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Maximal type test statistics based on conditional processes

Jan Beirlant<sup>a,\*</sup>, John H.J. Einmahl<sup>b</sup>

<sup>a</sup>*Department of Mathematics, Catholic University Leuven, Celestijnenlaan 200 B, 3001 Heverlee,  
Belgium*

<sup>b</sup>*Department of Mathematics and Computing Science, Eindhoven University of Technology,  
P.O. Box 513, 5600 MB Eindhoven, Netherlands*

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## Maximal type test statistics based on conditional processes

Jan Beirlant<sup>a,\*</sup>, John H.J. Einmahl<sup>b</sup>

<sup>a</sup>Department of Mathematics, Catholic University Leuven, Celestijnenlaan 200 B, 3001 Heverlee, Belgium

<sup>b</sup>Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, Netherlands

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### Abstract

A general methodology is presented for non-parametric testing of independence, location and dispersion in multiple regression. The proposed testing procedures are based on the concepts of conditional distribution function, conditional quantile, and conditional shortest  $t$ -fraction. Techniques involved come from empirical process and extreme-value theory. The asymptotic distributions are standard Gumbel.

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*Keywords:* Non-parametric regression; Empirical processes; Extreme-value theory

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### 1. Introduction and main results

Let  $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$  be i.i.d. random vectors from a distribution  $\tilde{\mu}$  on  $\mathbb{R}^{d+1}$ ,  $X_i \in \mathbb{R}^d$ ,  $Y_i \in \mathbb{R}$  ( $i = 1, \dots, n$ ). The marginal distribution of the  $X$ 's is denoted by  $\mu$ ; let  $S$  be the support of  $\mu$ . In this paper we are concerned with the conditional distribution of  $Y$  given  $X = x$ , determined by (a version of) the conditional distribution function (df)  $F_x$ . The corresponding conditional quantiles

$$Q_x(p) = \inf\{y : F_x(y) \geq p\}, \quad p \in (0, 1)$$

can be used to describe the location of  $Y$  given  $X = x$ , as employed in median regression. Dispersion characteristics will be measured by means of lengths of shortest  $t$ -fractions (shortt); see e.g. Rousseeuw and Leroy (1988), Grübel (1988), and Einmahl and Mason (1992). For any df  $G$  and any interval  $[c, d] \subset \mathbb{R}$  we use the notation  $G([c, d])$  for  $G(d) - G(c-)$ . The conditional length of a shortt is now defined by

$$U_x(t) = \inf\{b - a : F_x([a, b]) \geq t\}, \quad t \in (0, 1).$$

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\* Corresponding author. Tel: 32 16 32 2789; Fax: 32 16 32 2999; e-mail: Jan.Beirlant@wis.kuleuven.ac.be.

<sup>1</sup> Research performed while the author was a research fellow at the Eindhoven University of Technology.

It is our aim to provide new tests for independence, constant location, and homoscedasticity through  $F_x$ ,  $Q_x(p)$  and  $U_x(t)$ , respectively. More precisely, the following hypotheses will be considered for  $0 < p, t < 1$  fixed:

- $H_0^{(1)}$  :  $F_x$  is independent of  $x \in S$  ( $\mu$  a.e.);
- $H_0^{(2)}$  :  $Q_x(p)$  is independent of  $x \in S$  ( $\mu$  a.e.);
- $H_0^{(3)}$  :  $U_x(t)$  is independent of  $x \in S$  ( $\mu$  a.e.).

Our statistical test procedures will be based on an appropriately chosen partition  $\{A_{j,n} : j = 1, \dots, m_n\}$  of  $S$ , with for convenience,

$$\mu_j := \mu(A_{j,n}) \geq \mu(A_{j+1,n}) =: \mu_{j+1}, \quad \text{for all } 1 \leq j \leq m_n - 1.$$

Empirical estimates of

$$F_j(y) := P(Y \leq y \mid X \in A_{j,n}),$$

$$Q_j(p) := \inf\{y : F_j(y) \geq p\},$$

and

$$U_j(t) := \inf\{b - a : F_j([a, b]) \geq t\}$$

are given by

$$F_{j,n}(y) := \frac{\sum_{i=1}^n I_{A_{j,n} \times (-\infty, y]}(X_i, Y_i)}{\sum_{i=1}^n I_{A_{j,n}}(X_i)},$$

$$Q_{j,n}(p) := \inf\{y : F_{j,n}(y) \geq p\},$$

and

$$U_{j,n}(t) := \inf\{b - a : F_{j,n}([a, b]) \geq t\}.$$

Throughout we assume  $F_j$  ( $j = 1, \dots, m_n$ ) to be continuous on  $\mathbb{R}$ . Let  $\mu_n$  denote the empirical measure based on  $X_1, X_2, \dots, X_n$ , and set

$$\mu_{j,n} = \mu_n(A_{j,n}), \quad 1 \leq j \leq m_n.$$

Note that the common values of  $F_x, Q_x(p)$  under  $H_0^{(1)}, H_0^{(2)}$ , respectively, are equal to  $F, Q(p)$ , the marginal df and  $p$ th quantile of the  $Y$ -distribution. Hence they are appropriately estimated by  $F_n$  and  $Q_n(p)$ , with

$$F_n(y) = n^{-1} \sum_{i=1}^n I_{(-\infty, y]}(Y_i), \quad y \in \mathbb{R},$$

$$Q_n(p) = \inf\{y : F_n(y) \geq p\}.$$

Concerning the hypothesis  $H_0^{(3)}$ , observe that the common value of  $U_x(t)$ , denoted by  $U(t)$ , is not necessarily equal to the length of the marginal shortt of the  $Y$ -distribution

(e.g., consider a degenerate bivariate (normal) distribution on the line  $y = x$ ). We will estimate  $U.(t)$  by

$$U_{\cdot n}(t) = \sum_{j=1}^{m_n} \mu_{j,n} U_{j,n}(t).$$

Now we are ready to state our main results. Let

$$A(x) = \exp(-e^{-x}), \quad x \in \mathbb{R},$$

be the standard Gumbel df,  $\Gamma$  a rv with df  $A$ , and write

$$I_n = \sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} \sqrt{n \mu_{j,n}} |F_{j,n}(y) - F_n(y)|.$$

**Theorem 1.** *If  $n \mu_{m_n} / ((\log n)^2 \log m_n) \rightarrow \infty$  and  $\mu_1 \log m_n \rightarrow 0$  as  $n \rightarrow \infty$ , then we have under  $H_0^{(1)}$  that*

$$\sqrt{8 \log m_n} \left( I_n - \sqrt{\frac{1}{2} \log(2m_n)} \right) \xrightarrow{d} \Gamma.$$

Let  $c_\alpha$  be such that  $1 - A(c_\alpha) = \alpha$ ,  $\alpha \in (0, 1)$ . Our asymptotic test for independence can now be specified.

**Corollary 1.** *The test which rejects  $H_0^{(1)}$  when*

$$I_n \geq \sqrt{\frac{1}{2} \log(2m_n)} + c_\alpha / \sqrt{8 \log m_n}$$

*has asymptotic significance level  $\alpha$  if the assumptions of Theorem 1 are satisfied.*

The following corollary can be applied when the  $X$ -distribution is known and continuous.

**Corollary 2.** *If  $m_n \rightarrow \infty$ ,  $\mu_1 = \mu_{m_n}$ , and  $n \mu_1 / (\log n)^3 \rightarrow \infty$ , then*

$$\sqrt{8 \log m_n} \left( I_n - \sqrt{\frac{1}{2} \log(2m_n)} \right) \xrightarrow{d} \Gamma.$$

In the statement of our next result we make use of the following conditions:

(C.1) for some constant  $c_1 > 0$ ,

$$\limsup_{n \rightarrow \infty} \max_{1 \leq j \leq m_n} \sup_{y \in \mathbb{R}} f_j(y) < c_1$$

where  $f_j$  denotes the derivative of  $F_j$ ;

(C.2) the derivative  $f$  of  $F$  exists at  $Q(p)$  and satisfies  $f(Q(p)) > 0$ .

