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Robust Equilibria in Indefinite Linear-Quadratic Differential Games [1]

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Abstract. Equilibria in dynamic games are often formulated under the assumption that players have full knowledge of the dynamics they are subject to. Here we formulate equilibria in which players are looking for robustness and take model uncertainty explicitly into account in their decisions. Specifically we consider feedback Nash equilibria in indefinite linear-quadratic differential games on an infinite time horizon. Model uncertainty is represented by a malevolent input which is subject to a cost penalty or to a direct bound. We derive conditions for the existence of robust equilibria in terms of solutions of sets of algebraic Riccati equations.

Key Words. Feedback Nash equilibrium, robust design, linear-quadratic differential games, soft-constrained differential games, risk sensitivity.

1 Introduction

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. In this paper we add a fourth aspect, namely robustness with respect to variability in the environment. In usual formulations of dynamic games, a set of differential or difference equations is specified including input functions that are controlled by the players, and players are assumed to optimize a criterion over time. The dynamic model is supposed to be an exact representation of the environment in which the players act; optimization takes place with no regard of possible deviations. It can safely be assumed, however, that agents in reality follow a different strategy. If an accurate model can be formed at all, it would in general be complicated and difficult to handle. Moreover it may be unwise to optimize on the basis of a too detailed model, in view of possible changes in dynamics that may take place in the course of time and that may be hard to predict. It makes more sense for agents to work on the basis of a relatively simple model and to look for strategies that are robust with respect to deviations between the model and reality. In an economic context, the importance of incorporating aversion to specification uncertainty has been stressed for instance by Ref. 1.

In control theory, an extensive theory of robust design is already in place; see Ref. 2 for a survey. We use this background to arrive at suitable ways of describing aversion to model risk in a dynamic game context. We assume linear dynamics and quadratic cost functions. These assumptions are reasonable for situations of dynamic quasi-equilibrium, where no large excursions of the state vector are to be expected; also from the point of view of development of theory, the linear-quadratic case is a natural place to start. Following a pattern that has become standard in control theory, we introduce a malevolent disturbance input that will be used in the modeling of aversion to specification uncertainty. Our dynamic model is therefore the following:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) + Ew(t), \;\; x(0) = x_0 \tag{1}$$

where N is the number of players, x is the n-dimensional state of the system, u_i contains the m_i (control) variables that are chosen by player i, w is a q-dimensional disturbance vector affecting the system, x_0 is the initial state of the system, and A, B_i, and E are constant matrices containing system parameters. By combining (1) with an equation that expresses the disturbance $w(t)$ as a function of the state $x(t)$, one can express deviations from the nominal dynamics represented by the matrix A.

We have to specify the strategy space and the information structure available to players. In this paper we will assume a full state information structure, and we restrict the players to stabilizing constant linear feedback strategies. So we shall only consider controls u_i of the type $u_i = F_i x$, with $F_i \in I\!R^{m_i \times n}$, and where $(F_1, \cdots, F_N)$ belongs to the set

$$\mathcal{F} := \{F = (F_1, \cdots, F_N) \mid A + \textstyle\sum_{i=1}^{N} B_i F_i \text{ is stable}\}.$$

The stabilization constraint is imposed to ensure the finiteness of the infinite-horizon cost integrals that we will consider; also, the assumption helps to justify our basic supposition that the state vector remains close to the origin. Obviously the constraint is a bit unwieldy since it introduces dependence between the strategy spaces of the players. However, we will focus below on equilibria in which the inequalities that ensure the stability property are inactive constraints. It will be a standing assumption that the set $\mathcal{F}$ is non-empty; a necessary and sufficient condition for this to hold is that the matrix pair $(A, [B_1 \cdots B_N])$ is stabilizable. Given that we work below with an infinite horizon, restraining the players to constant feedback strategies seems reasonable; to prescribe linearity may also seem natural in the linear-quadratic context that we assume, although there is no way to exclude *a priori* equilibria in nonlinear feedback strategies. Questions regarding the existence of such equilibria are outside the scope of this paper.

We now come to the formulation of the objective functions of the players. Our starting point is the usual quadratic criterion which assigns to player i the cost function

$$J_i := \int_0^\infty \{x(t)^T Q_i x(t) + \sum_{j=1}^N u_j(t)^T R_{ij} u_j(t)\} dt. \tag{2}$$

Here, Q_i is symmetric and R_{ii} is positive definite for all $i = 1, \ldots, N$. In many applications, state changes that are beneficial to some players may be harmful to other players, and so we allow for the state weighting matrices Q_i to be indefinite. This is in contrast with the bulk of the control literature, in which the state weighting matrix is assumed to be positive definite. Allowing the matrices Q_i to be indefinite brings considerable technical complications, but we believe that in the multi-player context this generality is natural. In particular we are able in this way to formulate two-person games that are zero-sum as far as the state variable is concerned. On the other hand, the term $u_i^T(t) R_{ii} u_i(t)$ is interpreted as a measure of the effort expended by player i, and so we let R_{ii} be positive definite. Here we stay in line with standard control theory.[5] Under our assumption that the players use constant linear feedbacks, the criterion in (2) may be rewritten as

$$J_i := \int_0^\infty \{x^T (Q_i + \sum_{j=1}^N F_j^T R_{ij} F_j) x\} dt \tag{3}$$

where F_i is the feedback chosen by player i. Written in the above form, the criterion may be looked at as a function of the initial condition x_0 and the state feedbacks F_i.

The description of the players' objectives given above needs to be modified in order to express a desire for robustness. Here we consider two alternatives, which

[5] We note though that for some problems (where the control weighting term is not interpreted as spent energy) it may be natural to let the matrix R_{ii} be indefinite. It has been shown recently in a stochastic context that the resulting control problem may still be well-posed (Ref. 3). Here we do not consider this generalization, however.

both are well known in control theory. The first alternative consists of modifying the criterion (3) to

$$\bar{J}_i^{SC}(F_1, \cdots, F_N, x_0) := \sup_{w \in L_2^q(0,\infty)} J_i(F_1, \cdots, F_N, w, x_0) \tag{4}$$

where

$$J_i(F_1, \cdots, F_N, w, x_0) := \int_0^\infty \{x^T(Q_i + \sum_{j=1}^{N} F_j^T R_{ij} F_j)x - w^T V_i w\} dt. \tag{5}$$

The weighting matrix V_i is symmetric and positive definite for all $i = 1, \ldots, N$. Because it occurs with a minus sign in (5), this matrix constrains the disturbance vector w in an indirect way so that it can be used to describe the aversion to model risk of player i. Specifically, if the quantity $w^T V_i w$ is large for a vector $w \in R^q$, this means that player i does not expect large deviations of the nominal dynamics in the direction of Ew. In most of the paper we use this so-called "soft-constrained" formulation, which has been used extensively in control theory. Note that since we do not assume positive definiteness of the state weighting matrix, our development extends even in the one-player case the standard results that may be found for instance in Refs. 2, 4, 5, 6 (Section 20.2), 7, 8 (Section 6.6).

We also spend attention on a second way of describing aversion to model risk: again a minmax problem is solved, but the disturbance is not restrained by a cost term but simply by a direct norm bound. This formulation is sometimes referred to as the disturbance attenuation problem, or the problem with hard-bounded uncertainty; see Refs. 2 and 9. In control theory (see for instance Ref. 5), this problem is usually considered for a zero initial state. Here we carry out an analysis allowing a nonzero initial state, extending earlier results by Ref. 9 to the infinite-horizon and multiple-player context.

The remainder of the paper is organized as follows. The next section considers some preliminaries. Section 3 treats the soft-constrained case whereas Section 4 considers the hard-bounded case. The paper ends with some concluding remarks.

2 Preliminaries

The following notations and terminologies will be used throughout this paper.

- To indicate that a symmetric matrix P is positive (semi) definite, we write $P > 0$ ($P \geq 0$).

- Given a positive definite matrix P of size $n \times n$, the P-norm of a vector $a \in R^n$ is denoted by $\|a\|_P := (a^T P a)^{1/2}$.

- For an N-tuple $\gamma = (\gamma_1, \ldots, \gamma_N) \in \Gamma_1 \times \ldots \Gamma_N$ for given sets Γ_i, we write $\gamma_{-i}(\alpha) := (\gamma_1, \ldots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \ldots, \gamma_N)$, with $\alpha \in \Gamma_i$.

- We use $\sum_{i \neq j}^N a_i$ as an abbreviation for $a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_N$.

