^2}} \leqslant \sum_{i\in S} \frac{am_i^2}{\sqrt{\sum_{j\in S} m_j^2}} = a\sqrt{\sum_{j\in S} m_j^2} = c_o(S).$$

Second, this proportional rule can be reached through a *population monotonic allocation scheme*, in short, a PMAS. These schemes were introduced in Sprumont (1990) and defined as follows. A vector $y = \{y_{iS}\}$, $i \in S$, $S \subset N$, $S \neq \emptyset$, is a population monotonic allocation scheme of the cost game (N, c) if and only if it satisfies the following two conditions. Firstly, it should hold that $\sum_{i\in S} y_{iS} = c(S)$ for all nonempty coalitions S of N. Secondly, for all nonempty coalitions S and T of N and for all $i \in S$ it should hold that $S \subset T$ implies $y_{iS} \geqslant y_{iT}$. Also from Sprumont (1990) it follows that, since each ordering cost game (N, c_o) is concave and since $\pi(c_o) \in C(c_o)$, there exists a PMAS $y = \{y_{iS}\}$, $i \in S$, $S \subset N$, $S \neq \emptyset$ of the game (N, c_o) such that $y_{iN} = \pi_i(c_o)$ for all $i \in N$. Define for all $i \in S$, $S \subset N$ and $S \neq \emptyset$,

$$y_{iS} = \frac{am_i^2}{\sqrt{\sum_{j\in S} m_j^2}}.$$

Then for all $S \subset N$, $S \neq \emptyset$,

$$\sum_{i\in S} y_{iS} = \sum_{i\in S} \frac{am_i^2}{\sqrt{\sum_{j\in S} m_j^2}} = a\sqrt{\sum_{j\in S} m_j^2} = c_o(S)$$

and for all $S, U \subset N$, $S, U \neq \emptyset$, such that $S \subset U$ and for all $i \in S$,

$$y_{iS} = \frac{am_i^2}{\sqrt{\sum_{j\in S} m_j^2}} \geqslant \frac{am_i^2}{\sqrt{\sum_{j\in U} m_j^2}} = y_{iU}.$$

Finally, we see that $y_{iN} = \pi_i(c_o)$ for all $i \in N$. So, the rule $\pi(c_0)$ can be reached through the PMAS y.

We will now introduce three properties for solution rules on the class of ordering cost games. Let f be a solution rule on the class of ordering cost games. Then $f_i(c_o) \in \mathbb{R}$ denotes the cost allocated to player $i \in N$ according to this rule in the game c_o and $f(c_o) = (f_i(c_o))_{i\in N} \in \mathbb{R}^N$. Let (N, c_o) and $(N, \bar{c}_o)$ be ordering cost games. The rule f satisfies *efficiency* if $\sum_{i\in N} f_i(c_o) = c_o(N)$. It satisfies *symmetry* if $f_i(c_o) = f_j(c_o)$ when the players i and j are symmetric, that is, when $c_o(S \cup \{i\}) = c_o(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$. Finally, the rule f satisfies *monotonicity* if for all $i \in N$ such that $c_o(\{i\}) \geqslant \bar{c}_o(\{i\})$ it holds that $c_o(N) f_i(c_o) \geqslant \bar{c}_o \times (N) f_i(\bar{c}_o)$.

This monotonicity property, which resembles strong monotonicity as defined by Young (1985), starts from the following assumption: "if $c_o(\{i\}) \geqslant \bar{c}_o(\{i\})$ and $c_o(N) = \bar{c}_o(N)$ then $f_i(c_o) \geqslant f_i(\bar{c}_o)$". That is, if we have two inventory situations with the same total cost to share and a player generates more cost on his own in one situation than in the other, then he should pay more in the former than in the latter situation. This assumption is equivalent to: if $c_o(\{i\}) \geqslant \bar{c}_o(\{i\})$ and $c_o(N) = \bar{c}_o(N)$ then $c_o(N) f_i(c_o) \geqslant \bar{c}_o(N) f_i(\bar{c}_o)$. However, we want to go even further. If $c_o(\{i\}) \geqslant \bar{c}_o(\{i\})$ and $c_o(N) \neq \bar{c}_o(N)$ then we demand that the above inequality also holds and so $f_i(c_o)$ has to be greater than

[4] We thank a referee for pointing out to us that the proportional rule coincides with the Aumann–Shapley value (Aumann and Shapley, 1974) in a slightly changed version of the game. This version is such that the firms are free to choose the level of demand that they wish to cover.