Furthermore, let

$$c_{\alpha,n} = \sqrt{2 \log m_n} + \left( c_\alpha - \frac{1}{2} (\log \log m_n + \log \pi) \right) / \sqrt{2 \log m_n}.$$

**Theorem 2.** Let  $p \in (0, 1)$  be fixed. The test which rejects  $H_0^{(2)}$  when for some  $j \in \{1, 2, \dots, m_n\}$

$$Q_n(p) \notin \left[ Q_{j,n} \left( p - c_{\alpha,n} \sqrt{\frac{p(1-p)}{n\mu_{j,n}}} \right), Q_{j,n} \left( p + c_{\alpha,n} \sqrt{\frac{p(1-p)}{n\mu_{j,n}}} \right) \right)$$

has asymptotic significance level  $\alpha$  if (C.1) and (C.2) are satisfied and if  $n\mu_{m_n}/((\log n)^2 \log m_n) \rightarrow \infty$  and  $\mu_1 \log m_n \rightarrow 0$ .

In order to establish our last result some additional regularity conditions are required. The first one reads as follows:

(C.3) for large  $n$ , every  $F_j$  ( $1 \leq j \leq m_n$ ) has a density  $f_j$  which is continuous on  $\mathbb{R}$  and has support  $(\beta_j, \gamma_j)$ ,  $-\infty \leq \beta_j < \gamma_j \leq \infty$ , is strictly increasing on  $(\beta_j, y_{0,j}]$  and strictly decreasing on  $[y_{0,j}, \gamma_j)$  for some  $y_{0,j} \in (\beta_j, \gamma_j)$ . Moreover, every  $f_x$ ,  $x \in S$ , satisfies this unimodality assumption.

Let  $t \in (0, 1)$  be fixed. Under (C.3) we have for large  $n$  that there exists a unique interval  $[a_{j,t}, b_{j,t}]$  (the shortt) such that  $F_j([a_{j,t}, b_{j,t}]) = t$ ,  $f_j(a_{j,t}) = f_j(b_{j,t})$ , and  $f_j(y) > f_j(a_{j,t})$  for every  $y \in (a_{j,t}, b_{j,t})$  ( $1 \leq j \leq m_n$ ).

We also need that

(C.4) there exist constants  $c_2, \delta_2 > 0$  such that the derivatives  $f'_j$  of  $f_j$  satisfy

$$\liminf_{n \rightarrow \infty} \min_{1 \leq j \leq m_n} \inf_{y \in [a_{j,t}, b_{j,t}] \setminus [a_{j,t} + \delta_2, b_{j,t} - \delta_2]} |f'_j(y)| > c_2.$$

Introducing the derivative  $u_j$  of  $U_j$  ( $1 \leq j \leq m_n$ ) we assume

(C.5) there exist constants  $c_3, c_4 > 0$  such that for every  $s \in (0, 1)$

$$\limsup_{n \rightarrow \infty} \max_{1 \leq j \leq m_n} |u_j(s) - u_j(t)| \leq c_3 |s - t|,$$

$$\limsup_{n \rightarrow \infty} \max_{1 \leq j \leq m_n} u_j(t) < c_4.$$

Finally, we will assume

$$(C.6) \quad \sqrt{n\mu_1 \log m_n} \max_{1 \leq j \leq m_n} \left( U_j(t) - \sup_{x \in A_{j,n}} U_x(t) \right)^+ \rightarrow 0.$$

**Theorem 3.** Let  $t \in (0, 1)$  be fixed. The test which rejects  $H_0^{(3)}$  when for some  $j \in \{1, 2, \dots, m_n\}$

$$U_n(t) \notin \left[ U_{j,n} \left( t - c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right), U_{j,n} \left( t + c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right) \right)$$

has asymptotic significance level  $\alpha$  if (C.1), (C.3)–(C.6) are satisfied and if  $\mu_1 \log m_n \rightarrow 0$ ,  $\mu_1^4 (\sum_{j=1}^{m_n} \mu_j^{1/2})^8 (\log n)^4 (\log m_n)^5 / (n\mu_{m_n}) \rightarrow 0$ .

For any  $x \in S$ , let  $m_t(x)$  be defined as the midpoint of the interval pertaining to  $U_x(t)$ . This robust regression curve is strongly related to the least median of squares

regression estimator introduced in Rousseeuw (1984) (see also Rousseeuw and Leroy, 1988). The following smoothness conditions on  $m_t$  and  $F_x$  ( $x \in S$ ) can be used instead of assumption (C.6), as shown by the following corollaries.

(C.7) for some constant  $c_5 > 0$ ,

$$|m_t(x_1) - m_t(x_2)| \leq c_5 \|x_1 - x_2\|$$

for any  $x_1, x_2 \in S$ ;

(C.8) the second-order derivatives  $f'_x$  of  $F_x$  exist, and for some  $c_6 > 0$ ,

$$\sup_{x \in S} \sup_{y \in \mathbb{R}} |f'_x(y)| < c_6.$$

Let  $\text{diam}(A) := \sup\{\|x_1 - x_2\| : x_1, x_2 \in A\}$ , where  $\|x_1 - x_2\|$  denotes the Euclidean distance between  $x_1$  and  $x_2$ .

**Corollary 3.** *The test which rejects  $H_0^{(3)}$  when for some  $j \in \{1, 2, \dots, m_n\}$*

$$U_{.n}(t) \notin \left[ U_{j,n} \left( t - c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right), U_{j,n} \left( t + c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right) \right)$$

*has asymptotic significance level  $\alpha$  if (C.1), (C.3)–(C.5), (C.7) and (C.8) are satisfied, and if*

$$n\mu_1 \log m_n \left( \max_{1 \leq j \leq m_n} \text{diam}(A_{j,n}) \right)^4 \rightarrow 0, \quad \mu_1 \log m_n \rightarrow 0 \quad \text{and}$$

$$\mu_1^4 \left( \sum_{j=1}^{m_n} \mu_j^{1/2} \right)^8 (\log n)^4 (\log m_n)^5 / (n\mu_{m_n}) \rightarrow 0.$$

**Corollary 4.** *If  $m_n \rightarrow \infty$ ,  $\mu_1 = \mu_{m_n}$ ,  $n\mu_1 / (\log n)^9 \rightarrow \infty$ , and  $n\mu_1 \log m_n (\max_{1 \leq j \leq m_n} \text{diam}(A_{j,n}))^4 \rightarrow 0$ , then it follows under (C.1), (C.3)–(C.5), (C.7) and (C.8) that the test which rejects  $H_0^{(3)}$  when for some  $j \in \{1, 2, \dots, m_n\}$*

$$U_{.n}(t) \notin \left[ U_{j,n} \left( t - c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right), U_{j,n} \left( t + c_{\alpha,n} \sqrt{\frac{t(1-t)}{n\mu_{j,n}}} \right) \right)$$

*has asymptotic significance level  $\alpha$ .*

**Remark.** (1) The choice of  $Q_x(p)$ , resp.  $U_x(t)$ , rather than  $m_p(x)$ , resp. the interquartile range  $Q_x(\frac{1}{2}(1+t)) - Q_x(\frac{1}{2}(1-t))$ , to produce tests for  $H_0^{(2)}$ , resp.  $H_0^{(3)}$ , was motivated in part by considerations of statistical relevance. Indeed,  $m_p(x)$  ( $x \in A_{j,n}$ ) can only be estimated at a rate of  $(n\mu_j)^{-1/3}$  (see e.g. Kim and Pollard, 1990), whereas interquartile ranges have a lower breakdown point than the corresponding short measures when  $t > \frac{1}{3}$  (see Rousseeuw and Leroy (1988) for the case  $t = \frac{1}{2}$ ). The techniques we use to derive our results however, can also be applied to other testing procedures, e.g. those based on  $m_p(x)$  and interquartile ranges.