- The space of $\mathbb{R}^k$-valued functions that are quadratically integrable on $(0, \infty)$ is denoted by $L_2^k(0, \infty)$. With the usual inner product and induced norm, we denote the norm of a vector $v \in L_2^k(0, \infty)$ by $\|v\|$.

- A matrix A is called stable if all its eigenvalues are in the open left-half complex plane.

- For matrices A and B_i, $i = 1, \ldots, N$, the set $\mathcal{F}_N$ is defined by

$$\mathcal{F}_N := \{(F_1, \ldots, F_N) \mid A + \sum_{i=1}^{N} B_i F_i \text{ is stable}\}.$$

- For matrices B_i, R_{ij}, E and V_i matrices S_i, S_{ij} and M_i are defined as follows:

$$S_i := B_i R_{ii}^{-1} B_i^T, \quad S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T, \quad M_i := E V_i^{-1} E^T.$$

- Consider the algebraic Riccati equation

$$Q + A^T X + XA + XPX = 0 \qquad \text{(ARE)}$$

where Q and P are symmetric. A symmetric solution X is called a *stabilizing* solution of (ARE) if $A + PX$ is stable. It is well-known (see e.g. Ref. 7, Theorem 13.5) that if such a solution exists, it is unique.

3 Soft-Constrained Nash Equilibria

The robust equilibrium concepts to be introduced in this and the next section are both inspired by the game-theoretic approach to H_∞ control theory. In that theory the uncertainty in a system is expressed by an additive disturbance term in the differential equation. As outlined in the introduction we take a similar approach in an N-player context, i.e. we consider the differential equation

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i + Ew, \quad x(0) = x_0 \qquad (6)$$

where $w \in L_2^q(0, \infty)$ represents the unknown disturbance. We assume that the information structure of the players is a feedback pattern and that they are restricted to linear time-invariant stabilizing strategies, i.e. their control functions are of the form

$$u_i = F_i x, \quad (F_1, \ldots, F_N) \in \mathcal{F}_N. \qquad (7)$$

As motivated in the introduction we consider in this section the cost functions (4). These adjusted cost functions do not depend on the disturbance term. They only depend on the strategies and the initial state. According to the feedback information structure a set of equilibrium strategies should be independent of the initial state. Furthermore, the strategies should satisfy the usual equilibrium inequalities.

A formal definition is given below.

Definition 3.1
An N-tuple $\bar{F} = (\bar{F}_1, \dots, \bar{F}_N) \in \mathcal{F}_N$ is called a *soft-constrained Nash equilibrium* if for each $i = 1, \dots, N$ the following inequality holds:

$$\bar{J}_i^{SC}(\bar{F}, x_0) \leq \bar{J}_i^{SC}(\bar{F}_{-i}(F), x_0) \tag{8}$$

for all $x_0 \in \mathbb{R}^n$ and for all $F \in \mathbb{R}^{m_i \times n}$ that satisfy $\bar{F}_{-i}(F) \in \mathcal{F}_N$. ¤

In the next subsection we will discuss the one-player case. The results obtained for that particular case are the basis for the derivation of results for the general N-player case. We stress here the point again that in contrast to the usual H_∞-approach (see e.g. Ref. 5) we consider a cost criterion without assuming the state weighting matrix to be positive semidefinite. The general N-player case is dealt with in Subsection 3.2. In Subsection 3.3 we treat the scalar case in more detail.

3.1 One-Player Case

In this subsection we study the one-player case, i.e. we consider

$$\dot{x} = (A + BF)x + Ew, \; x(0) = x_0, \tag{9}$$

with (A, B) stabilizable, $F \in \mathcal{F}$ and

$$J(F, w, x_0) = \int_0^\infty (x^T(Q + F^T RF)x - w^T Vw)dt. \tag{10}$$

The matrices Q, R and V are symmetric, $R > 0$, and $V > 0$. The problem is to determine for each $x_0 \in \mathbb{R}^n$ the value

$$\inf_{F \in \mathcal{F}} \sup_{w \in L_2^q(0,\infty)} J(F, w, x_0). \tag{11}$$

Furthermore, if the infimum is finite, it is of interest to determine whether there is a feedback matrix $\bar{F} \in \mathcal{F}$ that achieves the infimum, and to determine all matrices that have this property. This soft-constrained differential game can also be interpreted as a model for a situation where the controller designer is minimizing the criterion (10) by choosing an appropriate $F \in \mathcal{F}$, while the uncertainty is maximizing the same criterion by choosing an appropriate $w \in L_2^q(0, \infty)$.

A necessary condition for the expression in (11) to be finite is that the supremum $\sup_{w \in L_2^q(0,\infty)} J(F, w, x_0)$ is finite for at least one $F \in \mathcal{F}$. This condition is not sufficient (see Remark 3.1 (iii) below). We now first present a lemma that gives necessary and sufficient conditions for the supremum in (11) to attain a finite value for a given stabilizing feedback matrix F. The lemma will be used later on in Theorem 3.1, which provides a sufficient condition under which the soft-constrained differential game associated to (9)–(10) has a saddle point.

Lemma 3.1 Let A be stable. Consider the system

$$\dot{x} = Ax + Ew \tag{12}$$

and the corresponding cost functional

$$\phi(w, x_0) := \int_0^\infty (x^\top Qx - w^\top Vw)dt,\ x(0) = x_0,$$

with $Q = Q^\top$ and $V > 0$. Let $M := EV^{-1}E^\top$. The following conditions are equivalent.

(i) For each $x_0 \in I\!R^n$ there exists a $\bar{w} \in L_2^q(0, \infty)$ such that $\phi(w, x_0) \leq \phi(\bar{w}, x_0)$.

(ii) The Hamiltonian matrix

$$H := \begin{pmatrix} A & M \\ -Q & -A^\top \end{pmatrix}$$

has no eigenvalues on the imaginary axis.

(iii) The algebraic Riccati equation

$$Q + A^\top X + XA + XMX = 0 \tag{13}$$

has a stabilizing solution (see (ARE)).

If these conditions hold, the maximum of $\phi(w, x_0)$ is uniquely attained by

$$\bar{w}(t) := V^{-1}E^\top Xe^{t(A+MX)}x_0$$

where X is the stabilizing solution of (13). Furthermore we have $\phi(\bar{w}, x_0) = x_0^\top Xx_0$.

Proof We will show the following implications: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. The second part of the lemma follows from the proof that (iii) implies (i).

$(i) \Rightarrow (ii)$: Denote the state trajectory corresponding to $\bar{w}$ by $\bar{x}$. Then the maximum principle (see e.g. Ref. 10) implies that there exists a costate variable p such that

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + E\bar{w}, \quad \bar{x}(0) = x_0 \\ \dot{p} &= -Q\bar{x} - A^\top p \\ \bar{w}(t) &= \arg\max_{w \in I\!R^q} (\bar{x}^\top Q\bar{x} - w^\top Vw + 2p^\top (A\bar{x} + Ew)). \end{aligned}$$

A completion of squares shows that

$$-w^\top Vw + 2p^\top Ew = -(w - V^{-1}E^\top p)^\top V(w - V^{-1}E^\top p) - p^\top Mp.$$

Since $V > 0$, it follows that $\bar{w} = V^{-1}E^\top p$. Hence

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & M \\ -Q & -A^\top \end{pmatrix} \begin{pmatrix} \bar{x} \\ p \end{pmatrix} = H \begin{pmatrix} \bar{x} \\ p \end{pmatrix}, \quad \bar{x}(0) = x_0.$$

Since $\bar{w} \in L_2^q(0,\infty)$ and A is stable, $\bar{x}(t) \to 0$ for $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. This shows that the spectral subspace corresponding to the eigenvalues in the open left-half plane has at least dimension n. Since H is an Hamiltonian matrix this implies that H has no eigenvalues on the imaginary axis.

(ii) $\Rightarrow$ *(iii)*: This implication follows from e.g. Ref. 7, Theorem 13.6.

(iii) $\Rightarrow$ *(i)*: Let $w \in L_2^q(0,\infty)$ and x be generated by (12). Since A is stable we have that $x(t) \to 0$ for $t \to \infty$. Hence a completion of the squares shows that

$$\begin{aligned}\phi(w,x_0) &= \int_0^\infty (x^T Q x - w^T V w + \frac{d}{dt}x^T X x - \frac{d}{dt}x^T X x)dt \\ &= x_0^T X x_0 - \int_0^\infty \|w - V^{-1}E^T X x\|_V^2 \, dt.\end{aligned}$$

Hence $\phi(w,x_0) \le x_0^T X x_0$ and equality holds if and only if $w = V^{-1}E^T X x$. Substituting this in (12) shows that $\phi(\cdot, x_0)$ is uniquely maximized by $\bar{w}$. ¤

Remark 3.1
(i) Note that the lemma does not imply that if the Hamiltonian matrix H has eigenvalues on the imaginary axis, the cost will be unbounded. Consider e.g. $a = -1; q = r = e = v = 1$. Then $X = 1$ is the unique (though not stabilizing) solution of (13). A completion of squares (see proof above) shows that $\phi(w,x_0) \le x_0^2$. Furthermore, it is easily verified that with $w = (1-\varepsilon)x$, for an arbitrarily small positive ε, we can approach this cost arbitrarily closely.

