Strategic Capital Budgeting:  
Asset Replacement under Market Uncertainty* 

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Abstract 

In this paper the impact of product market uncertainty on the optimal replacement timing of a production facility is studied. The existing production facility can be replaced by a technologically more advanced and thus more cost-effective one. Strategic interactions among the firms competing in the product market are taken into account by analyzing the problem in a duopolistic setting. We calculate the value of each firm and show that i) a preemptive (simultaneous) replacement occurs when the associated sunk cost is low (high), ii) despite the preemption effect uncertainty always raises the expected time to replace, and iii) the relationship between the probability of optimal replacement within a given time interval and uncertainty is decreasing for long time intervals and humped for short time intervals. Furthermore it is shown that result ii) carries over to the case where firms have to decide about starting production rather than about replacing existing facilities. 

Keywords: capital budgeting, real options, first passage time, product market uncertainty, Cournot duopoly 

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1 Introduction

Present value of growth opportunities often constitutes a significant part of a firm's value. Fama and French [5] estimate that on average 42% of the corporate value in mid-1990s can be attributed to growth opportunities. Translated into dollars, this means that the average listed US firm holds a $410 million portfolio of growth options. Consequently, proper management of such a portfolio via an appropriately designed capital budgeting process becomes crucial for maximizing shareholder value and satisfying return-on-capital requirements.

The extensive process of deregulation taking place in the last decade, combined with the wave of mergers and acquisitions, has resulted in an oligopolistic structure of a large number of sectors. A shift towards such a structure takes place not only in traditional regulated markets (telecommunications, energy, transportation) but also in more competitive industries (fast-moving consumer goods, car manufacturing, pharmaceuticals). Consequently, models designed for optimizing capital budgeting decisions while ignoring competitors or under perfect competition may cease to be valid. Imperfect competition in the firm's product market requires that strategic interactions with other firm(s) are taken into account (cf. Zingales [25]).

Furthermore, the volatile economic environment that firms face these days calls for an appropriate identification of the sources of uncertainty underlying their real activities. Therefore, there is considerable scope for a structural modeling of the product market, where the relationship between the uncertain factor and the cash flow of the firm is explicitly accounted for.

Keeping these considerations in mind, in the paper we address a number of issues, which we relate to the problem of production facility replacement with a more cost-effective one. The questions we endeavor to answer are as follows:

- What is the impact of strategic interactions on optimal capital budgeting strategies?
- How does the demand level that triggers the optimal replacement depend on uncertainty, and on the fact whether the firm is the market leader, has the role of the follower, or acts identically as its competitor?
- How does demand uncertainty affect the optimal threshold corresponding to new market entry?
- Is the relationship between the uncertainty and probability of optimal replacement monotonic?
- What are the implications of uncertainty and strategic interactions on the value of the firm?

We consider a continuous-time model in which the firm makes an investment decision under product market uncertainty and imperfect competition. The model follows Smets [20] and Grenadier [7] in assuming that i) there are two identical firms competing in the product market, and ii) the value of the
firm depends on the value of a stochastic process but is otherwise time independent. The payoff functions are derived from the firm’s reaction curves in the oligopolistic market.\textsuperscript{1} Moreover, we calculate the expected replacement timing and determine the probabilities of making optimal replacement within given time intervals.

Under either perfect competition or monopolistic market structure, modern theory of investment under uncertainty (cf. McDonald and Siegel \cite{15}, Dixit and Pindyck \cite{4}, Ch. 2) predicts that the firm will wait with investing for a higher level of demand if uncertainty is higher. This is due to the fact that investment is irreversible and the firm has an option to postpone it until some uncertainty is resolved. However, if (i) more than one firm holds the investment opportunity, and (ii) the firm’s investment decision directly influences payoffs of its competitor(s), opposite effects with respect to the investment timing can arise. First, increasing uncertainty enhances the value of the option to wait. Second, the value of an early strategic investment (made in order to achieve the first mover advantage) can significantly increase as well. Huisman and Kort \cite{10} show that in a continuous-time duopoly model with profit uncertainty (cf. Smets \cite{20} and Grenadier \cite{7}) the effect of a change in value of the option to wait on the optimal investment threshold is always stronger than the impact of strategic interactions. This implies a negative relationship between uncertainty of the firm’s profit flow and investment. On the contrary, Kulatilaka and Perotti \cite{13} find that product market uncertainty may, in some cases, stimulate investment. The latter authors analyze a two-period setting in which (one of the) duopolistic firms can invest in a cost-reducing technology. The payoff from investment is convex in the size of the demand since an increase of demand has a more-than-proportional effect on the realized duopolistic profits (firms are responding to higher demand by increasing both output and price). Taking into account Jensen’s inequality, Kulatilaka and Perotti \cite{13} conclude that higher volatility of the product market can accelerate investment.\textsuperscript{2}