(2) In the cases considered in Theorems 2 and 3, similar results on sup-norm statistics where  $t, p$  vary over non-degenerate intervals can be obtained with the technique of proof introduced in the next section.

(3) Our statistic  $I_n$  discussed in Theorem 1 is somewhat similar to the  $V$ -quantities in Kiefer (1959) to test equality of distributions in a one-way layout of several populations (see also the references in that paper). The situation considered here provides a generalization of Kiefer's result to the case where the number of groups increases with the sample size.

(4) Somewhat related papers are Bhattacharya and Gangopadhyay (1990), Stute (1986) and Ruymgaart (1994).

(5) In a non-regression setting an analogue of our type of test statistics is the goodness-of-fit test statistic in Dijkstra et al. (1984). In case  $S$  is compact, these authors propose to reject uniformity on  $S$  when  $P_n = \max_{1 \leq j \leq m_n} \mu_{j,n}$  becomes too large, where the partition is taken to be such that under the null hypothesis the  $\mu_j$  are all equal. Their simulation study shows that the power of this test is at least comparable to the power of the classical  $\chi^2$ -test for uniformity against peaked alternatives.

A 'continuous' version of this 'peak-test' is given by the scan statistic (see e.g. Naus, 1966, 1982; Cressie, 1980, 1987) which uses a maximal type statistic obtained from continuous scanning of  $S$  with a fixed window. In case  $d = 1$ , Deheuvels and Révész (1987) derived asymptotics for the scan statistic using a similar condition as in Corollary 2; i.e.  $(na_n)/(\log n)^3 \rightarrow \infty$ , where  $a_n$  is the window length.

When  $\mu_1 = \mu_{m_n}$  one can also derive the following result for  $P_n$ : if  $(n\mu_1)/(\log n)^3 \rightarrow \infty$  and  $\mu_1 \rightarrow 0$ , we have under the hypothesis of uniformity that

$$\sqrt{2 \log m_n} \left\{ \sqrt{\frac{n}{\mu_1}} (P_n - \mu_1) - \sqrt{2 \log m_n} + \frac{1}{2} (\log \log m_n + \log 4\pi) / (2 \log m_n)^{1/2} \right\} \xrightarrow{d} \Gamma.$$

(6) The condition  $n\mu_1(\log m_n)(\max_{1 \leq j \leq m_n} \text{diam}(A_{j,n}))^4 \rightarrow 0$  specifies to  $nh_n^{4+d} \log 1/h_n \rightarrow 0$  in case  $X$  possesses a uniform distribution on  $[0, 1]^d$ , say, and the partition is taken to be cubic with  $\text{diam}(A_{j,n}) \sim h_n$  ( $j = 1, \dots, m_n$ ). This rate condition of  $h_n$  lies close to the optimal rate of the window size in kernel density estimation when minimizing the mean squared error.

(7) If one wants to restrict attention to a subset of the support  $S$  of  $X$ , all of our results can still be used by translating them in terms of conditional distributions given  $X$  belongs to that subset.

## 2. Proofs

The proofs of our main results rely on the following important proposition which states that jointly over all elements  $A_{j,n}$  of the partition of  $S$ , we can approximate the different empirical processes

$$\alpha_{j,n} = \sqrt{n\mu_{j,n}}(F_{j,n} - F_j), \quad j = 1, \dots, m_n,$$

by independent Gaussian processes, and this, per  $j$ , at a rate which is comparable to the one attained by the Komlós–Major–Tusnády (1975) approximation of the (one-dimensional) uniform empirical process.

Denoting the joint distribution of  $(X, Y)$  by  $\tilde{\mu}$ , and the empirical measure based on  $(X_1, Y_1), \dots, (X_n, Y_n)$  by  $\tilde{\mu}_n$ , we will use the following quantities:

$$\tilde{\mu}_j(y) = P(X \in A_{j,n} \text{ and } Y \leq y) = \tilde{\mu}(A_{j,n} \times (-\infty, y]),$$

$$\tilde{\mu}_{j,n}(y) = \tilde{\mu}_n(A_{j,n} \times (-\infty, y]),$$

so that

$$F_{j,n}(y) = \frac{\tilde{\mu}_{j,n}(y)}{\mu_{j,n}}.$$

**Proposition.** *If  $m_n \rightarrow \infty$  and  $(n\mu_{m_n})/(\log n)^2 \rightarrow \infty$ , then there exists a triangular scheme of rowwise independent Brownian bridges  $\{B_{j,n}(t), 0 \leq t \leq 1\}$  ( $1 \leq j \leq m_n, n \geq 1$ ) such that*

$$\sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} |\alpha_{j,n}(y) - B_{j,n}(F_j(y))| = O_P\left(\frac{\log n}{\sqrt{n\mu_{m_n}}}\right).$$

**Proof.** We consider the transformation from  $S \times \mathbb{R}$  to  $[0, 1]$

$$(x, y) \rightarrow T(x, y) = \sum_{j=1}^{m_n} 1_{A_{j,n}}(x) \left[ \sum_{k=1}^{j-1} \mu_k + \mu_j F_j(y) \right],$$

and the transformed rv's

$$Z_i = T(X_i, Y_i), \quad i = 1, 2, \dots, n.$$

One easily checks that  $Z_1, Z_2, \dots, Z_n$  are independent uniformly  $(0,1)$  distributed rv's. Let  $\{e_n(t), 0 \leq t \leq 1\}$  denote the empirical process based on  $Z_1, Z_2, \dots, Z_n$ . The approximation theorem of Komlós et al. (1975) entails then the existence of a sequence of Brownian bridges  $\{\tilde{B}_n(t), 0 \leq t \leq 1\}$  such that as  $n \rightarrow \infty$

$$\sup_{0 \leq t \leq 1} |e_n(t) - \tilde{B}_n(t)| = O_P\left(\frac{\log n}{\sqrt{n}}\right).$$

Setting  $T(A_{j,n} \times (-\infty, y]) := \{T(x, v) : x \in A_{j,n}, v \leq y\}$ ,  $y \in \mathbb{R} \cup \{\infty\}$ , we have  $\lambda(T(A_{j,n} \times (-\infty, y])) = \mu_j F_j(y)$ , with  $\lambda$  denoting Lebesgue measure; also for a closed, half-open, or open interval  $A$  with endpoints  $a \leq b$  we set  $\tilde{B}_n(A) := \tilde{B}_n(b) - \tilde{B}_n(a)$ .