$f_i(\bar{c}_o)$ except for a correction with respect to the cost due to the other players.

Efficiency, symmetry and monotonicity characterize the proportional rule on the class of ordering cost games, as the next theorem shows.

Theorem 1. *There exists a unique rule on the class of ordering cost games satisfying efficiency, symmetry and monotonicity. It is the proportional rule.*

Proof. It is clear that the proportional rule satisfies efficiency, symmetry and monotonicity.

To show the converse, we take a rule f on the class of ordering cost games that satisfies efficiency, symmetry and monotonicity. Note that monotonicity implies that for all ordering cost games (N, c'_o) and (N, c''_o),

$$c'_o(\{i\}) = c''_o(\{i\}) \Rightarrow c'_o(N)f_i(c'_o) = c''_o(N)f_i(c''_o). \tag{7}$$

Define the ordering cost game (N, c^0_o) by $c^0_o(S) = 0$ for all $S \subset N$. By efficiency and symmetry [5] it follows that $f_i(c^0_o) = 0$ for all $i \in N$. Take an ordering cost game (N, c_o). If for some $i \in N$ it holds that $c_o(\{i\}) = 0$ then $c_o(\{i\}) = c^0_o(\{i\})$. When $c_o(N) = 0$ then $c_o = c^0_o$, because ordering cost games are monotone, and so $f_i(c_o) = 0$. Otherwise, when $c_o(N) > 0$ then it follows from (7) that $c_o(N) \times f_i(c_o) = c^0_o(N)f_i(c^0_o) = 0$ and thus $f_i(c_o) = 0$. We conclude that

$$\text{if } c_o(\{i\}) = 0 \text{ then } f_i(c_o) = 0. \tag{8}$$

Define the number $I(c_o)$ to be the number of players $i \in N$ with $c_o(\{i\}) > 0$. We show that $f_i(c_o) = \pi_i(c_o)$ for all $i \in N$ by induction on $I(c_o)$.

If $I(c_o) = 0$ then by (8), $f_i(c_o) = 0$ for all $i \in N$.

If $I(c_o) = 1$ then there is a single player $k \in N$ with $c_o(\{k\}) > 0$. For all $i \in N \setminus \{k\}$ $c_o(\{i\}) = 0$, so by (8), $f_i(c_o) = 0 = \pi_i(c_o)$. By efficiency it follows that $f_k(c_o) = c_o(N) - \sum_{i \neq k} f_i(c_o) = c_o(N) - \sum_{i \neq k} \pi_i(c_o) = \pi_k(c_o)$.

Assume now that $f(c_o) = \pi(c_o)$ for all ordering cost games (N, c_o) with $I(c_o) \leqslant I$, $I \leqslant n-1$. Consider an ordering cost game $(N, \bar{c}_o)$ corresponding to $\langle N, \bar{a}, \{\bar{m}_i\}_{i \in N}\rangle$ with $I(\bar{c}_o) = I+1$. Without loss of generality assume that $c_o(\{i\}) > 0$ for the players $i = 1, 2, \ldots, I+1$. Define the game (N, c_o) to be corresponding to $\langle N, a, \{m_i\}_{i \in N}\rangle$ where $a = \bar{a}$, $m_j = \bar{m}_j$ for all $j \in N \setminus \{I+1\}$ and $m_{I+1} = 0$. Then $I(c_o) = I$ and $f(c_o) = \pi(c_o)$. Since $c_o(\{k\}) = \bar{c}_o \times (\{k\}) > 0$ for all $k = 1, 2, \ldots, I$ it follows by (7) that $\bar{c}_o(N)f_k(\bar{c}_o) = c_o(N)f_k(c_o) = c_o(N)\pi_k(c_o)$. By (2) and (6),