(ii) From the above example we also immediately learn that if there exists a $\bar{F} \in \mathcal{F}$ such that $\sup_{w \in L_2^q(0,\infty)} J(F,w,x_0)$ is finite, this does not imply that there is an open neighborhood of $\bar{F} \in \mathcal{F}$ for which the supremum is also finite. Take e.g. $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $\bar{f} = -1$, $q = r = e = v = 1$. Then for every $\varepsilon > 0$, $\sup_{w \in L_2^q(0,\infty)} J((\bar{f}+\varepsilon), w, x_0)$ is infinite.

(iii) Since we did not assume that the state weighting Q in (10) is nonnegative definite, it may well happen that the value of the expression in (11) is $-\infty$. For a simple example, consider the scalar case with $E = 0$, $A = -1$, $B = R = V = 1$, and $Q = -2$. ¤

Motivated by this result we define for each $F \in \mathcal{F}$ the Hamiltonian matrix

$$H_F := \begin{pmatrix} A + BF & M \\ -Q - F^T R F & -(A+BF)^T \end{pmatrix}. \tag{14}$$

and introduce the set

$$\bar{\mathcal{F}} := \{F \in \mathcal{F} \mid H_F \text{ has no eigenvalues on the imaginary axis}\}. \tag{15}$$

The following lemma provides a convenient expression for the objective function of the game that we consider.

Lemma 3.2 Consider (9)–(10) with $F \in \mathcal{F}$ and $w \in L_2^q(0,\infty)$. Let X be an arbitrary symmetric matrix; then

$$J(F,w,x_0) = x_0^\top X x_0 + \int_0^\infty (x^\top (Q + A^\top X + XA - XSX + XMX)x + \|(F + R^{-1}B^\top X)x\|_R^2 - \|w - V^{-1}E^\top Xx\|_V^2)\, dt \tag{16}$$

where $S := BR^{-1}B^\top$ and $M := EV^{-1}E^\top$.

Proof Since $F \in \mathcal{F}$ and $w \in L_2^q(0,\infty)$, $x(t) \to 0$ for $t \to \infty$. Thus

$$\begin{aligned} J(F,w,x_0) &= \int_0^\infty (x^\top (Q + F^\top RF)x - w^\top Vw + \frac{d}{dt}x^\top Xx - \frac{d}{dt}x^\top Xx)\, dt \\ &= x_0^\top X x_0 + \int_0^\infty (x^\top (Q + A^\top X + XA)x + x^\top F^\top RFx \\ &\qquad + 2x^\top F^\top B^\top Xx - w^\top Vw + 2w^\top E^\top Xx)\, dt. \end{aligned}$$

Hence, the two completions of the squares

$$x^\top F^\top RFx + 2x^\top F^\top B^\top Xx = \|(F + R^{-1}B^\top X)x\|_R^2 - x^\top XSXx$$

and

$$-w^\top Vw + 2w^\top E^\top Xx = -\|w - V^{-1}E^\top Xx\|_V^2 + x^\top XMXx$$

show that (16) holds. ¤

The above lemma shows that if X satisfies the algebraic Riccati equation (18) below, an optimal choice for the minimizing player is $-R^{-1}B^\top X$, which is an admissible choice if X is the stabilizing solution of this equation. If the maximizing player would be restricted to choose linear state feedback matrices as well, his optimal choice would be the state feedback matrix $V^{-1}E^\top X$. The following theorem shows that under the open-loop information structure, the optimal choice for the maximizing player, given that the minimizing player chooses $-R^{-1}B^\top X$, can indeed be obtained from the feedback law $x \to V^{-1}E^\top Xx$. This theorem provides a set of sufficient conditions for a saddlepoint solution to exist. Consequently, it also generates a solution of problem (11).

To motivate the conditions in the theorem, consider for the moment the scalar case, without going into too much detail. We replace the upper case symbols for matrices by their lower case equivalents to emphasize that these matrices are now just real numbers. Under the assumption that the conditions of Lemma 3.1 are satisfied, the equation (cf. (13))

$$mx^2 + 2(a + bf)x + 2fr = 0 \tag{17}$$

holds for each f, and we have $\sup_{w \in L_2^q(0,\infty)} J(f,w,x_0) = x(f)x_0^2$. In particular, the minimizing $\bar{f}$ satisfies $x'(\bar{f}) = 0$. Differentiation of (17) with respect to f then yields that $\bar{f} = -bx/r$. Substitution of this relationship into (17) shows that x should be a stabilizing solution of $(m-s)x^2 + 2ax + q = 0$ (see (18)). On the other hand, to guarantee that $\bar{f}$ indeed yields a minimum, the condition $-a/b \notin \bar{\mathcal{F}}$ suffices; this is equivalent to $a^2 + qs > 0$. This requirement is the scalar version of condition (19) below.

Theorem 3.1 Consider (9)–(10) and let the matrices S and M be defined as in Lemma 3.2. Assume that the algebraic Riccati equation

$$Q + A^\mathsf{T} X + XA - XSX + XMX = 0 \tag{18}$$

has a stabilizing solution X and that additionally $A - SX$ is stable. Furthermore, assume that there exists a real symmetric $n \times n$ symmetric matrix Y that satisfies the matrix inequality

$$Q + A^\mathsf{T} Y + YA - YSY \geq 0. \tag{19}$$

Define $\bar{F} := -R^{-1}B^\mathsf{T} X$ and $\bar{w}(t) := V^{-1}E^\mathsf{T} X e^{t(A-SX+MX)} x_0$. Then the matrix $\bar{F}$ belongs to $\bar{\mathcal{F}}$, the function $\bar{w}$ is in $L_2^q(0,\infty)$, and for all $F \in \mathcal{F}$ and $w \in L_2^q(0,\infty)$ we have

$$J(\bar{F}, w, x_0) \leq J(\bar{F}, \bar{w}, x_0) \leq J(F, \bar{w}, x_0).$$

Moreover, $J(\bar{F}, \bar{w}, x_0) = x_0^\mathsf{T} X x_0$.

Proof The matrices $A - SX$ and $A - SX + MX$ are stable by assumption, which implies that $\bar{F} \in \mathcal{F}$ and $\bar{w} \in L_2^q(0,\infty)$, respectively. According to Lemma 3.2 we have

$$J(F, w, x_0) = x_0^\mathsf{T} X x_0 + \int_0^\infty (\|(F - \bar{F})x\|_R^2 - \|w - V^{-1}E^\mathsf{T} X x\|_V^2) dt.$$

From this it follows that

$$J(\bar{F}, w, x_0) \;=\; x_0^\mathsf{T} X x_0 - \int_0^\infty (\|w - V^{-1}E^\mathsf{T} X \tilde{x}\|_V^2) dt \;\leq\; x_0^\mathsf{T} X x_0,$$

where $\tilde{x}$ is generated by $\dot{\tilde{x}} = (A + B\bar{F})\tilde{x} + Ew$, $\tilde{x}(0) = x_0$. Furthermore, if $J(\bar{F}, w, x_0) = x_0^\mathsf{T} X x_0$ then $w = \bar{w}$. Hence $J(\bar{F}, w, x_0) < x_0^\mathsf{T} X x_0$ for all $w \neq \bar{w}$, and $J(\bar{F}, \bar{w}, x_0) = x_0^\mathsf{T} X x_0$. This, obviously, implies also that $\bar{F} \in \bar{\mathcal{F}}$.