It now follows that

$$\max_{1 \leq j \leq m_n} |\sqrt{n}(\mu_{j,n} - \mu_j) - \tilde{B}_n(T(A_{j,n} \times (-\infty, \infty]))| = O_P\left(\frac{\log n}{\sqrt{n}}\right)$$

and that

$$\max_{1 \leq j \leq m_n} \sup_{y \in \mathbb{R}} |\sqrt{n}(\tilde{\mu}_{j,n}(y) - \tilde{\mu}_j(y)) - \tilde{B}_n(T(A_{j,n} \times (-\infty, y]))| = O_P\left(\frac{\log n}{\sqrt{n}}\right).$$

Now uniformly in  $j \in \{1, \dots, m_n\}$  and  $y \in \mathbb{R}$  we have

$$\begin{aligned} \alpha_{j,n}(y) &= \sqrt{n\mu_{j,n}} \left( \frac{\tilde{\mu}_{j,n}(y)}{\mu_{j,n}} - \frac{\tilde{\mu}_j(y)}{\mu_j} \right) \\ &= \sqrt{n}(\tilde{\mu}_{j,n}(y) - \tilde{\mu}_j(y))/\sqrt{\mu_{j,n}} - \sqrt{n}(\mu_{j,n} - \mu_j)(\tilde{\mu}_j(y)/(\mu_j\sqrt{\mu_{j,n}})) \\ &= \left\{ \left\{ \tilde{B}_n(T(A_{j,n} \times (-\infty, y])) + O_P\left(\frac{\log n}{\sqrt{n}}\right) \right\} \mu_j^{-1/2} \right. \\ &\quad \left. - \left\{ \tilde{B}_n(T(A_{j,n} \times (-\infty, \infty))) + O_P\left(\frac{\log n}{\sqrt{n}}\right) \right\} F_j(y)\mu_j^{-1/2} \right\} \tau_{j,n}^{-1/2}, \end{aligned}$$

where

$$\tau_{j,n} = \mu_{j,n}/\mu_j = 1 + n^{-1/2} \mu_j^{-1} \tilde{B}_n(T(A_{j,n} \times (-\infty, \infty))) + \mu_j^{-1} O_P\left(\frac{\log n}{n}\right). \quad (2.1)$$

We can define a sequence of Wiener processes  $\{W_n(t), 0 \leq t \leq 1\}$  such that  $\tilde{B}_n = W_n - IW_n(1)$ , where  $I$  denotes the identity function. Hence, we find that

$$\begin{aligned} \alpha_{j,n}(y) &= \left\{ \left\{ W_n(T(A_{j,n} \times (-\infty, y])) - F_j(y)W_n(T(A_{j,n} \times (-\infty, \infty))) \right\} \mu_j^{-1/2} \right. \\ &\quad \left. + \mu_j^{-1/2} O_P\left(\frac{\log n}{\sqrt{n}}\right) \right\} \tau_{j,n}^{-1/2} \quad (n \rightarrow \infty). \end{aligned}$$

We now set

$$B_{j,n}(F_j(y)) := \mu_j^{-1/2} \{W_n(T(A_{j,n} \times (-\infty, y])) - F_j(y)W_n(T(A_{j,n} \times (-\infty, \infty)))\}.$$

One easily checks that the  $B_{j,n}$  are indeed independent in  $j \in \{1, 2, \dots, m_n\}$  and distributed as Brownian bridges.

Now as  $n \rightarrow \infty$

$$|\alpha_{j,n}(y) - B_{j,n}(F_j(y))| \leq |B_{j,n}(F_j(y))| (\tau_{j,n}^{-1/2} - 1) + \mu_j^{-1/2} \tau_{j,n}^{-1/2} O_P\left(\frac{\log n}{\sqrt{n}}\right).$$

For a function  $\varphi$  on  $[0, 1]$ , write  $\|\varphi\| = \sup_{0 \leq t \leq 1} |\varphi(t)|$ . First remark that as the  $F_j$  are assumed to be continuous:

$$\max_{1 \leq j \leq m_n} \sup_{y \in \mathbb{R}} |B_{j,n}(F_j(y))| = \max_{1 \leq j \leq m_n} \|B_{j,n}\| = O_P(\sqrt{\log m_n}),$$

as, because of the independence of the rv's  $\|B_{j,n}\|$  ( $1 \leq j \leq m_n$ ), we have for any  $M > 0$ , that

$$P\left(\max_{1 \leq j \leq m_n} \|B_{j,n}\| > M\sqrt{\log m_n}\right) \leq 2m_n e^{-2M^2 \log m_n} = 2m_n^{1-2M^2},$$

which tends to zero as  $m_n \rightarrow \infty$  when  $M > 2^{-1/2}$ . (Here we also used the fact that for a Brownian bridge  $B$  we have  $P(\|B\| > u) \leq 2e^{-2u^2}$ .)

Furthermore,

$$\begin{aligned} & n^{-1/2} \max_{1 \leq j \leq m_n} \mu_j^{-1} \left| \tilde{B}_n \left( \left( \sum_{k=1}^{j-1} \mu_k, \sum_{k=1}^j \mu_k \right) \right) \right| \\ & \leq n^{-1/2} \mu_{m_n}^{-1/2} \max_{1 \leq j \leq m_n} \left( \mu_j^{-1/2} \left| W_n \left( \left( \sum_{k=1}^{j-1} \mu_k, \sum_{k=1}^j \mu_k \right) \right) \right| + \mu_{m_n}^{1/2} |W_n(1)| \right) \\ & = (n\mu_{m_n})^{-1/2} O_p(\sqrt{\log m_n}), \end{aligned}$$

since  $W_n((\sum_{k=1}^{j-1} \mu_k, \sum_{k=1}^j \mu_k))/\sqrt{\mu_j}$  ( $1 \leq j \leq m_n$ ) are  $m_n$  independent standard normal rv's whose maximum is well known to be of order  $O_p(\sqrt{\log m_n})$  as  $n \rightarrow \infty$ .

Hence,

$$\max_{1 \leq j \leq m_n} |\tau_{j,n} - 1| = O_p \left( \sqrt{\frac{\log n}{n\mu_{m_n}}} + \frac{\log n}{n\mu_{m_n}} \right) \tag{2.2}$$

and

$$\max_{1 \leq j \leq m_n} \sup_{y \in \mathbb{R}} |B_{j,n}(F_j(y))| |\tau_{j,n}^{-1/2} - 1| = O_p \left( \frac{\log n}{\sqrt{n\mu_{m_n}}} + \frac{(\log n)^{3/2}}{n\mu_{m_n}} \right) (n \rightarrow \infty).$$

Finally, with

$$\left( \max_{1 \leq j \leq m_n} (\mu_j \tau_{j,n})^{-1/2} \right) O_p \left( \frac{\log n}{\sqrt{n}} \right) = O_p \left( \frac{\log n}{\sqrt{n\mu_{m_n}}} \right),$$

the result follows.  $\square$

**Proof of Theorem 1.** First remark that by the well-known fact that

$$\sqrt{n} \sup_{y \in \mathbb{R}} |F_n(y) - F(y)| = O_p(1) \quad (n \rightarrow \infty),$$

we have

$$\sqrt{\log m_n} \sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}} |F_n(y) - F(y)| = (\mu_1 \log m_n)^{1/2} O_p(1) \quad (n \rightarrow \infty)$$

since, as in the proof of the proposition we find that uniformly in  $j \in \{1, \dots, m_n\}$

$$\mu_{j,n}^{1/2} = \mu_j^{1/2} \left( 1 + O_p \left( \sqrt{\frac{\log m_n}{n\mu_{m_n}}} \right) \right).$$

Hence, since  $\mu_1 \log m_n \rightarrow 0$ , it suffices to show that, under  $H_0^{(1)}$ ,

$$\begin{aligned} & \sqrt{8 \log m_n} \left( \sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}} |F_{j,n}(y) - F(y)| - \sqrt{\frac{1}{2} \log(2m_n)} \right) \xrightarrow{d} \Gamma \\ & (n \rightarrow \infty). \end{aligned}$$

Under  $H_0^{(1)}$  it now follows from the proposition that

$$\begin{aligned} & \sqrt{8 \log m_n} \left| \sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} \sqrt{n \mu_{j,n}} |F_{j,n}(y) - F(y)| - \sup_{y \in \mathbb{R}} \max_{1 \leq j \leq m_n} |B_{j,n}(F_j(y))| \right| \\ &= O_p \left( \frac{\log n \sqrt{\log m_n}}{\sqrt{n \mu_{m_n}}} \right) = o_p(1) \end{aligned}$$

if  $n \rightarrow \infty$  and  $n \mu_{m_n} / ((\log n)^2 \log m_n) \rightarrow \infty$ .