We begin the analysis by describing the equilibrium strategies that occur in the considered real option game. We show that, contrary to the models based on profit uncertainty, the type of equilibrium depends on the investment cost: if this cost is sufficiently low (high), a preemptive (simultaneous) equilibrium occurs.\textsuperscript{3} Furthermore, we prove that the minimal demand level triggering the optimal asset replacement increases with uncertainty for both firms, despite

\textsuperscript{1}To our best knowledge, the product market structure in a 2-player game is explicitly modeled only by Kulatilaka and Perotti \cite{13}, and Perotti and Rossetto \cite{17}.

\textsuperscript{2}Other strategic real options models in continuous time include Williams \cite{24}, Grenadier \cite{8}, Lambrachi and Fernández \cite{14}, Décamps and Mariotti \cite{2}, Perotti and Rossetto \cite{17}, and Mason and Weeds \cite{16}. A discrete time analysis of a strategic real options exercise is presented, next to Kulatilaka and Perotti \cite{13}, by Smit and Ankum \cite{21}, whereas Ringsmuth \cite{18}, and Fudenberg and Tirole \cite{6} provide the game-theoretical foundations within a deterministic framework.

\textsuperscript{3}The overall type of equilibria can easily be affected by e.g., the authority. A recent rule imposed by Germany’s telecoms regulator enabling six companies which acquired the third generation mobile-phone licenses to share the costs of building a new infrastructure may serve as an example of such an action. See The Economist, June 9th 2001.
the strategic effect encouraging earlier investment of the first mover (leader).
This results also holds for the case in which the firms have to decide when to
start up production. Moreover, we show that the expected timing of replace-
mament increases with uncertainty. Finally, we analyze the probability of asset
replacement within a given time interval. It turns out that the replacement
probability decreases with uncertainty for time intervals that include the opti-
mal time to invest in the deterministic case. For shorter intervals there are two
opposite effects which leads to a humped relationship between uncertainty and
the probability of replacement (cf. Sarkar [19]).

The model is presented in Section 2 and the value functions as well as
replacement thresholds are derived in Section 3. Section 4 contains the de-
scription of the equilibria and in Section 5 the effect of uncertainty on replace-
ment thresholds is determined. In Section 6 the decision to start production in a new
market is analyzed. Section 7 examines how these results can be translated into
conclusions with respect to replacement timing, and Section 8 concludes.

2 Framework of the Model

Consider a risk-neutral firm that has an investment opportunity to replace
its existing production facility with a technologically superior one. The firm
operates in a duopoly, in which, in line with basic microeconomic theory, the
following inverse linear demand function holds

\[ p_t = A_t - Q_t, \] (1)

\( p_t \) is the price of a non-durable good/service offered by the firm and can be
interpreted as the instantaneous cash flow per unit sold, \( A_t \) is a measure of
the size of the demand and \( Q_t \) is the total amount of the good supplied to the
market at a given instant.\(^4\) Uncertainty in demand is generated by a geometric
Brownian motion

\[ dA_t = \alpha A_t dt + \sigma A_t dw_t, \] (2)

which is defined on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P}) \), where \( \alpha \) is the
instantaneous drift parameter, \( \sigma \) is the instantaneous standard deviation, \( dt \)
isa time increment and \( dw_t \) is the Wiener increment. The other firm is
identical to the first, both are profit-maximizers and compete in quantities (à
la Cournot).\(^5\)

The initial constant marginal cost of supplying a unit of the good is
\( K \) and implementing the new production facility reduces this cost from \( K \) to
\( k \). In order to start using the new technology, Firm \( i, i \in \{1, 2\} \), has to incur
an irreversible cost \( I \). Simple algebraic manipulation results in the following

\(^4\)Introducing a linear demand function is not crucial for our results but allows for obtaining
analytical expressions for the optimal investment thresholds and the firms' values.