Next, we show that $J(F, \bar{w}, x_0) \geq J(\bar{F}, \bar{w}, x_0)$ for all $F \in \mathcal{F}$. Let $\hat{x}$ and $\bar{x}$ be generated by

$$\dot{\hat{x}} = (A + BF)\hat{x} + E\bar{w}, \quad \hat{x}(0) = x_0$$

and

$$\dot{\bar{x}} = (A + B\bar{F})\bar{x} + E\bar{w}, \quad \bar{x}(0) = x_0$$

respectively. Define furthermore

$$\nu := (\bar{F} - F)\hat{x}, \quad \zeta := \bar{w} - V^{-1}E^\mathsf{T} X \hat{x}.$$

Then $J(F,\bar{w},x_0) - J(\bar{F},\bar{w},x_0) = \int_0^\infty (\|\nu\|_R^2 - \|\zeta\|_V^2)dt$. Introducing $\xi := \bar{x} - \hat{x}$ we have that

$$\dot{\xi} = (A + B\bar{F})\xi + B\nu \tag{20}$$

with $\xi(0) = 0$, and $\zeta = V^{-1}E^\top X\xi$. Since both $\hat{x}$ and $\bar{x}$ belong to $L_2^n(0,\infty)$ it follows that ξ and ν are quadratically integrable as well, which implies that $\xi(t) \to 0$ for $t \to \infty$. So, we conclude that $\int_0^\infty \frac{d}{dt}\xi^\top X\xi dt = 0$. Hence

$$\begin{aligned} J(F,\bar{w},x_0) - J(\bar{F},\bar{w},x_0) &= \int_0^\infty [(\|\nu\|_R^2 - \|\zeta\|_V^2) - \frac{d}{dt}\xi^\top X\xi]\,dt = \\ &= \int_0^\infty (\|\nu\|_R^2 - 2\nu^\top B^\top X\xi - \xi^\top(A^\top X + XA - 2XSX + XMX)\xi)\,dt \\ &= \int_0^\infty (\|\nu - R^{-1}B^\top X\xi\|_R^2 - \xi^\top(A^\top X + XA - XSX + XMX)\xi)\,dt \\ &= \int_0^\infty (\|\nu + \bar{F}\xi\|_R^2 + \xi^\top Q\xi)\,dt. \end{aligned}$$

Next, define $w := \nu + \bar{F}\xi = \bar{F}\bar{x} - F\hat{x}$. Then, (20) shows that $\dot{\xi} = A\xi + Bw$. Since $\xi(0) = 0$ and $\xi(t) \to 0$ for $t \to \infty$ we also have $\int_0^\infty (\frac{d}{dt}\xi^\top Y\xi)dt = 0$. Hence

$$\begin{aligned} J(F,\bar{w},x_0) - J(\bar{F},\bar{w},x_0) &= \int_0^\infty [(\|w\|_R^2 + \|\xi\|_Q^2) + \frac{d}{dt}\xi^\top Y\xi]\,dt = \\ &= \int_0^\infty (w^\top Rw + 2w^\top B^\top Y\xi + \xi^\top(Q + A^\top Y + YA)\xi)\,dt \\ &= \int_0^\infty (\|w + R^{-1}B^\top Y\xi\|_R^2 + \xi^\top(Q + A^\top Y + YA - YSY)\xi)\,dt \geq 0 \end{aligned}$$

where the last inequality follows by assumption. ¤

Note that if $Q \geq 0$, condition (19) is trivially satisfied by choosing $Y = 0$. The following corollary summarizes the consequences of Theorem 3.1 for the problem posed at the beginning of this subsection.

Corollary 3.1 Let the assumptions of Theorem 3.1 hold and let X, $\bar{F}$, and $\bar{w}$ be as in the theorem. We have

$$\min_{F\in\mathcal{F}} \sup_{w\in L_2^q(0,\infty)} J(F,w,x_0) = \max_{w\in L_2^q(0,\infty)} J(\bar{F},w,x_0) = x_0^\top X x_0 \tag{21}$$

and

$$\max_{w\in L_2^q(0,\infty)} \inf_{F\in\mathcal{F}} J(F,w,x_0) = \min_{F\in\mathcal{F}} J(F,\bar{w},x_0) = x_0^\top X x_0. \tag{22}$$

¤

The rest of this section is concerned with the question to what extent it is necessary for the expression (11) to be finite that the algebraic Riccati equation (18) has a symmetric solution such that both $A + MX - SX$ and $A - SX$ are stable. The theorem below shows that this condition must hold if the infimum in (11) is achieved at some $\bar{F} \in \bar{\mathcal{F}}$.

Theorem 3.2 Assume there exists an $\bar{F} \in \bar{\mathcal{F}}$ such that for each $x_0 \in I\!R^n$

$$\min_{F \in \bar{\mathcal{F}}} \sup_{w \in L_2^q(0,\infty)} J(F, w, x_0) = \max_{w \in L_2^q(0,\infty)} J(\bar{F}, w, x_0).$$

Then the algebraic Riccati equation (18) has a stabilizing solution X. Furthermore, the matrix $A - SX$ is stable.

Proof From the assumption it follows that $\bar{F} \in \mathcal{F}$ is such that the Hamiltonian matrix $H_{\bar{F}}$ defined in (14) has no eigenvalues on the imaginary axis. This implies that there is an open neighborhood $O_{\bar{F}} \subset \mathcal{F}$ of $\bar{F}$ such that for all $F \in O_{\bar{F}}$, H_F has no eigenvalues on the imaginary axis. Let $F \in O_{\bar{F}}$ be an arbitrary element. This implies that 3.1.(ii) holds with A, Q and $\phi(w, x_0)$ replaced by $A + BF$, $Q + F^T RF$ and $J(F, w, x_0)$, respectively. Hence, according to this lemma

$$\bar{J}(F, x_0) := \max_{w \in L_2^q(0,\infty)} J(F, w, x_0) = x_0^T \psi(F) x_0$$

where $\psi : O_{\bar{F}} \to I\!R^{n \times n}$ is defined by $\psi(F) := X$, where X is the stabilizing solution of

$$Q + F^T RF + (A + BF)^T X + X(A + BF) + XMX = 0.$$

In Ref. 6, Section 11.3, it is shown that the maximal solution of

$$\tilde{X}(\mu)\tilde{D}(\mu)\tilde{X}(\mu) - \tilde{X}(\mu)\tilde{A}(\mu) - \tilde{A}^T(\mu)\tilde{X}(\mu) - \tilde{C}(\mu) = 0 \tag{23}$$

is a real-analytic function of k real variables $\mu \in \Omega$, where Ω is an open connected set in $I\!R^k$ if (i) $\tilde{A}(\mu)$, $\tilde{C}(\mu)$ and $\tilde{D}(\mu)$ are real-analytic functions of μ, (ii) $\tilde{D}(\mu) \geq 0$, (iii) $(\tilde{A}(\mu), \tilde{D}(\mu))$ is stabilizable, and (iv) the matrix

$$\begin{pmatrix} -\tilde{A}(\mu) & \tilde{D}(\mu) \\ \tilde{C}(\mu) & \tilde{A}^T(\mu) \end{pmatrix}$$

has no eigenvalues on the imaginary axis for all $\mu \in \Omega$. Under the conditions (ii) and (iii), the maximal solution of (23) coincides with the unique solution of (23) for which the spectrum of $\tilde{A}(\mu) - \tilde{D}(\mu)\tilde{X}(\mu)$ lies in the closed left-half plane (see e.g. Ref. 6, Theorem 7.9.3). Note that $-X$ is the maximal solution of (23) with

$$\tilde{A}(\mu) = A + BF,\ \tilde{C}(\mu) = -Q - F^T RF,\ \tilde{D}(\mu) = M \text{ and } \mu = \operatorname{vec} F$$

($\operatorname{vec} F$ denotes the vector obtained from F by stacking the columns of F). Clearly, condition (i) and (ii) hold; condition (iii) follows from the stability of $A + BF$ and condition (iv) follows from the easily verifiable fact that the matrices H_F and

$$\begin{pmatrix} -A - BF & M \\ -Q - F^T RF & (A + BF)^T \end{pmatrix}$$

have the same spectrum. Hence, ψ is an analytic function of F in any open connected subset of $\bar{\mathcal{F}}$. In particular $\bar{J}$ is differentiable with respect to F in such a set. Since $\bar{J}$ attains its minimum at $\bar{F} \in \mathcal{F}$, for each $x_0 \in \mathbb{R}^n$, a differentiation argument shows (see Ref. 11 for details) that the Fréchet derivative $\partial\psi(\bar{F}) = 0$. Next, define the transformation $\Psi : \bar{\mathcal{F}} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$\Psi(F, X) := Q + F^T RF + (A + BF)^T X + X(A + BF) + XMX.$$

By definition, we have $\Psi(F, \psi(F)) = 0$ for all $F \in O_{\bar{F}}$. Taking the derivative of this equality at $F = \bar{F}$ shows that $\bar{F} = -R^{-1}B^T\psi(\bar{F})$ (see again Ref. 11 for details). Substituting this in $\Psi(\bar{F}, \psi(\bar{F})) = 0$ yields

$$Q + A^T\psi(\bar{F}) + \psi(\bar{F})A - \psi(\bar{F})S\psi(\bar{F}) + \psi(\bar{F})M\psi(\bar{F}) = 0.$$

This shows that $\psi(\bar{F})$ satisfies (18) and furthermore, since it is the stabilizing solution of the equation $\Psi(\bar{F}, X) = 0$ it follows that $A + B\bar{F} + M\psi(\bar{F}) = A - S\psi(\bar{F}) + M\psi(\bar{F})$ is stable. Finally, since $\bar{F} \in \bar{\mathcal{F}}$, the matrix $A - S\psi(\bar{F})$ is stable. ¤

3.2 N-Player Case

From Corollary 3.1, a sufficient condition for the existence of a soft-constrained feedback Nash equilibrium follows in a straightforward way.