$$\pi_k(c_o) = am_k^2 \Big/ \sqrt{\sum_{j \in N} m_j^2} = c_o^2(\{k\})/c_o(N),$$

so using induction,

$$\begin{aligned}\bar{c}_o(N)f_k(\bar{c}_o) &= c_o(N)\pi_k(c_o) = c_o(N)\frac{c_o^2(\{k\})}{c_o(N)} \\ &= c_o^2(\{k\}) = \bar{c}_o^2(\{k\}).\end{aligned}$$

From this it follows that $f_k(\bar{c}_o) = \bar{c}_o^2(\{k\})/\bar{c}_o(N) = \pi_k(\bar{c}_o)$. We also have $c_o(\{j\}) = \bar{c}_o(\{j\}) = 0$ for all $j = I+2, \ldots, n-1, n$, so by (8) $f_j(c_o) = 0 = \pi_j(c_o)$. Finally, efficiency implies that

$$\begin{aligned}f_{I+1}(\bar{c}_o) &= \bar{c}_o(N) - \sum_{k \neq I+1} f_k(\bar{c}_o) \\ &= \bar{c}_o(N) - \sum_{k \neq I+1} \pi_k(\bar{c}_o) = \pi_{I+1}(\bar{c}_o),\end{aligned}$$

which concludes the proof. □

The minimal cost for coalition N, including holding cost, equals, as we have seen before, $2am_N = 2a\sqrt{\sum_{i \in N} m_i^2}$, since the holding cost equals the ordering cost in the optimum. We define the corresponding *inventory cost game* (N, c_v) to be the game with the cost of coalition S equal to the minimal cost it can obtain on its own, that is, $c_v(S) = 2a\sqrt{\sum_{i \in S} m_i^2}$ and $c_v(\emptyset) = 0$. Thus, $c_v = 2c_o$. The properties for ordering cost games also hold for inventory cost games, so these games are concave. Furthermore, based on the proportional rule for the ordering cost game, we can find a core allocation of the inventory cost game.

In the ordering cost game, the proportional rule divides the total ordering cost of the grand coali-

[5] This is the only instance where we use the symmetry property. In fact, it could be weakened to symmetry only for the zero game. Alternatively, one could consider another property like dummy player or positivity.

tion among the players. In an inventory cost game, we have to divide ordering and holding cost. Define the distribution rule $r(c_v)$ as follows. Firm i has to pay its part of the ordering cost according to the proportional rule and its private holding cost, so that $r_i(c_v) = \pi_i(c_o) + h_i\widehat{Q}_i/2$, where $\widehat{Q}_i$ is the optimal order size for firm i when he cooperates with all the other firms.

Theorem 2. *If (N, c_v) is an inventory cost game, then $r(c_v) \in C(c_v)$ and $r(c_v)$ can be reached through a PMAS.*

Proof. Let (N, c_v) be an inventory cost game. First, we show that $\pi_i(c_o) = h_i\widehat{Q}_i/2$. If we solve the cost minimization problem for coalition N, we get that

$$\widehat{Q}_i = \frac{d_i}{m_N} = \frac{2am_i^2}{h_i m_N} = \frac{2am_i^2}{h_i\sqrt{\sum_{j\in N} m_j^2}}.$$