\(^5\)Quantity competition yields the same output as a two-stage game in which the capacities
are chosen first and, subsequently, the firms are competing in prices (see Tirole [25], p. 216).
instantaneous profits of Firm $i$ (the other firm is denoted by $j$, $j \neq i$):\[\pi_{t}^{00} = \frac{1}{9} (A_t - K)^2, \tag{3}\]
\[\pi_{t}^{10} = \frac{1}{9} (A_t + K - 2k)^2, \tag{4}\]
\[\pi_{t}^{01} = \frac{1}{9} (A_t - 2K + k)^2, \tag{5}\]
\[\pi_{t}^{11} = \frac{1}{9} (A_t - k)^2. \tag{6}\]

where superscript 1 (0) in $\pi_{ij}^{ij}$ indicates which firm replaced (did not replace) its production facility. It is seen immediately that $\pi_{t}^{10} > \pi_{t}^{11} > \pi_{t}^{00} > \pi_{t}^{01}$.\[\tag{7}\]

Let us consider the value of Firm $i$ before replacement and denote it by $V_i$.\[\tag{7}\]

Using standard dynamic programming methodology (see Dixit and Pindyck [4]) we arrive at the following Bellman equation
\[rV = \frac{1}{2} \sigma^2 A_t^2 V'' + \alpha A_t V' + \pi_{t}^{0j}, \tag{8}\]

where $\pi_{t}^{0j}$ denotes the instantaneous profit flow before replacement. If the firm replaces its production facility first, $\pi_{t}^{0j}$ is equal to $\pi_{t}^{00}$ (see (3)). If the other firm has already implemented the new technology, $\pi_{t}^{0j}$ equals $\pi_{t}^{01}$ (cf. (5)). Solving the differential equation (8) for $j = 1$ gives
\[V = \frac{CA\beta}{\sigma^2} + \frac{1}{9} \left( \frac{A_t^2}{r - 2\alpha - \sigma^2} - \frac{2(2K - k)A_t}{r - \alpha} + \frac{(2K - k)^2}{r} \right), \tag{9}\]

where $C$ is a constant and $\beta$ is the positive root of the following equation (cf. Dixit and Pindyck [4])\[\tag{10}\]
\[\frac{1}{2} \sigma^2 \beta (\beta - 1) + \alpha \beta - r = 0. \]

From (9) it can be seen that there are two components contributing to the value of the firm. The first component corresponds to the value of the option to replace the production facility. The remainder of the LHS of (9) reflects the present value of the expected cash flow given that the firm produces with existing technology for ever. Convexity of the value of the firm in $A_t$ implies that a finite valuation is obtained only if the condition $r - 2\alpha - \sigma^2 > 0$ is satisfied.

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6 We assume that $K \ll A_0$, so that the probability weighted discount factor associated with the event $\{A_t < 2K - k\}$ equals zero. Waiving this assumption would not contribute to our results and would be done at the expense of explicit analytical formulae for the optimal investment thresholds (cf. Dixit and Pindyck [4] p. 191).

7 Since the firms are identical, we omit subscripts $i, j$ to simplify notation.

8 Note that the boundary condition $V(0) = 0$ implies that the negative root of (10) can be ignored.
3 Value Functions and Replacement Thresholds

In this section we establish the value of the firms and their optimal replacement thresholds. Consider the case of the firm that replaces as second (follower). Since the other firm (leader) has already replaced its production facility, the follower’s replacement decision is not affected by the result of strategic interactions (the follower chooses its optimal threshold as if the roles of the firms are preassigned). From (5) and (6) it is obtained that the value of the follower at \( t < T^F \), where \( T^F \) is the stochastic time of replacing the production facility by the follower, equals

\[
V^F(A_t) = E\left[\int_t^{T^F} \frac{1}{9}(A_s - 2K + k)^2 e^{-r(s-t)} ds \right]
\]

(11)

The first row of (11) is the expected discounted cash flow received until replacement. At \( T^F \) the follower makes the replacement and from now on produces against a lower marginal cost \( k \). The expected discounted cash flow after replacement is captured by the second row of (11).

In investment problems of this type (cf. Dixit and Pindyck [4]) a threshold value of \( A_t \) exists at which the firm is indifferent between investing and refraining from investment. Analogously, the value of the firm is maximized as soon as replacement of the production facility takes place at this realization of \( A_t \). To derive the optimal replacement threshold we apply the value-matching and smooth-pasting conditions (see Dixit and Pindyck [4]) to (9). This leads to

\[
C A^\beta_t = \frac{1}{9} \left( \frac{4(K - k) A_t}{r - \alpha} - \frac{4K(K - k)}{r} \right) - I,
\]

(12)

\[
\beta C A^{\beta-1}_t = \frac{4K - k}{9} \frac{K - k}{r - \alpha}.
\]

(13)

From (12) and (13) we obtain the optimal replacement threshold of the follower

\[
A^F = \frac{\beta}{\beta - 1} \frac{I + \frac{4K}{9} (K - k)}{\frac{4}{9} (K - k) (r - \alpha)} (r - \alpha).
\]

(14)

9 The case where immediate investment is optimal since \( A_0 \) exceeds the optimal replacement threshold [as defined here] is analyzed in Section 4.