Theorem 3.3 Consider the differential game defined by (1), (4) and (5). Assume there exist N real symmetric $n \times n$ matrices X_i and N real symmetric $n \times n$ matrices Y_i such that

$$Q_i + A^T X_i + X_i A - \sum_{j\neq i}^{N}(X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j\neq i}^{N} X_j S_{ij} X_j + X_i M_i X_i = 0 \quad (24)$$

$$A - \sum_{j=1}^{N} S_j X_j + M_i X_i \text{ is stable for } i = 1, \dots, N \quad (25)$$

$$A - \sum_{j=1}^{N} S_j X_j \text{ is stable} \quad (26)$$

$$Q_i + A^T Y_i + Y_i A - \sum_{j\neq i}^{N}(Y_i S_j X_j + X_j S_j Y_i) - Y_i S_i Y_i + \sum_{j\neq i}^{N} X_j S_{ij} X_j \geq 0. \quad (27)$$

Define the N-tuple $\bar{F} = (\bar{F}_1, \dots, \bar{F}_N)$ by

$$\bar{F}_i := -R_{ii}^{-1} B_i^T X_i. \quad (28)$$

Then $\bar{F} \in \mathcal{F}_N$, and this N-tuple is a soft-constrained Nash equilibrium. Furthermore

$$\bar{J}_i^{SC}(\bar{F}_1, \dots, \bar{F}_N, x_0) = x_0^T X_i x_0. \quad (29)$$

Proof The assumption (26) immediately implies that $\bar{F} \in \mathcal{F}_N$. Let $x_0 \in I\!R^n$ and $1 \le i \le N$. Let the functional ϕ be defined by

$$\phi : \{F \in I\!R^{m_i \times n} \mid \bar{F}_{-i}(F) \in \mathcal{F}_N\} \to I\!R, \quad \phi(F) = \bar{J}_i^{SC}(\bar{F}_{-i}(F), x_0).$$

We need to show that this functional is minimal at $F = \bar{F}_i$. We have

$$\phi(F) = \sup_{w \in L_2^q(0,\infty)} \int_0^\infty \left(x^T \left(Q_i + \sum_{j \ne i}^N X_j S_{ij} X_j + F^T R_{ii} F \right) x - w^T V_i w \right) dt$$

where x follows from

$$\dot{x} = \left(A - \sum_{j \ne i}^N S_j X_j + B_i F \right) x + Ew, \quad x(0) = x_0.$$

Note that the functional ϕ coincides with the functional J, as defined in Theorem 3.1, with A replaced by $A - \sum_{j \ne i}^N S_j X_j$, $B := B_i$, $Q := Q_i + \sum_{j \ne i}^N X_j S_{ij} X_j$, $R := R_{ii}$, $V = V_i$, and the same values for E and x_0. It is easily seen that the conditions (24)–(27) guarantee that the conditions of Theorem 3.1 are satisfied with $X := X_i$ and $Y = Y_i$. So, according to Theorem 3.1, the functional ϕ is minimal at $F = -R_{ii}^{-1} B_i^T X_i = \bar{F}_i$, and the minimal value is equal to $x_0^T X_i x_0$. ¤

Remark 3.2 If $Q_i \ge 0$ for all $i = 1, \ldots, N$, the matrix inequality (27) is trivially satisfied with $Y_i = 0$. ¤

3.3 Scalar Case

In this subsection we consider the scalar case of the one-player problem (9)–(10) in some more detail. Specifically we are interested in the meaning of condition (19) which plays a role only when the state weighting matrix in the objective function is indefinite. First we obtain necessary and sufficient conditions for

$$\bar{J} := \inf_{f \in \mathcal{F}} \sup_{w \in L_2^q(0,\infty)} J(f, w, x_0) \tag{30}$$

to be finite. We begin with necessary and sufficient conditions under which the supremum takes a finite value.

Lemma 3.3 Let $f \in \mathcal{F}$ be fixed. Then $\sup_{w \in L_2^q(0,\infty)} J(f, w, x_0)$ is finite if and only if $g(f) := (a + bf)^2 - m(q + f^2 r) \ge 0$. Furthermore, the value of the supremum is $-\frac{1}{2}(q + f^2 r)x_0^2/(a + bf)$ if $e = 0$, and is $-(a + bf + \sqrt{g(f)})x_0^2/m$ otherwise.

Proof If $e = 0$, the supremum is achieved at $w = 0$ and so it is finite for any $f \in \mathcal{F}$. In this case we also have $m = e^2/v = 0$ and so the condition of the lemma holds. If $e \ne 0$, the pair $(a + bf, e)$ is controllable. Using Ref. 12 (or, in this scalar case, elementary analysis), we have that the supremum is finite if and only if the algebraic Riccati equation

$$mx^2 + 2(a + bf)x + q + f^2r = 0 \tag{31}$$

has a real solution. Furthermore, the value of the supremum is $x_0^2 x_\mathsf{s}$, where x_s is the smallest solution of (31). From this, the above statement follows directly. ¤

Next, we consider the outer minimization. From the above lemma it is clear that the case $e = 0$ is a special one. Therefore, we analyse this case first.

Proposition 3.1 Suppose that in the scalar version of (9)–(10) we have $e = 0$, and write $t := a^2 + sq$. If $b \neq 0$, the following holds.

(i) If $t > 0$, then $\bar{J} = (qs + (a + \sqrt{t})^2)/2s\sqrt{t}$ and $\bar{f} = -(a + t)/b$.

(ii) If $t = 0$, then $\bar{J} = a/s$ and the infimum in problem (30) is not achieved (actually, the infimum is attained at $f = -a/b$).

(iii) If $t < 0$, then $\bar{J} = -\infty$.

If $b = 0$, then necessarily $a < 0$ and the minimum $\bar{J} = -q/(2a)$ is attained at $f = 0$.

Proof All statements follow by an elementary analysis of the function $\mathcal{F} \ni f \mapsto -\frac{1}{2}(q + f^2r)/(a + bf)$ (see Lemma 3.3). If $t > 0$ this function has a unique minimum at $\bar{f}$; if $t = 0$ its graph is a line; if $t < 0$ it is a monotonic function that has a vertical asymptote at $f = -a/b$. ¤

Next, consider the case $e \neq 0$ or, equivalently, $m \neq 0$. Let

$$\bar{\mathcal{F}}_\mathsf{e} := \{f \in \mathcal{F} \mid g(f) = (a + bf)^2 - m(q + f^2r) \geq 0\} \tag{32}$$

(see Lemma 3.3). That is, $\bar{\mathcal{F}}_\mathsf{e}$ is the set of all stabilizing feedback matrices for which $\sup_{\mathsf{w} \in \mathsf{L}_2^\mathsf{q}(0,\infty)} J(f, w, x_0)$ is finite. From Lemma 3.3 we know that the supremum equals $x_0^2 x_\mathsf{s}(f)$, where $x_\mathsf{s}(f)$ is given by

$$x_\mathsf{s}(f) = -\frac{a + bf + \sqrt{g(f)}}{m}. \tag{33}$$

We are looking for the minimum of $x_\mathsf{s}(f)$ over all f in the set $\bar{\mathcal{F}}_\mathsf{e}$. To perform this minimization we first consider the domain $\bar{\mathcal{F}}_\mathsf{e}$ in some more detail.

Lemma 3.4 The set $\bar{\mathcal{F}}_\mathsf{e}$ defined in (32) is either

- empty,
- a single point,
- a halfline,

- a bounded interval, or
- the union of a halfline and a bounded interval.