So, the holding cost for firm i equals

$$h_i\frac{\widehat{Q}_i}{2} = \frac{h_i}{2}\frac{2am_i^2}{h_i\sqrt{\sum_{j\in N} m_j^2}} = \frac{am_i^2}{\sqrt{\sum_{j\in N} m_j^2}} = \pi_i(c_o)$$

for all $i \in N$, where the last equality follows from (6). Next, we show that $r(c_v)$ is an element of the core. From the first part of this proof it follows that $r_i(c_v) = 2\pi_i(c_o)$ for all $i \in N$. Furthermore, it holds that

$$\sum_{i\in N} 2\pi_i(c_o) = 2\sum_{i\in N}\pi_i(c_o) = 2c_o(N) = c_v(N)$$

and for all $S \subset N$, $S \neq \emptyset$, it holds that

$$\sum_{i\in S} 2\pi_i(c_o) = 2\sum_{i\in S}\pi_i(c_o) \leqslant 2c_o(S) = c_v(S).$$

Hence, $r(c_v) \in C(c_v)$.

Just as in the case of ordering cost games, we can show that the rule $r(c_v)$ can be reached through the PMAS $2y$ where y is defined as before. □

What would happen to these results if we had full disclosure of information, i.e., if each firm would reveal its demand and holding cost? Nothing. This is not very surprising since knowing other firm's d_i and h_i is not valuable for determining the optimal order quantity. The value of each coalition remains unchanged. What does change is that Q_i^* and $\widehat{Q}_i$ become public information for all $i \in N$. Furthermore, it is possible to define rules to divide the cost of the grand coalition based on this new information. For example, one could think of a division rule based on the demand d_i of each firm i.

4. Ordering and holding cost

In this section we will consider situations in which there is full disclosure of information. Each firm $i \in N$ reveals its demand d_i, holding cost h_i, its individual optimal number of orders m_i and its individual optimal order size Q_i^*. If we assume that there are no limits to storage capacities, transport cost equal to zero and deterministic transport times, then we can consider coordination with regard to holding cost. If a member of a coalition has a very low holding cost, then this coalition can reduce its cost if it stores its inventory in the storehouse of this member.

The average cost per time unit for a coalition S of firms consists of ordering and holding cost. Just as before, the total cost is minimized if all cycle lengths are equal, so it should hold that $Q_i/d_i = Q_j/d_j$ for all $i, j \in S$. Without loss of generality we assume that firm 1 is a member of coalition S. Now we can express Q_i as a function of Q_1 for all $i \in S$: $Q_i = d_iQ_1/d_1$. In each cycle the coalition places one joint order at cost a, so the average ordering cost per time unit equals ad_1/Q_1. All goods will be stored in the warehouse of the firm with lowest holding cost. Define $h_S := \min_{i\in S} h_i$. The average inventory level of firm $i \in S$ equals $Q_i/2$ per time unit and $h_SQ_i/2$ denotes the average holding cost per time unit. Putting things together we see that the average cost per time unit for the firms in S equals

$$a\frac{d_1}{Q_1} + \sum_{i\in S} h_S\frac{Q_i}{2}.$$

When we substitute $Q_i = d_iQ_1/d_1$ we express the cost as a function of Q_1 and we get

$$a\frac{d_1}{Q_1} + \sum_{i\in S} h_S\frac{d_iQ_1}{2d_1}.$$

This cost will be minimized if

$$Q_1 = \sqrt{\frac{2ad_1^2}{h_S \sum_{j\in S} d_j}}$$

so,

$$Q_i = \frac{d_i}{d_1} Q_1 = \sqrt{\frac{2ad_i^2}{h_S \sum_{j\in S} d_j}}$$

for all $i \in S$. The minimal cost per time unit for coalition S now equals

$$\sqrt{2ah_S \sum_{i\in S} d_i}. \tag{9}$$

A holding cost situation is described by the tuple $\langle N, a, \{h_i, d_i\}_{i\in N}\rangle$. Given a holding cost situation we can define the corresponding *holding cost game* (N, c_h) as the game that assigns to coalition $S \subset N$ its minimal cost as in (9) and $c_h(\emptyset) = 0$. These games are *subadditive*, i.e., for all coalitions S and T in N such that $S \cap T = \emptyset$ it holds that $c_h(S) + c_h(T) \geqslant c_h(S \cup T)$, but not necessarily concave, as the following example shows.