10 The value matching condition equates the value of the firm before the replacement (therefore including the replacement option), as in (9), with the value after the replacement net of the associated sunk cost. Upon observing that the value after the replacement corresponds to the expected cash flow from new assets in place and equals,

\[
\frac{1}{9} \left( \frac{A^2_t}{r - 2\alpha - \sigma^2} - \frac{2kA_t}{r - \alpha} + \frac{k^2}{r} \right),
\]

condition (12) is obtained. Condition (13) is obtained by taking first derivatives of (12).
Note that the optimal threshold (14) is increasing in uncertainty and in the wedge $r - \alpha$. Now we are able to define the optimal time of replacement made by the follower as

$$T^F = \inf \{ t | A_t \geq A^F \}.$$  \hfill (15)

The value of the follower can now be calculated by substituting $C$ derived from (12) and (13) into (9). Derivation of the value functions of the leader and in case of simultaneous replacement can be performed in an analogous way.

The relevant results are reported in Table 1 (below). The first row corresponds to the case of the follower. Its value consist of two components: cash flow from the existing production asset plus the option to replace it with a cost-efficient facility. This option-like component is a product of the future value of the incremental cash flow (at the replacement time) and the stochastic discount factor. The value of the leader (second row) is determined for the moment at which the leader replaces its asset.\textsuperscript{12} It reflects the expected net present value of cash flow based on the marginal cost $k$ reduced by the discounted future value of cash flow lost due to the follower's replacement (cf. (7)). The value of the firm when the simultaneous replacement is made optimally (third row) has a structure closely related to the one of the follower and is equal to the expectation of the present value of cash flow based on the current production facility plus the discounted expectation of the change in cash flow resulting from the future simultaneous asset replacement of both firms. Immediate simultaneous investment (last row) corresponds to the net present value of cash flow when both firms produce at the marginal cost $k$.

4 Equilibria

Since both firms are ex ante identical, it is natural to consider symmetric replacement strategies and assume the endogeneity of the firms' roles, i.e. that it is not determined beforehand which firm will be the first to replace. There are two types of equilibria that can occur under this choice of strategies. We start by presenting the preemptive equilibrium which is followed by a description of the simultaneous equilibrium.

\textsuperscript{11}Increasing wedge $r - \alpha$ has also an indirect effect via increasing $\beta$ but that effect is dominated.

\textsuperscript{12}Such formulation is convenient while formulating the optimal investment strategies. For the same reason, immediate simultaneous replacement is also of interest.
[insert Table 1]
4.1 Preemptive Equilibrium

The first type of equilibrium is a preemptive equilibrium where Firm $i$ is the leader and Firm $j$ is the follower. Figure 1 (below) depicts the payoffs associated with the preemptive equilibrium. Let us define $A^p$ to be the smallest root of

$$
\xi(A_t) = V^L(A_t) - V^F(A_t).
$$

Assume for the moment that $A_0 < A^P$. Since on the interval $(A^P, A^F)$ the payoff of the leader is higher than the payoff of the follower (cf. Figure 1), each firm will have an incentive to be the leader at the moment that $A_t \in (A^P, A^F)$. In the search for equilibrium we reason backwards in terms of the values of $A_t$ (note that Equation (2) does not imply that $A_t$ increases monotonically over time). At $A^F$ the firms are indifferent between being the leader and the follower. However, for a smaller value of $A_t$, say at $A^F - \varepsilon$, the leader’s payoff is higher than the payoff of the follower. This implies that (without loss of generality) Firm $i$ has an incentive to be the first investor there. Firm $j$ anticipates this and would invest at $A^F - 2\varepsilon$. Repeating this reasoning we reach an equilibrium in which Firm $i$ invests at $A^P$ and Firm $j$ waits until demand exceeds $A^F$.