Proof Define $\mathcal{G} := \{f \mid (a+bf)^2 - m(q+f^2r) \geq 0\}$. Then $\bar{\mathcal{F}}_{\mathsf{e}} = \mathcal{G} \cap \mathcal{F}$. Note that $\mathcal{F}$ is an open halfline. To determine $\mathcal{G}$, we consider the graph of $g(f) := (a+bf)^2 - m(q+f^2r), f \in I\!R$. If g is concave, $\mathcal{G}$ is a (possibly empty) closed interval or just a single point. So $\bar{\mathcal{F}}_{\mathsf{e}}$ is a (possibly empty) interval or single point too. In case g is convex, $\mathcal{G}$ consists of either the whole real line or the union of two closed halflines. From this the other possibilities mentioned in the lemma are easily established. ¤

From this lemma we conclude that whenever $\bar{\mathcal{F}}_{\mathsf{e}}$ is not empty or consists of a single point, we can use differentiation arguments to investigate the finiteness of $\bar{J}$. Therefore, we first analyse these two cases.

Proposition 3.2

(I.) $\bar{\mathcal{F}}_{\mathsf{e}} = \emptyset$ if and only if $s < m$ and either i) $a^2+q(s-m) < 0$ or ii) $a^2+q(s-m) \geq 0$; $a \geq 0$; and $a^2+qs \geq 0$. In this case, $\bar{J} = \infty$.

(II.) $\bar{\mathcal{F}}_{\mathsf{e}}$ consists of only one point if and only if simultaneously $s-m < 0$, $a^2+q(s-m) = 0$, and $-ma/(s-m) < 0$ hold. Then, $\bar{J} = a/(s-m)$ and $\bar{f} = -ab/(r(s-m))$.

Proof

(I.) $\bar{\mathcal{F}}_{\mathsf{e}} = \emptyset$ if and only if (see Lemma 3.4) either $\mathcal{G}$ is empty, or the intersection of $\mathcal{F}$ with $\mathcal{G}$ (with $\mathcal{G}$ a bounded interval) is empty. The first case occurs if both $s-m < 0$ and $a^2+q(s-m) < 0$. The second case occurs if $s-m < 0$; $a^2 + q(s-m) \geq 0$ and (assume without loss of generality $b > 0$) $-a/b \leq -(ab/r + \sqrt{m/r}\sqrt{a^2+q(s-m)})/(s-m)$. This holds if and only if $a \geq 0$ and $a^2+qs \geq 0$.

(II.) $\bar{\mathcal{F}}_{\mathsf{e}}$ consists of only one point if and only if $g(f) = 0$ has exact one solution in $\mathcal{F}$. Elementary calculations then show the stated result. ¤

The next case we consider is $\mathcal{G} = I\!R$. This corresponds to the case $s > m$ and $a^2+q(s-m) \leq 0$. It can be shown that under these conditions the derivative of $x_{\mathsf{s}}(f)$ is negative. So, the infimum is finite, but is attained at the boundary of $\mathcal{F}$. From this and Proposition 3.2 we see that the only case for which $a^2+q(s-m) \leq 0$ we did not treat yet is the case $s = m$. Obviously, this case only occurs if $a = 0$. It is easily verified that $x_{\mathsf{s}}'(f) < 0$ again, so the same conclusion as above holds.

Finally, consider the case that $a^2+q(s-m) > 0$. Elementary calculations show that in that case the derivative of $x_{\mathsf{s}}(f)$ has a unique zero f^*. This zero coincides with $-(b/r)x^*$, where x^* is the smallest solution of the algebraic Riccati equation (18). Furthermore, $y_{\mathsf{f}}''(f^*) < 0$, so $x_{\mathsf{s}}(f)$ has a minimum at f^*. Moreover, $g(f^*) = (a - sx^* + mx^*)^2 \geq 0$. So, $f^* \in \mathcal{G}$. If additionally $a - sx^* \in \mathcal{F}$, then $x_{\mathsf{s}}(f)$ has a minimum in $\bar{\mathcal{F}}_{\mathsf{e}}$ which, moreover, is a global minimum if e.g. $\bar{\mathcal{F}}_{\mathsf{e}}$ is connected. On the other hand it is clear that if $a - sx^* \notin \mathcal{F}$ the infimum value is again attained at the boundary of $\mathcal{F}$. The following example illustrates the case in which $\bar{\mathcal{F}}_e$ is not connected.

Example 3.1 Let $a = 5$, $b = 1$, $m = 1$, $r = \frac{1}{9}$ and $q = -3$. Then, $\bar{\mathcal{F}}_e = (-\infty, -6) \cup (-5\frac{1}{4}, -5)$. Moreover, $J(-5, \bar{w}, x_0) = -\frac{1}{3}\sqrt{2}x_0^2$, $f^* = -6\frac{3}{4}$ and $J(f^*, \bar{w}, x_0) = \frac{3}{4}x_0^2$. In this case the infimum is not achieved. Note that if $f = -5$ the worst case action (from the player's point of view) the disturbance can take is to stabilize the system since the player's aim is to maximize the revenues x (subject to the constraint that the undisturbed closed-loop system must be stable). ¤

The next lemma gives conditions, in terms of the problem parameters, under which a nonempty set $\bar{\mathcal{F}}_e$ is not connected.

Lemma 3.5 Assume that $\bar{\mathcal{F}}_e \neq \emptyset$. Then $\bar{\mathcal{F}}_e$ is not connected if and only if the following four conditions are satisfied:

(i) $s - m > 0$

(ii) $a^2 + q(s - m) \geq 0$

(iii) $a^2 + qs < 0$

(iv) $a > 0$.

Proof If g is concave (see proof of Lemma 3.4), the set $\bar{\mathcal{F}}_e$ is an interval and is thus connected. It is easily verified that this situation occurs if and only if $s - m \leq 0$.

Next consider the case that g is convex. If g has no zeros it is obvious that $\bar{\mathcal{F}}_e$ is connected. This occurs if and only if $a^2 + (s - m)q < 0$. Otherwise, $\mathcal{G} = (-\infty, a_0) \cup (a_1, \infty)$. Then, $\bar{\mathcal{F}}_e$ is connected if and only if (assume without loss of generality $b > 0$) $-a/b \leq -(ab + \sqrt{a^2b^2 - (a^2 - mq)(b^2 - mr)})/(r(s - m))$. This condition holds if and only if either $a \leq 0$ or $a^2 + qs \geq 0$. ¤

If $\bar{\mathcal{F}}_e$ is not connected, $J(f, \bar{w}(f), x_0)$ does not have a global minimum since

$$J(-a/b, \bar{w}(-a/b), x_0) = -(\sqrt{-m(q + a^2/s)}/m)x_0^2 < 0$$
$$< (a + \sqrt{a^2 + q(s - m)})/(s - m)x_0^2 = J(f^*, \bar{w}(f^*), x_0).$$

Actually one can show that $x_s(f)$ attains again an infimum at $-a/b$. So we conclude the following.

Theorem 3.4 Consider the scalar version of the one-player game (9)–(10). Assume that the set $\bar{\mathcal{F}}_e$ defined in (32) has more than one element and that $e \neq 0$; then the following statements hold.

(i) The one-player game has a solution if and only if either one of the four conditions in Lemma 3.5 is violated and (18) has a stabilizing solution x^* for which additionally $a - sx^*$ is stable. In that case the solution is provided by Theorem 3.1.

(ii) Otherwise, the infimum in (30) is achieved and $\bar{J} = J(-\frac{a}{b}, \bar{w}(-\frac{a}{b}), x_0)$.

¤

Remark 3.3 The assumption that there exists a number y such that $q+2ay-sy^2 \geq 0$ (see Theorem 3.1) is equivalent to the assumption that $a^2 + qs \geq 0$. So, this condition indeed implies in the scalar case that $\bar{\mathcal{F}}_e$ is connected. ¤

We end this subsection by noting that for the two-player case one can study the number of solutions to the algebraic Riccati equations like in Ref. 13. Rewriting $m_i =: \alpha_i s_i$ for some positive α_i, and using the same notation, one has to study the solution set of the following (in)equalities:

$$(1+\alpha_i)\kappa_i^2 - 2\kappa_3\kappa_i + \sigma_i = 0, \quad i = 1, 2 \tag{34}$$

$$\kappa_3 := -a + \kappa_1 + \kappa_2 > 0 \tag{35}$$

$$\kappa_3 - \alpha_i \kappa_i > 0, \quad i = 1, 2. \tag{36}$$

Equations (34)–(35) can be analysed similarly as in Ref. 13. By taking α_i small it is clear that the number of equilibria can vary again between zero and three.