Example 1. Consider the holding cost situation with player set $N = \{1, 2, 3\}$, $a = 0.5$, holding cost $h_1 = 10$, $h_2 = 10$, $h_3 = 30$ and demand equal to 1 for each player. Then

$$c_h(\{1,3\}) - c_h(\{3\}) = \sqrt{20} - \sqrt{30} < 0$$

and

$$c_h(\{1,2,3\}) - c_h(\{2,3\}) = \sqrt{30} - \sqrt{20} > 0.$$

So, this holding cost game is not concave.

As in the case of ordering cost games, we can define a proportional rule to allocate the cost of the grand coalition. The rule $p(c_h)$ divides the cost of the grand coalition proportionally to the demands. [6] This means that for each $i \in N$,

[6] Again, we thank a referee for pointing out to us that also here the proportional rule coincides with the Aumann–Shapley value (Aumann and Shapley, 1974) in a slightly changed version of the game. This version is such that the firms are free to choose the level of demand that they wish to cover.

$$p_i(c_h) = \frac{d_i}{\sum_{j\in N} d_j} c_h(N) = \frac{d_i}{\sum_{j\in N} d_j} \sqrt{2ah_N \sum_{j\in N} d_j}.$$

Theorem 3. *Let $\langle N, a, \{h_i, d_i\}_{i\in N}\rangle$ be a holding cost situation. Then the proportional rule $p(c_h)$ is a core-allocation of the corresponding holding cost game and can be reached through a PMAS y.*

Proof. By definition of the proportional rule $p(c_h)$ it holds that $\sum_{i\in N} p_i(c_h) = c_h(N)$. It also holds that

$$\begin{aligned}\sum_{i\in S} p_i(c_h) &= \frac{\sum_{i\in S} d_i}{\sum_{j\in N} d_j} \sqrt{2ah_N \sum_{j\in N} d_j} \\ &= \sum_{i\in S} d_i \sqrt{\frac{2ah_N}{\sum_{j\in N} d_j}} \leqslant \sum_{i\in S} d_i \sqrt{\frac{2ah_N}{\sum_{j\in S} d_j}} \\ &= \sqrt{2ah_N \sum_{j\in S} d_j} \leqslant \sqrt{2ah_S \sum_{j\in S} d_j} = c_h(S).\end{aligned}$$

Hence, $p(c_h) \in C(c_h)$. Similar to the proof in the previous paragraph we can define a PMAS y such that $y_{iN} = p_i(c_h)$ for all $i \in N$. □

If a cost game is concave, then it follows from Shapley (1971) that all its marginal vectors belong to the core. Since holding cost games are not necessarily concave, there may be marginal vectors that lie outside the core. However, we will show that holding cost games are permutationally concave games from which it follows that there is at least one marginal vector in the core.

Permutationally concave games were introduced in Granot and Huberman (1982) and studied in Driessen (1988) from which the following definitions are taken. Let $\Pi(N)$ denote the set of all permutations of the player set N. For all $\sigma \in \Pi(N)$, $\sigma(i)$ denotes the position of player $i \in N$ in the ordering σ. Let P_i^σ be the set of players who precede player i with respect to the ordering σ. Further, the set $\overline{P}_i^\sigma$ is obtained from P_i^σ by adding player i. Thus, $P_i^\sigma = \{j \in N | \sigma(j) < \sigma(i)\}$ and $\overline{P}_i^\sigma = \{j \in N | \sigma(j) \leqslant \sigma(i)\} = P_i^\sigma \cup \{i\}$. Define for all $\sigma \in \Pi(N)$, $\sigma(0) = 0$ and $P_0^\sigma = \emptyset$.