Note that if both firms invest at $A^P$ with probability one, they end up with the low payoff $V^J(A^P, A^P)$ (see Figure 1). At $A_t = A^F$ simultaneous investment is not profitable because demand is insufficient. Therefore, as in the seminal work Fudenberg and Tirole [6], the firms use mixed strategies in which the expected payoff is equal to the payoff of the follower (let us recall that the roles of the firms are not predetermined and the firms are risk-neutral). The firms are identical, so that they both have equal probability of becoming leader or follower. As in Huisman and Kort [11] it can be shown that for $A_t \in [A^P, A^F)$ the probability of a firm to become leader, $P^L$ (or follower, $P^F$) equals

$$
P^L = P^F = \frac{1 - p(A_t)}{2 - p(A_t)},
$$

where $p(A_t)$, being the probability of investment at the demand level $A_t$, equals

$$
p(A_t) = \frac{V^L(A_t) - V^F(A_t)}{V^L(A_t) - V^J(A_t)}.
$$

Consequently, since all probabilities add up to one, from (17) it follows that the probability of joint investment leading to the low payoff $V^J(A_t)$ is $\frac{p(A_t)}{2 - p(A_t)}$.

If $A_t < A^P$, the leader payoff curve lies below the follower curve which implies that it is optimal for both firms to refrain from investment. For $A_t = A^P$, the leader and the follower values are equal. Therefore (17) and (18) yield the probability of being the leader (or follower) equal to $\frac{1}{2}$. The probability of simultaneous investment at $A_t = A^P$ is therefore equal to zero. The leader invests at the moment that $A_t = A^P$, which is the smallest solution of $V^L(A_t) = \frac{p(A_t)}{2 - p(A_t)}$.

$V^F(A_t)$, and the follower waits until $A^F$ is reached. If the stochastic process starts at $A_0 > A^F$, the optimal investment behavior is to apply the mixed strategy described by (17) and (18), thus both firms invest with probability $p(A_0)$. In this case, according to (17) and (18), $p(A_0) > 0$ since the payoff of the leader exceeds the payoff of the follower. This makes the probability of investing jointly, and ending up with a low payoff of $V^J(A_0, A_0)$, become positive.

Figure 1. The values of the leader, $V^L$, optimal simultaneous replacement, $V^S$, and early simultaneous replacement, $V^J$, relative to the value function of the follower, $V^F$, for the set of parameter values: $K = 2$, $k = 0$, $r = 0.05$, $\alpha = 0.015$, $\sigma = 0.1$, and $I = 60$. For $A_0 < A^P$ the set of input parameters results in a preemptive replacement at $A^P$ (leader) and $A^F$ (follower).

4.2 Simultaneous Equilibrium

Another type of outcome that can occur in the analyzed real option game is the simultaneous replacement equilibrium. In such a case, the firms replace their production facilities at the same point in time defined by $T^S = \inf \{t | A_t \geq A^S\}$. No firm has an incentive to deviate from this equilibrium since the payoff of this strategy exceeds all other payoffs. A graphical illustration of the simultaneous equilibrium is depicted in Figure 2 below.

14 Of course, the payoffs resulting from the preemptive equilibrium in Section 4.1 may be lower than those associated with the optimal joint replacement. However, the occurrence of the preemptive equilibrium, as in Section 4.1, is due to the fact that such values of $A_t$ exist that the corresponding leader payoff exceeds the value from the joint replacement strategy. It is the lack of coordination among the firms (with possible transfer of excess value) that leads to ex post Pareto-inefficient outcomes. In the case of the simultaneous equilibrium the payoff of the leader never exceeds the payoff from joint optimal replacement and therefore the preemptive equilibrium, while still existent, is Pareto-dominated (see Fudenberg and Tirole [6]).
Figure 2. The values of the leader, $V^L$, optimal simultaneous replacement, $V^S$, and early simultaneous replacement, $V^J$, relative to the value function of the follower, $V^F$, for the set of parameter values: $K = 2$, $k = 0$, $r = 0.05$, $\alpha = 0.015$, $\sigma = 0.1$, and $I = 120$. The set of input parameters results in the optimality of a simultaneous replacement at $A^S$.

The occurrence of a particular type of equilibrium is determined by the relative payoffs. The preemptive equilibrium occurs when

$$\exists A_t \in (A_0, A^F) \text{ such that } V^L (A_t) > V^S (A_t),$$

i.e. when for some $A_t$ it is more profitable to become the leader than to replace production facilities simultaneously. Otherwise simultaneous replacement is the Pareto-dominant equilibrium. Proposition 1 implies that firms replace their production facilities simultaneously if the investment cost is sufficiently high.

**Proposition 1** A unique $I^*$ exists such that $\forall I > I^*$ simultaneous replacement is the Pareto-dominant equilibrium.