Another interesting point is that the incorporation of noise by players into their decision making may result in the fact that a situation of no equilibrium changes into a situation in which an equilibrium does exist. Take e.g. $q_i = -1$; $b_i = r_i = v_i = e = 1$ and $a = -\frac{3}{2}$. For these parameters the undisturbed game has no equilibrium (see Ref. 13, Theorem 3) whereas the disturbed game has the equilibrium $\kappa_i = -\frac{1}{2}$, $i = 1, 2$; $\kappa_3 = \frac{1}{2}$.

Furthermore, using the implicit function theorem, one can analyse the consequences of a change in the α_i parameters on the equilibrium location. Assuming that the equilibrium (κ_1^*, κ_2^*) can be described locally as a function $h(\alpha_1, \alpha_2)$, it is easily verified that

$$h' = \frac{-1}{2(p_1 p_2 - \kappa_1^* \kappa_2^*)} \begin{pmatrix} -p_2 & \kappa_1^* \\ \kappa_2^* & -p_1 \end{pmatrix} \begin{pmatrix} \kappa_1^{*2} & 0 \\ 0 & \kappa_2^{*2} \end{pmatrix}$$

where $p_i := \kappa_3^* - \alpha_i \kappa_i^* > 0$ (see (36)). From this it is immediately clear, for example, that at a positive equilibrium an increase in α_1 will have an opposite effect on the entries of the equilibrium location. One entry will increase, the other will decrease. That is, the response to a more risk-averse behavior by one player is a more risk-seeking behavior by the other player. We do not undertake a more detailed analysis here since such an analysis can be carried out best in the context of a specific application.

4 Hard-Bounded Nash Equilibria

In this section we consider again the system

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i + Ew, \quad x(0) = x_0, \tag{37}$$

with strategies

$$u_i = F_i x \tag{38}$$

and cost functions

$$J_i^{HB}(F_1, \ldots, F_N, w, x_0) = \int_0^\infty x^T \left(Q_i + \sum_{j=1}^N F_j^T R_{ij} F_j \right) x \, dt. \tag{39}$$

In the hard-bounded modeling approach we consider as objective functions for the players the adjusted cost functions

$$\bar{J}_i^{HB}(F_1, \ldots, F_N, x_0) = \sup_{\|w\| \le r_i} J_i^{HB}(F_1, \ldots, F_N, w, x_0). \tag{40}$$

The numbers r_i express the players' degrees of aversion against model risk; in this sense their role is similar to that of the matrices V_i in the soft-constrained problem given by the objective functions (5).

Definition 4.1 An N-tuple $\bar{F} = (\bar{F}_1, \ldots, \bar{F}_N) \in \mathcal{F}_N$ is called a *hard-bounded Nash equilibrium* if for each $i = 1, \ldots, N$ the following inequality holds:

$$\bar{J}_i^{HB}(\bar{F}, x_0) \;\le\; \bar{J}_i^{HB}(\bar{F}_{-i}(F), x_0) \tag{41}$$

for all $F \in I\!R^{m_i \times n}$ that satisfy $\bar{F}_{-i}(F) \in \mathcal{F}_N$. ¤

Note that here, in contrast to Section 3, players have a memoryless perfect state information structure (see Ref. 8), so that the linear stabilizing feedback control may depend on the initial state. A sufficient condition for the existence of hard-bounded Nash equilibria is presented in Theorem 4.2 below. We present first a result on a minmax problem, corresponding to a one-player situation, on which the proof of the theorem will be based. For $x_0 = 0$, the minmax problem reduces to the state feedback H_∞ control problem. For nonzero unknown initial state, the corresponding maxmin problem has been studied in Ref. 15. Here we study the minmax problem with known $x_0 \neq 0$, i.e. we study the worst-case disturbance attenuation problem with known nonzero initial state. The problem addressed here is a direct generalization of Ref. 9.

Theorem 4.1 Let $r > 0$, $x_0 \in I\!R^n$, $A, Q \in I\!R^{n \times n}$, $B \in I\!R^{n \times m}$, $E \in I\!R^{n \times q}$, $R \in I\!R^{m \times m}$, with (A, B) stabilizable, $Q \ge 0$ and $R > 0$. Define

$$S := BR^{-1}B^T \tag{42}$$

$$\mathcal{F} := \{F \in I\!R^{m \times n} \mid A + BF \text{ is stable}\}. \tag{43}$$

Assume there exists a real symmetric $n \times n$ matrix X, an $n \times n$ matrix P, and a real number $\gamma \neq 0$, such that

$$X \geq 0 \tag{44}$$

$$Q + A^{\mathsf{T}} X + XA - XSX + \gamma^{-2} XEE^{\mathsf{T}} X = 0 \tag{45}$$

$$A_0 := A - SX + \gamma^{-2} EE^{\mathsf{T}} X \text{ is stable} \tag{46}$$

$$A_0^{\mathsf{T}} P + PA_0 = -XEE^{\mathsf{T}} X \tag{47}$$

$$\gamma^{-4} x_0^{\mathsf{T}} P x_0 = r^2. \tag{48}$$

Define $\bar{F} := -R^{-1} B^{\mathsf{T}} X$. Then $\bar{F} \in \mathcal{F}$ and the functional $J : \mathcal{F} \to I\!R$, defined by

$$J : F \mapsto \sup_{\|w\| \leq r} \int_0^\infty x^{\mathsf{T}} (Q + F^{\mathsf{T}} RF) x \, dt \tag{49}$$

where x follows from

$$\dot{x} = (A + BF)x + Ew, \quad x(0) = x_0, \tag{50}$$

is minimal at $F = \bar{F}$. Furthermore, the corresponding minimal value is equal to $x_0^{\mathsf{T}} X x_0 + \gamma^2 r^2$.

Proof Note that the matrix X is the stabilizing solution of the ARE (45). Since X is positive semidefinite, it follows from Ref. 7, Lemma 16.6, that $A - SX$ is stable. Hence $\bar{F} \in \mathcal{F}$. Next, define the functional $\phi : \mathcal{F} \times L_2^q(0, \infty) \to I\!R$ by

$$\phi(F, w) = \int_0^\infty x^{\mathsf{T}} (Q + F^{\mathsf{T}} RF) x \, dt$$

where x follows from (50). Furthermore, define $\bar{w} \in L_2^q(0, \infty)$ by

$$\bar{w}(t) := \gamma^{-2} E^{\mathsf{T}} X e^{tA_0} x_0. \tag{51}$$

Due to the stability of the matrix A_0 we have indeed $\bar{w} \in L_2^q(0, \infty)$. Furthermore,

$$\|\bar{w}\|^2 = \gamma^{-4} x_0^{\mathsf{T}} \left(\int_0^\infty e^{tA_0^{\mathsf{T}}} XEE^{\mathsf{T}} X e^{tA_0} dt \right) x_0 = \gamma^{-4} x_0^{\mathsf{T}} P x_0 = r^2.$$

Next, we will show that[6]

$$\phi(\bar{F}, w) \leq \phi(\bar{F}, \bar{w}) \leq \phi(F, \bar{w}) \tag{52}$$

[6] The set of inequalities (52) is equivalent to stating that the pair $(\bar{F}, \bar{w})$ is a saddlepoint solution of the zero-sum game with minimization set $\mathcal{F}$, maximization set $\{w \in L_2^q(0, \infty) | \ \|w\| \leq r\}$, and objective function ϕ (see for instance Ref. 5, Chapter 2, for a brief overview of zero-sum game theory).

for all $F \in \mathcal{F}$ and for all $w \in L_2^q(0,\infty)$ with $\|w\| \leq r$. In order to prove these inequalities, we apply two completions of the squares which allow to rewrite $\phi(F,w)$ as

$$\begin{aligned}\phi(F,w) &= \int_0^\infty \left(x^\top (Q + F^\top RF)x + \frac{d}{dt}x^\top Xx - \frac{d}{dt}x^\top Xx \right) dt = \\ &= x_0^\top Xx_0 + \gamma^2\|w\|^2 + \int_0^\infty \|(F-\bar{F})x\|_R^2\, dt - \gamma^2\|w - \gamma^{-2}E^\top Xx\|^2. \quad (53)\end{aligned}$$