A cost game (N, c) is called *permutationally concave with respect to the ordering $\sigma \in \Pi(N)$* if it satisfies

$$c(\overline{P}_i^\sigma \cup R) - c(\overline{P}_i^\sigma) \geqslant c(\overline{P}_j^\sigma \cup R) - c(\overline{P}_j^\sigma) \quad (10)$$

for all $i, j \in N \cup \{0\}$ and all $R \subset N$ such that $\sigma(i) \leqslant \sigma(j)$ and $R \subset N \setminus \overline{P}_j^\sigma$. A game is said to be *permutationally concave* if there exists an ordering $\sigma \in \Pi(N)$ such that the game is permutationally concave with respect to the ordering σ. The marginal vector $x^\sigma(c) \in \mathbb{R}^N$ with respect to the ordering σ in the cost game (N, c) is given by $x_i^\sigma(c) = c(\overline{P}_i^\sigma) - c(P_i^\sigma)$ for all $i \in N$. Granot and Huberman (1982) showed that if the game (N, c) is permutationally concave with respect to the ordering $\sigma \in \Pi(N)$ then $x^\sigma(c) \in C(c)$. If we show that holding cost games are permutationally concave then it follows from this result that there is at least one marginal vector in the core.

Theorem 4. *Holding cost games are permutationally concave games.*

Proof. Let (N, c_h) be a holding cost game. Without loss of generality we number all players from 1 to n, $N = \{1, 2, \ldots, n\}$, in such a way that the holding cost per time unit of all players forms a non-decreasing sequence, i.e., $h_1 \leqslant h_2 \leqslant \cdots \leqslant h_n$. Take $\sigma \in \Pi(N)$ such that $\sigma(i) = i$ for all $i \in N$. We show that (N, c_h) is permutationally concave with respect to this ordering and thus that (N, c_h) is permutationally concave.

Let $i, j \in N \cup \{0\}$, $\sigma(i) \leqslant \sigma(j)$ and $R \subset N \setminus \overline{P}_j^\sigma$. Then $i \leqslant j$ since $\sigma(k) = k$ for all $k \in N \cup \{0\}$. The game $(N, \bar{c})$ where $\bar{c}(S) = \sqrt{\sum_{j \in S} d_j}$ for all $S \subset N$, is a concave game (cf. Proposition 1), that is,

$$\bar{c}(S \cup U) - \bar{c}(S) \geqslant \bar{c}(T \cup U) - \bar{c}(T)$$

for all $S \subset T \subset N$ and for all $U \subset N \setminus T$. Take $S = \overline{P}_i^\sigma$, $T = \overline{P}_j^\sigma$ and $U = R$. Then it holds that $S \subset T$ since $\sigma(i) \leqslant \sigma(j)$, $U \subset N \setminus T$ and (cf. the proof of Proposition 1)

$$\sqrt{\sum_{k \in \overline{P}_i^\sigma \cup R} d_k} - \sqrt{\sum_{k \in \overline{P}_i^\sigma} d_k} \geqslant \sqrt{\sum_{k \in \overline{P}_j^\sigma \cup R} d_k} - \sqrt{\sum_{k \in \overline{P}_j^\sigma} d_k}. \quad (11)$$

We have to show that

$$c_h(\overline{P}_i^\sigma \cup R) - c_h(\overline{P}_i^\sigma) \geqslant c_h(\overline{P}_j^\sigma \cup R) - c_h(\overline{P}_j^\sigma).$$

We distinguish three cases. If $i = 0$ and $j = 0$ then $\overline{P}_i^\sigma = \overline{P}_j^\sigma = \emptyset$ and

$$c_h(\overline{P}_i^\sigma \cup R) - c_h(\overline{P}_i^\sigma) = c_h(R) - c_h(\emptyset) = c_h(\overline{P}_j^\sigma \cup R) - c_h(\overline{P}_j^\sigma).$$