Finally, remark that by the independence of the  $\|B_{j,n}\|$  ( $1 \leq j \leq m_n$ ) we can apply standard extreme value theory to show that

$$\sqrt{8 \log m_n} \left\{ \max_{1 \leq j \leq m_n} \|B_{j,n}\| - \sqrt{\frac{1}{2} \log(2m_n)} \right\} \xrightarrow{d} \Gamma \tag{2.3}$$

since  $P(\|B_{j,n}\| > u) \sim 2e^{-2u^2}$  (see Resnick, 1987, Proposition 1.19).  $\square$

**Proof of Corollary 2.** If  $\mu_1 = \mu_2 = \dots = \mu_{m_n}$ , then  $m_n = \mu_1^{-1}$ .

The condition  $\mu_1 \log m_n \rightarrow 0$  is then automatically satisfied when  $m_n \rightarrow \infty$ .  $\square$

**Proof of Theorem 2.** Observe that, under  $H_0^{(2)}$ ,

$$P(Q_n(p) \notin \left[ Q_{j,n} \left( p - c_{\alpha,n} \sqrt{\frac{p(1-p)}{n \mu_{j,n}}} \right), Q_{j,n} \left( p + c_{\alpha,n} \sqrt{\frac{p(1-p)}{n \mu_{j,n}}} \right) \right], \text{ for some } j \in \{1, \dots, m_n\}) \rightarrow \alpha \text{ as } n \rightarrow \infty$$

if

$$\begin{aligned} & \sqrt{2 \log m_n} \left\{ \max_{1 \leq j \leq m_n} \left( \sqrt{n \mu_{j,n}} |F_{j,n}(Q_n(p)) - p| / \sqrt{p(1-p)} \right) - \sqrt{2 \log m_n} \right. \\ & \left. + \frac{1}{2} (\log \log m_n + \log \pi) (2 \log m_n)^{-1/2} \right\} \xrightarrow{d} \Gamma. \end{aligned} \tag{2.4}$$

Indeed, for any df  $G$  on the real line and any  $p \in (0, 1)$  we have

$$G(x) \geq p \quad \text{if and only if} \quad G^{-1}(p) \leq x$$

and hence

$$G(x) < p \quad \text{if and only if} \quad G^{-1}(p) > x.$$

We first show that under (C.1), (C.2),  $n \mu_{m_n} / ((\log n)^2 \log m_n) \rightarrow \infty$  and  $\mu_1 \log m_n \rightarrow 0$

$$\sqrt{2 \log m_n} \left\{ \max_{1 \leq j \leq m_n} \sqrt{n \mu_{j,n}} |F_{j,n}(Q_n(p)) - p| - \max_{1 \leq j \leq m_n} |B_{j,n}(F_j(Q_n(p)))| \right\} \xrightarrow{p} 0 \tag{2.5}$$

where  $\{B_{j,n}\}$  ( $1 \leq j \leq m_n, n \geq 1$ ) is the sequence of Brownian bridges described in the Proposition. Now (2.5) follows from the Proposition if we can show that under our assumptions

$$\sqrt{\log m_n} \max_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}} |F_j(Q_n(p)) - p| \xrightarrow{P} 0. \tag{2.6}$$

The well-known central limit theorem for quantiles yields that under  $H_0^{(2)}$  and (C.2)

$$Q_n(p) = Q(p) + O_P(n^{-1/2}) = Q_j(p) + O_P(n^{-1/2})$$

when  $n \rightarrow \infty$ . Hence, by the mean value theorem we have under  $H_0^{(2)}$  that

$$F_j(Q_n(p)) = F_j(Q_j(p) + O_P(n^{-1/2})) = p + O_P(n^{-1/2})f_j(\tilde{Q}_{j,n}(p))$$

with  $\tilde{Q}_{j,n}(p) \in (Q_n(p) \wedge Q_j(p), Q_n(p) \vee Q_j(p))$  ( $1 \leq j \leq m_n$ ). Hence, by (C.1) and under  $H_0^{(2)}$ ,

$$\sqrt{n} \max_{1 \leq j \leq m_n} |F_j(Q_n(p)) - p| = O_P(1) \quad (n \rightarrow \infty), \tag{2.7}$$

so that it remains to check that  $(\log m_n)(\max_{1 \leq j \leq m_n} \mu_{j,n}) \xrightarrow{P} 0$  ( $n \rightarrow \infty$ ) for (2.6) (and hence (2.5)) to hold.

However, using  $\tau_{j,n}$  in (2.1) again, we get that

$$\log m_n \left( \max_{1 \leq j \leq m_n} \mu_{j,n} \right) \leq \mu_1(\log m_n) \left( \max_{1 \leq j \leq m_n} \tau_{j,n} \right),$$

which tends to zero in probability as  $n \rightarrow \infty$  and  $\mu_1 \log m_n \rightarrow 0$  because of (2.2).

Next, it follows from (2.7), and the modulus of continuity behaviour of Brownian bridges (see e.g. Csörgő and Révész, 1981, Lemma 1.1.1) that

$$\begin{aligned} \sqrt{\log m_n} \max_{1 \leq j \leq m_n} \left| |B_{j,n}(F_j(Q_n(p)))| - |B_{j,n}(p)| \right| &= O_P(n^{-1/4}((\log n)(\log m_n))^{1/2}) \\ &= o_P(1) \quad (n \rightarrow \infty). \end{aligned} \tag{2.8}$$

As  $B_{j,n}(p)$  ( $1 \leq j \leq m_n$ ) are independent  $\mathcal{N}(0, p(1-p))$  rv's, standard techniques from extreme value theory yield that

$$\begin{aligned} \sqrt{2 \log m_n} \left\{ (p(1-p))^{-1/2} \max_{1 \leq j \leq m_n} |B_{j,n}(p)| - \sqrt{2 \log m_n} \right. \\ \left. + \frac{1}{2}(\log \log m_n + \log \pi) \cdot (2 \log m_n)^{-1/2} \right\} \xrightarrow{d} \Gamma \quad (m_n \rightarrow \infty). \end{aligned} \tag{2.9}$$

Limit statement (2.4) now follows from (2.5), (2.8) and (2.9).  $\square$

**Proof of Theorem 3.** We introduce the functions

$$H_j(z) = \sup\{F_j([a, b]) : b - a \leq z\}.$$

Note that  $H_j$  is the inverse of  $U_j$  (for  $n$  large enough). The derivative of  $H_j$  is denoted by  $h_j$ . Remark that condition (C.1) implies

$$\limsup_{n \rightarrow \infty} \max_{1 \leq j \leq m_n} \sup_{z \geq 0} h_j(z) < \infty, \tag{2.10}$$

as for each  $j \in \{1, \dots, m_n\}$  we find that  $h_j$  is non-increasing and  $h_j(0) = \max_{y \in \mathbb{R}} f_j(y)$ . Analogously, we define the inverse function  $H_{j,n}$  of  $U_{j,n}$  by

$$H_{j,n}(z) = \inf\{t : U_{j,n}(t) \geq z\}$$

and note that

$$H_{j,n}(z) = \sup\{F_{j,n}([a, b]) : b - a \leq z\}, \quad 1 \leq j \leq m_n.$$