**Proof.** See Appendix.

This proposition is an important result with respect to the comparison between the real option exercise game with profit uncertainty and the situation where the firms face market uncertainty. In the first case the occurrence of either of the equilibria does not depend on the irreversible cost associated with the investment decision (see Huisman and Kort [11]). This results from the fact that the optimal threshold under profit uncertainty is proportional to investment cost $I$. This proportionality is a consequence of a multiplicative way in which uncertainty enters the profit function. Conversely, introducing market uncertainty via linking profit functions to a Cournot model results in the optimal threshold being no longer proportional to $I$. Therefore, the resulting equilibrium regions depends on the sunk cost.\(^{15}\)

\(^{15}\)In general, the investment cost affects the boundaries of the equilibrium regions. Therefore, the lack of such a relationship in a profit uncertainty model is rather a coincidence than a rule.
5 Uncertainty and Asset Replacement Thresholds

First, we investigate the impact of volatility on the optimal asset replacement thresholds of the follower and for simultaneous replacement. In these cases (see Table 1) the optimal thresholds, $A^{opt}$, can be expressed as

$$A^{opt} = \frac{\beta}{\beta - 1} f(I, K, k, r, \alpha).$$

(20)

It is straightforward to derive that

$$\frac{\partial A^{opt}}{\partial (\sigma^2)} = -\frac{1}{(\beta - 1)^2} f(I, K, k, r, \alpha) \frac{\partial \beta}{\partial (\sigma^2)} > 0,$$

(21)

so that the optimal replacement thresholds of the follower and in case of simultaneous replacement increase in uncertainty.

The impact of volatility on the production facility replacement threshold of the leader requires an additional analysis. Let us set the marginal cost $k$ to zero to simplify the notation. The replacement threshold of the leader equals

$$\max[A_0, A^F],$$

where $A^F$ is the smallest root of $\xi(A_t) = 0$. To find out the effect of market uncertainty on $A^F$, we calculate the derivative of $\xi(A_t)$ with respect to $\sigma^2$. The change of (16) resulting from a marginal increase in $\sigma^2$ can be decomposed as follows

$$\frac{d\xi(A_t)}{d(\sigma^2)} = \left(\frac{\partial \xi(A_t)}{\partial \beta} \frac{\partial \beta}{\partial (\sigma^2)} + \frac{\partial \xi(A_t)}{\partial A^F} \frac{dA^F}{d\beta} \frac{\partial \beta}{\partial (\sigma^2)}\right).$$

(22)

The derivative $\frac{\partial \xi(A_t)}{\partial \beta} \frac{\partial \beta}{\partial (\sigma^2)}$ directly measures the influence of uncertainty on the net benefit of being the leader. The product $\frac{\partial \xi(A_t)}{\partial A^F} \frac{dA^F}{d\beta} \frac{\partial \beta}{\partial (\sigma^2)}$ reflects the impact on the net benefit of being the leader of the fact that the follower replacement threshold increases with uncertainty.

It is easy to show that

$$\frac{\partial \xi(A_t)}{\partial \beta} \frac{\partial \beta}{\partial (\sigma^2)} < 0,$$

(23)

$$\frac{\partial \xi(A_t)}{\partial A^F} \frac{dA^F}{d\beta} \frac{\partial \beta}{\partial (\sigma^2)} > 0.$$

(24)

Therefore, at first sight, the joint impact of both effects is ambiguous. (23) represents the simple value of waiting argument: if uncertainty is large, it is more
It is possible to show that the direct effect captured by (23) dominates, irrespective of the values of the input parameters.

**Proposition 2** When uncertainty in the product market increases, the threshold value of the demand at which the leader replaces its production facility increases, too.

**Proof.** See Appendix.

From Proposition 2 it can be concluded that the leader threshold responds to volatility in a qualitatively similar way as a non-strategic threshold, i.e. it increases with uncertainty. The reason for this result is the following. First, in our model we introduced the possibility to postpone the replacement of the production facility. Increased uncertainty raises the profitability of replacement (because the follower replaces later), however, the value of the option to wait rises even more. Second, uncertainty could be beneficial for earlier replacement because of the convex shape of the net gain function, resulting in a power option-like type of payoff (Kulatilaka and Perotti [13]). Then, while performing a mean preserving spread, downside losses are more than compensated by upside gains. However, unlike the two-period framework of Kulatilaka and Perotti [13], in our continuous-time model the net gain function is always linear in the stochastic variable $A_t$. If the leader invests, the profit flow $\pi^{00}$ is replaced by the profit flow $\pi^{10}$, and it is clear from (3) and (4) that $\pi^{10} - \pi^{00}$ is linear. The same holds for the follower investment ($\pi^{11} - \pi^{01}$ linear) and simultaneous investment (linearity of $\pi^{11} - \pi^{00}$). To see whether the convexity argument could also work here, in Section 6 we consider the decision to start production. In this case the firms are not active initially and can start up production only upon investing. Consequently, the net gain flows for the leader and the follower are convex in $A_t$.