If $i = 0$ and $j > 0$ then $\overline{P}_i^\sigma = \emptyset$ and $\overline{P}_j^\sigma = \{1, 2, \ldots, j\}$. Since $1 \in \overline{P}_j^\sigma$ and $1 \notin R$ it holds that $h_{\overline{P}_j^\sigma} = h_{\overline{P}_j^\sigma \cup R} = h_1$ and $h_R \geqslant h_1$. Multiplying both sides of (11) by $\sqrt{2ah_R}$ gives

$$\begin{aligned}\sqrt{2ah_R \sum_{k \in R} d_k} - 0 &\geqslant \sqrt{2ah_R \sum_{k \in \overline{P}_j^\sigma \cup R} d_k} - \sqrt{2ah_R \sum_{k \in \overline{P}_j^\sigma} d_k} \\ &\geqslant \sqrt{2ah_1 \sum_{k \in \overline{P}_j^\sigma \cup R} d_k} - \sqrt{2ah_1 \sum_{k \in \overline{P}_j^\sigma} d_k}\end{aligned}$$

and this is equal to

$$c_h(R) - c_h(\emptyset) \geqslant c_h(\overline{P}_j^\sigma \cup R) - c_h(\overline{P}_j^\sigma).$$

Finally, if $0 < i \leqslant j$ then $1 \in \overline{P}_i^\sigma$ and $1 \in \overline{P}_j^\sigma$ so

$$h_{\overline{P}_i^\sigma} = h_{\overline{P}_i^\sigma \cup R} = h_{\overline{P}_j^\sigma} = h_{\overline{P}_j^\sigma \cup R} = h_1.$$

Multiplying both sides of (11) with $\sqrt{2ah_1}$ gives

$$\sqrt{2ah_1 \sum_{k \in \overline{P}_i^\sigma \cup R} d_k} - \sqrt{2ah_1 \sum_{k \in \overline{P}_i^\sigma} d_k} \geqslant \sqrt{2ah_1 \sum_{k \in \overline{P}_j^\sigma \cup R} d_k} - \sqrt{2ah_1 \sum_{k \in \overline{P}_j^\sigma} d_k}$$

which is $c_h(\overline{P}_i^\sigma \cup R) - c_h(\overline{P}_i^\sigma) \geqslant c_h(\overline{P}_j^\sigma \cup R) - c_h(\overline{P}_j^\sigma)$.

This shows that condition (10) is satisfied. □

5. An example

In this example, we consider three airline companies, Line1, Line2, and Line3 (in short: 1, 2 and 3), which operate in the same country. Airplanes can suffer from small defects that need repair. Each airline company would like to see that its airplanes are repaired as soon as possible so that no flights have to be canceled. To attain this goal, each airline company owns a warehouse in which it stores all the things their repairmen may need. One of the items stored in these warehouses are taillights. Over time, each firm has learned how much taillights are used on the average by the repairmen in a year. Line1 needs 500 taillights per year, Line2 300 and Line3 400 taillights per year. The holding cost to store one light for one year is, respectively, 9.6, 11 and 10 dollars. The individual demand and holding cost are private information. The cost of placing an order for taillights equals \$600. We can model this situation as an inventory situation.

If the airline companies work on their own, then Line1 will order $Q_1^* = \sqrt{2ad_1/h_1} = 250$ taillights per cycle of length $Q_1^*/d_1 = 0.5$ years and place $m_1 = d_1/Q_1^* = 2$ orders per year. Its annual cost equals \$2400.00. Note that most numbers in this section are approximations. Line2 will order $Q_2^* = 180.9$ taillights per cycle of length 0.61 years and it places $m_2 = 1.66$ orders per year. Its annual cost equals \$1989.98. Finally, Line3 will order $Q_3^* = 219.1$ taillights per cycle of length 0.55 years, so it places $m_3 = 1.83$ orders per year and its annual cost equals \$2190.89. The cost of the various coalitions in the inventory cost game equals (in dollars)

$$c_v(\{1\}) = 2400.00, \quad c_v(\{2\}) = 1989.98,$$
$$c_v(\{3\}) = 2190.89, \quad c_v(\{1,2\}) = 3117.69,$$
$$c_v(\{1,3\}) = 3249.62, \quad c_v(\{2,3\}) = 2959.73,$$
$$c_v(\{1,2,3\}) = 3810.51.$$