To prove Theorem 3 it now suffices to show that under  $H_0^{(3)}$

$$\begin{aligned} & \sup_{t \in (0,1)} \max_{1 \leq j \leq m_n} |\sqrt{n\mu_{j,n}}(H_{j,n}(U_{j,n}(t)) - t) - \tilde{B}_{j,n}(t)| \\ & = O_P((n\mu_{m_n})^{-1/8}(\log m_n)^{1/8}(\log n)^{1/2}) \end{aligned} \tag{2.11}$$

for some triangular scheme of rowwise independent Brownian bridges  $\{\tilde{B}_{j,n}\}(1 \leq j \leq m_n, n \geq 1)$ ; cf. the proof of Theorem 2. We derive (2.11) in three steps by showing that under the given conditions

$$\begin{aligned} & \sup_{t \in (0,1)} \max_{1 \leq j \leq m_n} |\sqrt{n\mu_{j,n}}(H_{j,n}(U_j(t)) - t) - \tilde{B}_{j,n}(t)| \\ & = O_P((n\mu_{m_n})^{-1/8}(\log m_n)^{1/8}(\log n)^{1/2}), \end{aligned} \tag{2.12}$$

$$\sqrt{\log m_n} \max_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}} |H_j(U_{j,n}(t)) - t| \xrightarrow{P} 0 \tag{2.13}$$

and

$$\sqrt{\log m_n} \max_{1 \leq j \leq m_n} |\tilde{B}_{j,n}(H_j(U_{j,n}(t))) - \tilde{B}_{j,n}(t)| \xrightarrow{P} 0 \quad (n \rightarrow \infty). \tag{2.14}$$

First, we prove the existence of a sequence  $\{\tilde{B}_{j,n}\}$  of Brownian bridges for which (2.12) holds. Remark that from the Proposition it follows that

$$\sup_{[a,b]} \max_{1 \leq j \leq m_n} |\alpha_{j,n}([a, b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))| = O_P\left(\frac{\log n}{\sqrt{n\mu_{m_n}}}\right) \tag{2.15}$$

as  $B_{j,n}([F_j(a), F_j(b)]) = B_{j,n}(F_j(b)) - B_{j,n}(F_j(a))$ . To derive (2.12) from (2.15) we apply and refine the method of proof of Proposition 3.1 in Einmahl and Mason (1992). We define

$$\tilde{B}_{j,n}(t) = B_{j,n}(F_j(b_{j,t})) - B_{j,n}(F_j(a_{j,t})), \quad 1 \leq j \leq m_n.$$

As the intervals  $[a_{j,t}, b_{j,t}]$  are nested for different values of  $t$ , one easily checks that the  $\tilde{B}_{j,n}$  are distributed as Brownian bridges for every  $j \in \{1, 2, \dots, m_n\}$  and large  $n$ ;

moreover,  $\tilde{B}_{1,n}, \dots, \tilde{B}_{m_n,n}$  are clearly independent. Notice that for any  $j \in \{1, \dots, m_n\}$  and  $0 < t < 1$

$$\begin{aligned} \tilde{B}_{j,n}(t) - \sqrt{n\mu_{j,n}}(H_{j,n}(U_j(t)) - t) &\leq (B_{j,n}(F_j(b_{j,t})) - B_{j,n}(F_j(a_{j,t}))) \\ &\quad - \alpha_{j,n}([a_{j,t}, b_{j,t}]) \end{aligned} \tag{2.16}$$

which, by (2.15), is seen to be  $O_p(\log n/\sqrt{n\mu_{m_n}})$ , uniformly in  $j \in \{1, \dots, m_n\}$ .

Next, we also have for any  $j \in \{1, \dots, m_n\}$  and any sequence  $\varepsilon_n \downarrow 0$

$$\begin{aligned} &\sqrt{n\mu_{j,n}}(H_{j,n}(U_j(t)) - t) - \tilde{B}_{j,n}(t) \\ &\leq \left\{ \sqrt{n\mu_{j,n}} \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} (F_{j,n}([a,b]) - t) - \tilde{B}_{j,n}(t) \right\} \\ &\vee \left\{ \sqrt{n\mu_{j,n}} \sup_{F_j([a,b]) \leq t-\varepsilon_n} (F_{j,n}([a,b]) - t) - \tilde{B}_{j,n}(t) \right\}. \end{aligned} \tag{2.17}$$

The second term on the right-hand side of (2.17) is

$$\begin{aligned} &\leq \sqrt{n\mu_{j,n}} \sup_{F_j([a,b]) \leq t} (F_{j,n}([a,b]) - F_j([a,b])) + |\tilde{B}_{j,n}(t) - \varepsilon_n \sqrt{n\mu_{j,n}}| \\ &\leq 2 \max_{1 \leq j \leq m_n} \sup_{[c,d]} |B_{j,n}([c,d])| \\ &\quad + \max_{1 \leq j \leq m_n} \sup_{[a,b]} |\alpha_{j,n}([a,b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))| - \varepsilon_n \min_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}}. \end{aligned}$$

From (2.3), (2.15) and (2.2) it now follows that the second term on the right-hand side of (2.17) can be asymptotically bounded from above by 0 in probability, by making the appropriate choice

$$\varepsilon_n = M(\log m_n/(n\mu_{m_n}))^{1/2}$$

with  $M$  a large enough positive constant.

The first term on the right-hand side of (2.17) is

$$\begin{aligned} &\leq \sqrt{n\mu_{j,n}} \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} ((F_{j,n}(b) - F_{j,n}(a)) - (F_j(b) - F_j(a))) - \tilde{B}_{j,n}(t) \\ &\leq \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} |\alpha_{j,n}([a,b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))| \\ &\quad + \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a))) - \tilde{B}_{j,n}(t). \end{aligned} \tag{2.18}$$

The first term on the right-hand side of (2.18) is of order  $O_p(\log n/\sqrt{n\mu_{m_n}})$ , uniformly in  $j \in \{1, 2, \dots, m_n\}$ , by (2.15).



Finally, observe that for any  $j \in \{1, \dots, m_n\}$

$$\begin{aligned} & \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} \{B_{j,n}(F_j(b)) - B_{j,n}(F_j(a))\} - \tilde{B}_{j,n}(t) \\ & \leq \max_{1 \leq j \leq m_n} \sup_{\substack{b-a \leq U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leq t}} \{|B_{j,n}(F_j(b)) - B_{j,n}(F_j(b_{j,t}))| \\ & \quad + |B_{j,n}(F_j(a)) - B_{j,n}(F_j(a_{j,t}))|\}. \end{aligned} \tag{2.19}$$

For any interval  $[a, b]$  with  $b - a = U_j(t)$  and  $t - \varepsilon_n < F_j([a, b]) \leq t$ , we find by (C.4) that (uniformly in  $j$ )  $|a - a_{j,t}|$  and  $|b - b_{j,t}|$  become arbitrarily small as  $n \rightarrow \infty$ . We then find, with  $\delta_2$  as in (C.4), that eventually as  $n \rightarrow \infty$  whether  $a \in [a_{j,t}, a_{j,t} + \delta_2)$  or  $b \in [b_{j,t} - \delta_2, b_{j,t}]$ . In case  $a < a_{j,t} < y_{0,j} < b < b_{j,t}$ , we have that

$$F_j([b, b_{j,t}]) \leq |b_{j,t} - b| f_j(y_{0,j}) \leq c_1(b_{j,t} - b).$$

On the other hand, if  $\varepsilon_n \geq F_j([a_{j,t}, b_{j,t}]) - F_j([a, b]) \geq 0$ , then

$$\varepsilon_n \geq F_j([b, b_{j,t}]) - (b_{j,t} - b) f_j(b_{j,t}) = -((b_{j,t} - b)^2/2) f'_j(\tilde{b}_{j,t})$$

with  $\tilde{b}_{j,t} \in (b_{j,t} \wedge b, b_{j,t} \vee b)$ , so that (C.4) implies that for  $n$  large enough  $F_j(b_{j,t}) - F_j(b) \leq C\varepsilon_n^{1/2}$ , for some  $C > 0$ . Also in the other possible cases we can obtain this same bound for  $|F_j(b_{j,t}) - F_j(b)| \vee |F_j(a_{j,t}) - F_j(a)|$ . Hence the expression on the right-hand side of (2.19) can be bounded by

$$\omega(n, \varepsilon_n) := 2 \max_{1 \leq j \leq m_n} \sup_{0 \leq s \leq 1 - C\sqrt{\varepsilon_n}} \sup_{0 \leq t \leq C\sqrt{\varepsilon_n}} |B_{j,n}(s+t) - B_{j,n}(s)|. \tag{2.20}$$