## 6 Decision to Start Production

Consider two firms having a possibility to start production in a new market where there is no incumbent. The new market assumption implies, in contrast with Sections 3-4, that the firms can only start realizing profits after incurring
a sunk cost \( I \). It still holds that demand follows the stochastic process (2). Without loss of generality the marginal cost of a unit of output after starting production is set to \( k = 0 \).

First, we calculate the optimal threshold of the follower to start production. After, by now, familiar steps it is obtained that

\[
A^{FN} = 3 \sqrt{\frac{\beta}{\beta - 2}} I (r - 2\alpha - \sigma^2). \tag{25}
\]

It is straightforward to show that

\[
\frac{\partial A^{FN}}{\partial (\sigma^2)} > 0. \tag{26}
\]

The optimal follower threshold (25) exists only for \( \sigma^2 < r - 2\alpha \). For a relatively high degree of uncertainty, i.e. for \( \sigma^2 > r - 2\alpha \) (which corresponds to \( \beta \in (1, 2) \)), the follower will never start production since for such levels of uncertainty the value of the option to invest always exceeds the net present value of investment. In the limiting case, the optimal follower threshold (25) is equal to

\[
A^{FNlim} = \lim_{\sigma^2 \to r - 2\alpha} A^{FN} = 3 \sqrt{(3r - 4\alpha) I} \tag{27}
\]

(for a derivation see Appendix). (27) corresponds to the maximal value of \( A^{FN} \) provided that it is finite. Therefore, even if the correct estimate of uncertainty is unavailable to the corporate decision maker, it is still known that the investment in the interval \( A_t \in (A^{FNlim}, \infty) \) is never optimal. In such a case the optimal investment problem is solved by calculating the uncertainty implied by the threshold \( A^{FNlim} \). Subsequently, the decision maker can decide whether the true level of uncertainty is more likely to lie below or above the implied value.

Now, let us define \( \tau \) to be the moment at which the leader starts producing in the new market. The value of the follower at \( t \geq \tau \) is equal to

\[
V^{FN} (A_t) = \begin{cases} 
\left( \frac{\pi}{r - 2\alpha - \sigma^2} I \right) \left( \frac{A_t}{A^{FN}} \right)^\beta & \text{if } A_t \leq A^{FN}, \\
\frac{1}{9} \frac{A_t^2}{r - 2\alpha - \sigma^2} - I & \text{if } A_t > A^{FN}.
\end{cases} \tag{28}
\]

The value of the leader at \( \tau \) can be expressed as

\[
V^{LN} (A_\tau) = \begin{cases} 
\frac{1}{9} \frac{A_\tau^2}{r - 2\alpha - \sigma^2} - I & \text{if } A_\tau \leq A^{FN}, \\
\frac{1}{9} \frac{A_\tau^2}{r - 2\alpha - \sigma^2} - I & \text{if } A_\tau > A^{FN}.
\end{cases} \tag{29}
\]

From (28) and (29) it is obtained that indeed the leader and follower values are convex in \( A_t \). The threshold of the leader is the smallest solution of the following equation

\[
V^{LN} (A_t) - V^{FN} (A_t) = \frac{1}{4} \frac{A_t^2}{r - 2\alpha - \sigma^2} - I - I \left( \frac{9}{4} \frac{\beta}{\beta - 2} - 1 \right) \left( \frac{A_t}{A^{FN}} \right)^\beta = 0. \tag{30}
\]
The impact of uncertainty on the threshold of the leader is not straightforward. Similar as in the model with the firms initially competing on the product market, there are two effects: the effect of the waiting option and of the strategic option. Let us denote $V^{LN}(A_t) - V^{FN}(A_t)$ by $\xi^N(A_t)$. We have

$$
\frac{d\xi^N(A_t)}{d(\sigma^2)} = \frac{\partial \xi(A_t)}{\partial (\sigma^2)} + \frac{\partial \xi(A_t)}{\partial A^{FN}} \frac{dA^{FN}}{d(\sigma^2)} + \left( \frac{\partial \xi(A_t)}{\partial \beta} + \frac{\partial \xi(A_t)}{\partial A^{FN}} \frac{dA^{FN}}{d\beta} \right) \frac{\partial \beta}{\partial (\sigma^2)}. $$

Uncertainty affects the magnitude of each of the mentioned effects via parameter $\beta$, as in Section 5, and via an effective discount rate, $r - 2\alpha - \sigma^2$. The latter contribution results from convexity of the profit function, i.e. its proportionality to the square of the underlying stochastic variable $A_t$ (see (29)).