In case all airline companies work together, the cycle length equals 0.32 years, which is shorter than any individual optimal cycle length. The cost for a coalition in the ordering cost game is half its cost in the inventory cost game. The rule $r(c_v)$ assigns the total cost $c_v(N)$ proportionally to the square of the individual cost, so it assigns the cost \$(1511.61, 1039.23, 1259.67) to the airlines. This allocation lies in the core of the inventory cost game. The proportional rule in the ordering cost game assigns half of this cost to the airlines. Again, this results in a core-allocation. If there is full disclosure of information, then the values above will not change. All calculations are based on the individual optimal number of times to place an order, m_i, for all firms i in N. These m_i depend on the demand and holding cost of the corresponding firm since $m_i = \sqrt{d_i h_i/(2a)}$.

If we include coordination with respect to holding cost, then we see that Line1 owns a very attractive warehouse, since its holding cost is the lowest. The holding cost game (N, c_h) looks as follows:

$$c_h(\{1\}) = 2400.00, \quad c_h(\{2\}) = 1989.98,$$
$$c_h(\{3\}) = 2190.89, \quad c_h(\{1,2\}) = 3035.79,$$
$$c_h(\{1,3\}) = 3219.94, \quad c_h(\{2,3\}) = 2898.28,$$
$$c_h(\{1,2,3\}) = 3718.06.$$

The rule which assigns $c_h(N)$ proportionally to the demands, assigns \$(1549.19, 929.52, 1239.36) to the airlines. Line2 pays the smallest amount since its demand is smallest. The marginal vector x, which results in a core-element, corresponds to Line1 entering first, then Line3 and finally, Line2. So, $x = \$(2400, 498.13, 819.94)$. Notice that although all firms store their goods in the warehouse of Line1, this firm has to pay the greatest part of the total cost. This is caused by the fact that $x_1 = c(\{1\})$ and $x_i \leqslant c(\{i\})$ for all $i \neq 1$.

6. Concluding remarks

The model introduced in the second paragraph is called the basic inventory model since it forms the basis for a wide variety of inventory models. The basic inventory model is a simple model and extensions would make the model more realistic. Some possible extensions are a purchasing cost per unit of the good, a stochastic lead time, a finite supply rate for the ordered goods, individual ordering cost, allowing for stockouts, quantity discounts and non-constant demand. We will shortly discuss some of these extensions.

A purchasing cost c per unit of the good implies that next to the fixed cost per order firms also have to pay the variable cost cQ per order of Q units. Per time unit this implies an extra cost of $cQ \cdot d/Q = cd$, a constant cost. Since this extra cost is a constant, it will not influence the optimal order size or the cycle length. Only the cost will increase. Therefore this is not really an extension.

When we speak of a finite supply rate s, we assume that the amount ordered is not delivered all at once. We assume that the supplying process is continuous and takes place at a constant rate s until Q units are delivered to the stock and then it stops. This is only interesting if $s > d$.

Quantity discounts can be defined in at least two ways. First, we can think of quantity discounts for all units purchased. If we ordered a certain amount of goods then all units will have the same purchasing cost. Second, we can think of increasing quantity discounts. For example, the first 100 goods ordered have a unit price of 20 dollars, the next 100 a unit price of 15, and so on.

In case of non-deterministic demand we may think of D being the stochastic demand for the firm. The games arising from these inventory situations may fall within the class of inventory centralization games (see e.g. Hartman et al., 2000) where expected values are considered. Otherwise they may fall within the class of cooperative TU games with stochastic payoffs as considered in e.g. Suijs et al. (1999).

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