By Lemma 1.1.1 in Csörgő and Révész (1981), the representation of Brownian bridges in terms of Wiener processes, and the independence of the Brownian bridges  $B_{j,n}$  ( $1 \leq j \leq m_n$ ), we obtain that for any  $K > 0$  there exist constants  $K_1, K_2 > 0$  such that

$$P(\omega(n, \varepsilon_n) > K\gamma_n) \leq K_1 m_n \varepsilon_n^{-1/2} \exp(-K_2 K^2 \gamma_n^2 \varepsilon_n^{-1/2}).$$

Choosing

$$\gamma_n = \varepsilon_n^{1/4} (\log n)^{1/2} = M^{1/4} (\log m_n)^{1/8} (n\mu_{m_n})^{-1/8} (\log n)^{1/2},$$

one easily checks that

$$P(\omega(n, \varepsilon_n) > K\gamma_n) \rightarrow 0 \quad (n \rightarrow \infty)$$

choosing  $K > 0$  large enough. This together with (2.16)–(2.20) implies (2.12).

To derive (2.13), note that under  $H_0^{(3)}$  we have for any  $j \in \{1, \dots, m_n\}$

$$\begin{aligned} & |H_j(U_{\cdot n}(t)) - t| \\ & \leq |H_j(U_{\cdot n}(t)) - H_j(U_{\cdot}(t))| + |H_j(U_{\cdot}(t)) - H_j(U_j(t))| \\ & \leq |U_{\cdot n}(t) - U_{\cdot}(t)| h_j(\tilde{U}_n(t)) + |U_{\cdot}(t) - U_j(t)| h_j(\tilde{U}_j(t)), \end{aligned} \tag{2.21}$$

where  $\tilde{U}_n(t) \in (U_{\cdot n}(t) \wedge U_{\cdot}(t), U_{\cdot n}(t) \vee U_{\cdot}(t))$  and  $\tilde{U}_j(t) \in (U_j(t) \wedge U_{\cdot}(t), U_j(t) \vee U_{\cdot}(t))$ ,  $1 \leq j \leq m_n$ .

Now using (2.2) and (2.10), and the fact that under  $H_0^{(3)}$  from (C.6) and  $U_j(t) \geq U_{\cdot}(t)$  it follows that

$$\sqrt{n\mu_1 \log m_n} \max_{1 \leq j \leq m_n} |U_j(t) - U_{\cdot}(t)| \rightarrow 0 \quad (n \rightarrow \infty), \tag{2.22}$$

we now find that as  $n \rightarrow \infty$

$$\begin{aligned} & \sqrt{\log m_n} \max_{1 \leq j \leq m_n} (\sqrt{n\mu_{j,n}} |U_{\cdot}(t) - U_j(t)|) h_j(\tilde{U}_j(t)) \\ & = O_p \left( \sqrt{n\mu_1 \log m_n} \max_{1 \leq j \leq m_n} |U_j(t) - U_{\cdot}(t)| \right) = o_p(1). \end{aligned} \tag{2.23}$$

On the other hand, by (2.2) and (2.10), as  $n \rightarrow \infty$

$$\begin{aligned} & \sqrt{\log m_n} \max_{1 \leq j \leq m_n} (\sqrt{n\mu_{j,n}} h_j(\tilde{U}_n(t))) |U_{\cdot n}(t) - U_{\cdot}(t)| \\ & = O_p(1) \sqrt{n\mu_1 \log m_n} |U_{\cdot n}(t) - U_{\cdot}(t)|. \end{aligned} \tag{2.24}$$

Furthermore,

$$|U_{\cdot n}(t) - U_{\cdot}(t)| \leq \sum_{j=1}^{m_n} \mu_{j,n} |U_j(t) - U_{\cdot}(t)| + \left| \sum_{j=1}^{m_n} \mu_{j,n} (U_{j,n}(t) - U_j(t)) \right|. \tag{2.25}$$

Now

$$\sqrt{n\mu_1 \log m_n} \sum_{j=1}^{m_n} \mu_{j,n} |U_j(t) - U_{\cdot}(t)| \leq \sqrt{n\mu_1 \log m_n} \max_{1 \leq j \leq m_n} |U_j(t) - U_{\cdot}(t)| \tag{2.26}$$

which tends to zero by (2.22).

The mean value theorem yields that for some  $\tilde{t}_{j,n} \in (H_j(U_{j,n}(t)) \wedge t, H_j(U_{j,n}(t)) \vee t)$

$$\begin{aligned} \sum_{j=1}^{m_n} \mu_{j,n} (U_{j,n}(t) - U_j(t)) &= \sum_{j=1}^{m_n} \mu_{j,n} (U_j(H_j(U_{j,n}(t))) - U_j(t)) \\ &= \sum_{j=1}^{m_n} \mu_{j,n} u_j(\tilde{t}_{j,n})(H_j(U_{j,n}(t)) - t). \end{aligned} \tag{2.27}$$

We now show that in this last expression we can replace  $\mu_{j,n}u_j(\tilde{t}_{j,n})$  by  $\sqrt{\mu_j}u_j(t)\sqrt{\mu_{j,n}}$ . To this end we first remark that using (2.2) and (2.12) we have as  $n \rightarrow \infty$  that

$$\begin{aligned} \max_{1 \leq j \leq m_n} \sup_{t \in (0,1)} |H_j(U_{j,n}(t)) - t| &= \max_{1 \leq j \leq m_n} \sup_{t \in (0,1)} |H_{j,n}(U_j(t)) - t| \\ &= (n\mu_{m_n})^{-1/2} \max_{1 \leq j \leq m_n} \|\tilde{B}_{j,n}\| + O_P((n\mu_{m_n})^{-5/8}(\log m_n)^{1/8}(\log n)^{1/2}) \\ &= O_P((n\mu_{m_n})^{-1/2}(\log m_n)^{1/2}) + (n\mu_{m_n})^{-5/8}(\log m_n)^{1/8}(\log n)^{1/2} \\ &= O_P((n\mu_{m_n})^{-1/2}(\log m_n)^{1/2}). \end{aligned} \tag{2.28}$$

A similar argument yields that

$$\max_{1 \leq j \leq m_n} \sqrt{n\mu_{j,n}} \sup_{t \in (0,1)} |H_j(U_{j,n}(t)) - t| = O_P((\log m_n)^{1/2}) \quad (n \rightarrow \infty). \tag{2.29}$$

Using (C.5) we obtain that

$$\begin{aligned} |u_j(\tilde{t}_{j,n}) - u_j(t)| &\leq c_3 |\tilde{t}_{j,n} - t| \leq c_3 \max_{1 \leq j \leq m_n} |H_j(U_{j,n}(t)) - t| \\ &= O_P((n\mu_{m_n})^{-1/2}(\log m_n)^{1/2}) \quad (n \rightarrow \infty). \end{aligned}$$

Hence, with (2.29) and the rate condition in the statement of the theorem we have that

$$\begin{aligned} \sqrt{n\mu_1 \log m_n} \sum_{j=1}^{m_n} \mu_{j,n} |u_j(\tilde{t}_{j,n}) - u_j(t)| |H_j(U_{j,n}(t)) - t| \\ &= O_P(\mu_1^{1/2}(n\mu_{m_n})^{-1/2} \log m_n) \sum_{j=1}^{m_n} \mu_{j,n}^{1/2} (\sqrt{n\mu_{j,n}} |H_j(U_{j,n}(t)) - t|) \\ &= O_P \left( (\log m_n)^{3/2} (n\mu_{m_n})^{-1/2} \mu_1^{1/2} \left( \sum_{j=1}^{m_n} \mu_{j,n}^{1/2} \right) \right) \\ &= o_P(1) \quad (n \rightarrow \infty). \end{aligned} \tag{2.30}$$