Here we used the fact that $x(t) \to 0$ as $t \to \infty$, which holds because $x, \dot{x} \in L_2^n(0,\infty)$ (see for instance Ref. 15, Exercise 6.10). From (53) we deduce

$$\phi(\bar{F},\bar{w}) = x_0^\top Xx_0 + \gamma^2 r^2 - \|\bar{w} - \gamma^{-2}E^\top X\bar{x}\|^2$$

where $\bar{x}$ is defined by $\dot{\bar{x}} = (A + B\bar{F})\bar{x} + E\bar{w}$ and $\bar{x}(0) = x_0$. From this and (51), it is easily seen that $\bar{w} = \gamma^{-2}E^\top X\bar{x}$. Thus

$$\phi(\bar{F},\bar{w}) = x_0^\top Xx_0 + \gamma^2 r^2. \quad (54)$$

Furthermore, from (53) we also deduce for each $w \in L_2^q(0,\infty)$ with $\|w\| \leq r$ that

$$\phi(\bar{F},w) \leq x_0^\top Xx_0 + \gamma^2 r^2 = \phi(\bar{F},\bar{w})$$

which is the first inequality in (52). In order to show the second inequality, we introduce the variables $\bar{x}_w$, ν, ζ, and ξ by

$$\begin{aligned}\dot{\bar{x}}_w &= (A + BF)\bar{x}_w + E\bar{w}, \quad \bar{x}_w(0) = x_0 \\ \nu &:= (\bar{F} - F)\bar{x}_w \\ \zeta &:= \bar{w} - \gamma^{-2}E^\top X\bar{x}_w \\ \xi &:= \bar{x} - \bar{x}_w.\end{aligned}$$

Then, (53) and (54) imply that

$$\phi(F,\bar{w}) - \phi(\bar{F},\bar{w}) = \int_0^\infty \|\nu\|_R^2\, dt - \gamma^2\|\zeta\|^2 = \int_0^\infty \left(\nu^\top R\nu - \gamma^{-2}\xi^\top XEE^\top X\xi\right) dt.$$

It is easily seen that $\dot{\xi} = (A + B\bar{F})\xi + B\nu$, $\xi(0) = 0$, and $\xi(t) \to 0$ for $t \to \infty$, so that a completion of the squares yields

$$\begin{aligned}\phi(F,\bar{w}) - \phi(\bar{F},\bar{w}) &= \int_0^\infty \left(\nu^\top R\nu - \gamma^{-2}\xi^\top XEE^\top X\xi - \frac{d}{dt}\xi^\top X\xi\right) dt = \\ &= \int_0^\infty \left(\|\nu - R^{-1}B^\top X\xi\|_R^2 + \xi^\top Q\xi\right) dt.\end{aligned}$$

Since Q is positive semidefinite, this expression is clearly nonnegative. This completes the proof of the second inequality in (52). The inequalities (52) imply

$$J(\bar{F}) = \sup_{\|w\| \leq r} \phi(\bar{F}, w) = \phi(\bar{F}, \bar{w}) \leq \phi(F, \bar{w}) \leq \sup_{\|w\| \leq r} \phi(F, w) = J(F).$$

This shows that the functional J is minimal at $F = \bar{F}$. The last part of the theorem immediately follows from (54). ¤

Remark 4.1 The ARE (45) also appears in the context of H_∞ control theory, see e.g. Refs. 5, 7, 16 or 17. It is well-known that under the additional assumption that $(A, Q^{1/2})$ has no unobservable eigenvalues on the imaginary axis, a positive number γ^* exists such that there exists a unique real symmetric $n \times n$ matrix X satisfying (44)–(46) if and only if $\gamma > \gamma^*$. Hence, for each $\gamma > \gamma^*$, one can determine a matrix X from (44)–(46). Equation (47) is a Lyapunov equation and since the matrix A_0 is stable, a unique matrix P can easily be determined from this equation. Equation (48) requires a bit more care. Under some weak conditions, it can be shown that the function $\gamma \mapsto \gamma^{-4} x_0^T P x_0$ is strictly decreasing in γ for $\gamma > \gamma^*$. Furthermore, the expression $\gamma^{-4} x_0^T P x_0$ typically approaches infinity in the limit $\gamma \downarrow \gamma^*$. Thus in principle it is straightforward to construct a numerical scheme producing a triple (X, P, γ) satisfying (44)–(48). For a further discussion of these aspects we refer to Ref. 11, Section 6.3. ¤

On the basis of the one-player results derived above, the theorem below follows rather straightforwardly.

Theorem 4.2 Let $Q_i \geq 0$ for each $i = 1, \ldots, N$. Assume there exist N real symmetric $n \times n$ matrices X_i, N symmetric $n \times n$ matrices P_i, and N nonzero real numbers γ_i, such that

$$X_i \geq 0 \tag{55}$$

$$Q_i + A^T X_i + X_i A - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j + \gamma_i^{-2} X_i E E^T X_i = 0 \tag{56}$$

$$A_i := A - \sum_{j=1}^{N} S_j X_j + \gamma_i^{-2} E E^T X_i \text{ is stable for each } i = 1, \ldots, N \tag{57}$$

$$A_i^T P_i + P_i A_i = -X_i E E^T X_i \tag{58}$$

$$\gamma_i^{-4} x_0^T P_i x_0 = r_i^2. \tag{59}$$

Then the N-tuple $(\bar{F}_1, \ldots, \bar{F}_N)$ defined by

$$\bar{F}_i := -R_{ii}^{-1} B_i^T X_i \tag{60}$$

is a hard-bounded Nash equilibrium, and

$$\bar{J}_i^{HB}(\bar{F}_1, \dots, \bar{F}_N, x_0) = x_0^T X_i x_0 + \gamma_i^2 r_i^2. \tag{61}$$

Proof We have to show that the functional $\phi : F \mapsto \bar{J}_i^{HB}(\bar{F}_{-i}(F), x_0)$ is minimal in the set

$$\{F \in \mathbb{R}^{m_i \times n} \mid A - \textstyle\sum_{j \neq i}^{N} S_j X_j + B_i F \text{ is stable } \}.$$

if $F = \bar{F}_i$. We have

$$\bar{J}_i^{HB}(\bar{F}_{-i}(F), x_0) = \sup_{\|w\| \leq r_i} \int_0^\infty x^T \left(Q_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j + F^T R_{ii} F \right) x \, dt$$

where x follows from

$$\dot{x} = \left(A - \sum_{j \neq i}^{N} S_j X_j + B_j F \right) x + Ew, \quad x(0) = x_0.$$

Note that the functional ϕ coincides with the functional J, as defined in Theorem 4.1, with A replaced by $A - \sum_{j \neq i}^{N} S_j X_j$, $B := B_i$, $r := r_i$, $Q := Q_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j$, $R := R_{ii}$, and the same values for E and x_0. It is easily seen that the conditions (55)–(59) guarantee that the conditions (44)–(48) are satisfied with $X := X_i$, $P := P_i$, and $\gamma := \gamma_i$. So, according to Theorem 4.1, the functional ϕ is minimal if $F = -R_{ii}^{-1} B_i^T X_i = \bar{F}_i$, and the minimal value is equal to $x_0^T X_i x_0 + \gamma_i^2 r_i^2$. ¤

Note that the solvability of (59) is unclear. In the one-player case, the left-hand side of (59) is decreasing in γ for $\gamma > \gamma^*$. In the N-player case, we deal with a coupled system of N nonlinear equations in the unknowns $\gamma_1, \dots, \gamma_N$ which need to be solved in a set Γ, defined as the collection of N-tuples $(\gamma_1, \dots, \gamma_N)$ of nonzero real numbers with the property that there exists an N-tuple $X_1, \dots, X_N$ satisfying (55)–(57).

5 Concluding Remarks

In this paper we studied the existence of Nash equilibria in linear-quadratic differential games on an infinite planning horizon if the system is disturbed by deterministic noise and the strategy spaces are of the static linear feedback type. We considered the soft-constrained and hard-bounded cases. For the soft-constrained case we discussed the general indefinite control problem. For the hard-bounded case we just considered the definite control problem.

The soft-constrained problem has been extensively discussed for the one-player case. A set of sufficient conditions was given under which we can conclude that there exists a saddlepoint solution. Under a further restriction on the strategy spaces, some of these conditions were found to be necessary as well. In the scalar case, we have provided necessary and sufficient conditions for existence of a saddlepoint solution. It turns out that these conditions are intimately related to the question whether the outer optimization takes place over a connected set or not.

A sufficient condition was provided for the existence of soft-constrained equilibria in the N-player case, and it was argued that for the two-player scalar case one can expect that the corresponding algebraic Riccati equations again have from zero up to three solutions. For the definite control problem one can show (see Ref. 13) that the soft-constrained equilibria can also be interpreted in a stochastic environment as risk-sensitive equilibria.

Finally, we derived sufficient conditions for existence of hard-bounded equilibria. We indicated an algorithm to calculate such equilibria. Considerable development is still required to get efficient numerical methods for solving the systems of equations associated with the equilibria that we have discussed; this is left as an item of future research.

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