Next, using (C.5), (2.29), and  $\max_{1 \leq j \leq m_n} |\mu_{j,n}^{1/2} - \mu_j^{1/2}| = O_P((\log n)^{1/2} n^{-1/2})$  ( $n \rightarrow \infty$ ) we find

$$\begin{aligned} \sqrt{n\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_{j,n}^{1/2} (\mu_{j,n}^{1/2} - \mu_j^{1/2}) u_j(t) (H_j(U_{j,n}(t)) - t) \right| \\ &= O_P \left( \sqrt{\frac{\mu_1 \log m_n \log n}{n}} \sum_{j=1}^{m_n} u_j(t) |\sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - t)| \right) \\ &= O_P \left( \log m_n \sqrt{\frac{\mu_1 \log n}{n}} m_n \right) \\ &= O_P \left( \log m_n \sqrt{\frac{\mu_1 \log n}{n\mu_{m_n}}} \sum_{j=1}^{m_n} \mu_j^{1/2} \right) \quad (n \rightarrow \infty), \end{aligned} \tag{2.31}$$

which is  $o_p(1)$  as  $n \rightarrow \infty$  because of the rate conditions in the statement of the theorem.

From (2.27), (2.30) and (2.31) it now remains to show that

$$\sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - t) \right| \xrightarrow{P} 0 \tag{2.32}$$

as  $n \rightarrow \infty$  in order to verify (2.13).

To this end, as  $|H_{j,n}(U_{j,n}(t)) - t| \leq (n\mu_{j,n})^{-1}$  a.s., the expression on the left-hand side of (2.32) is equal to

$$\begin{aligned} & \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - H_{j,n}(U_{j,n}(t))) \right| \\ & + O_p \left( \left( \sum_{j=1}^{m_n} \mu_j^{1/2} \right) \sqrt{\frac{\mu_1 \log m_n}{n\mu_{m_n}}} \right) \quad (n \rightarrow \infty). \end{aligned} \tag{2.33}$$

Now, by (2.11),

$$\begin{aligned} & \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - H_{j,n}(U_{j,n}(t))) \right| \\ & = \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(H_j(U_{j,n}(t))) \right| \\ & + O_p \left( \mu_1^{1/2} \left( \sum_{j=1}^{m_n} \mu_j^{1/2} \right) (n\mu_{m_n})^{-1/8} (\log m_n)^{5/8} (\log n)^{1/2} \right). \end{aligned} \tag{2.34}$$

Using the modulus of continuity behaviour of Brownian bridges together with (2.27), we get

$$\begin{aligned} & \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(H_j(U_{j,n}(t))) \right| \\ & = \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(t) \right| \\ & + O_p \left( \mu_1^{1/2} \left( \sum_{j=1}^{m_n} \mu_j^{1/2} \right) (n\mu_{m_n})^{-1/4} (\log n)^{1/2} (\log m_n)^{3/4} \right). \end{aligned} \tag{2.35}$$

Observe that because of the independence of the  $\tilde{B}_{j,n}$  we have that

$$\sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(t) \sim \mathcal{N} \left( 0, t(1-t) \sum_{j=1}^{m_n} \mu_j u_j^2(t) \right).$$

With (C.5)

$$t(1-t) \sum_{j=1}^{m_n} \mu_j u_j^2(t) = O(1) \quad (n \rightarrow \infty)$$

and hence

$$\sqrt{\mu_1 \log m_n} \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(t) \xrightarrow{P} 0 \quad (n \rightarrow \infty). \tag{2.36}$$

Statements (2.33)–(2.36) yield (2.32), and (2.13) follows from (2.21)–(2.27) and (2.30)–(2.32).

Finally, statement (2.14) follows by (2.13), the behaviour of the modulus of continuity of Brownian bridges, and the independence of the  $\tilde{B}_{j,n}$  ( $j = 1, \dots, m_n$ ). This concludes the proof of Theorem 3.  $\square$

**Proof of Corollary 3.** It suffices to show that, under  $H_0^{(3)}$ ,  $\sqrt{n\mu_1 \log m_n} \max_{1 \leq j \leq m_n} (U_j(t) - U(t)) \rightarrow 0$  ( $n \rightarrow \infty$ ) is implied by (C.7), (C.8) and the rate  $n\mu_1 \log m_n (\max_{1 \leq j \leq m_n} \text{diam}(A_{j,n}))^4 \rightarrow 0$  ( $n \rightarrow \infty$ ).

Let  $K_x = [a_x, b_x]$  denote the shortt pertaining to  $F_x$ , let  $\alpha_j = \inf_{x \in A_{j,n}} a_x$ ,  $\tilde{\beta}_j = \sup_{x \in A_{j,n}} b_x$ , and set

$$\beta_j = \alpha_j + U(t), \quad \tilde{\alpha}_j = \tilde{\beta}_j - U(t).$$

Let  $a$  be such that  $\alpha_j \leq a < a + U(t) \leq \tilde{\beta}_j$ . A Taylor expansion, using  $f_x(a_x) = f_x(b_x)$  and (C.8), yields that for some  $\tilde{a}_x \in (a_x \wedge a, a_x \vee a)$  and  $\tilde{b}_x \in (b_x \wedge (a + U(t)), b_x \vee (a + U(t)))$  we have

$$\begin{aligned} t - F_x([a, a + U(t)]) &= (F_x(a) - F_x(a_x)) - (F_x(a + U(t)) - F_x(b_x)) \\ &= \frac{1}{2}(a - a_x)^2 f'_x(\tilde{a}_x) - \frac{1}{2}(a + U(t) - b_x)^2 f'_x(\tilde{b}_x) \\ &\leq c_6(\alpha_j - \tilde{\alpha}_j)^2, \end{aligned}$$

and hence,

$$t - F_j([a, a + U(t)]) \leq \max_{1 \leq j \leq m_n} (\alpha_j - \tilde{\alpha}_j)^2 c_6 =: v_n.$$

Set  $\eta = (U_j(t) - U(t))/v_n$ . Since  $U_j(t) \geq U(t)$ , we have  $\eta \geq 0$ . Observe that for  $y_1 \in [\tilde{\alpha}_j, \beta_j]$  and  $y_2 \leq \alpha_j$  or  $y_2 \geq \tilde{\beta}_j$  we have  $f_j(y_1) \geq f_j(y_2)$ . Hence, it readily follows that  $[\tilde{\alpha}_j, \beta_j] \subset K_j$ . This means that we can find an  $a$  as above such that  $K_j = [a - \eta v_n, a + U(t)]$  or such that  $K_j = [a, a + U(t) + \eta v_n]$ .

Without loss of generality assume the first equality holds. Observe that the second condition in (C.5) implies that

$$\liminf_{n \rightarrow \infty} \min_{1 \leq j \leq m_n} \inf_{y \in [a_n, b_{j,t}]} f_j(y) > 1/c_4.$$

Hence,

$$0 = t - F_j([a - \eta v_n, a + U.(t)]) \leq v_n - F_j([a - \eta v_n, a]) \leq v_n(1 - \eta/c_4),$$

which (when  $v_n > 0$ ) implies  $\eta \leq c_4$ . This, in combination with  $v_n \sqrt{n\mu_1 \log m_n} \rightarrow 0$  ( $n \rightarrow \infty$ ), completes the proof.  $\square$

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