After substituting the functional forms of $V^{LN}(A_t)$ and $V^{FN}(A_t)$ into $\xi^N(A_t)$ and calculating the derivative explicitly, the following result is obtained.

**Proposition 3** The threshold value of the demand at which the leader starts production increases in uncertainty.

**Proof.** See Appendix. ■

Analogous to the follower case, there exists a critical level of uncertainty, $\sigma^2 = r - 2\alpha$, above which it is optimal for the leader never to invest. In the limit, where $\sigma^2 \to r - 2\alpha$, the leader threshold is the smaller root of the equation (for the proof see Appendix)

$$
\left( \left( \frac{A_t}{A^{FN}} \right)^2 - 1 \right) I - \frac{A_t^2}{2(3r - 4\alpha)} \ln \left( \frac{A_t}{A^{FN}} \right) = 0.
$$

(32)

The conclusion is that also in the case of a new market, uncertainty raises the threshold levels of market demand at which it is optimal for firms to invest. Moreover, the resulting convexity of the payoff functions not only does not decrease the threshold of the firms but also results in a subset of parameters for which no replacement is optimal.

7 Uncertainty and Replacement Timing

Until now we analyze the impact of uncertainty and strategic interactions on the optimal replacement threshold of the firm. Although threshold values and timing have a lot to do with each other, it cannot be concluded in general that the relation between the two is monotonic (cf. Sarkar [19]). In this section we investigate the relationship between uncertainty, optimal threshold, expected timing of replacement and the probability with which the threshold is reached within a time interval of a given length.
First, let us observe that the expectation of the first passage time equals
\[ E_t [T^*] = \frac{1}{\alpha - \frac{1}{2}\sigma^2} \ln \frac{A^*(\sigma^2)}{A_t}, \]  
(33)
where \( A^*(\sigma^2) \) denotes the optimal replacement threshold as a function of uncertainty. We note that expectation (33) tends to infinity for \( \sigma^2 \to 2\alpha \) and does not exist for \( \sigma^2 > 2\alpha \).\(^{18}\) For \( \sigma^2 < 2\alpha \) it holds that
\[ \frac{\partial E_t [T^*]}{\partial (\sigma^2)} = \frac{1}{2 (\alpha - \frac{1}{2}\sigma^2)^2} \ln \frac{A^*(\sigma^2)}{A_t} + \frac{1}{\alpha - \frac{1}{2}\sigma^2} \frac{\partial A^*}{\partial (\sigma^2)} > 0. \]  
(34)
The expected timing of replacement increases in uncertainty due to two effects. First, for any given threshold, the associated expected first passage time is increasing in uncertainty (cf. the first component of the RHS of (34)). Second, for a fixed level of uncertainty, an increase in the optimal investment threshold leads to an increase in the expected time to reach (cf. second component of RHS of (34)). Based on (34) it can be concluded that whenever the threshold goes up due to more uncertainty, it also holds that the expected time to replace the production facility increases.

An alternative approach to measure the impact of uncertainty on the timing of replacement is to look at the probability with which the threshold is reached within a time interval of a given length, say \( \tau \). Contrary to the expected first passage time, this approach does not impose any restrictions on the values of \( \sigma \). The probabilities of optimal asset replacement within a given interval are particularly useful when this interval coincides with a budgeting period.\(^{19}\)

After substituting \( y = \ln \frac{A^*}{A_t} \) in the formula (8.11) in Harrison \[9\] and rearranging, we obtain
\[ P (T < \tau) = \Phi \left( \frac{- \ln \frac{A^*}{A_t} + (\alpha - \frac{1}{2}\sigma^2) \tau}{\sigma \sqrt{\tau}} \right) + \left( \frac{A^*}{A_t} \right)^{\frac{2\alpha}{\sigma^2} - 1} \Phi \left( \frac{- \ln \frac{A^*}{A_t} - (\alpha - \frac{1}{2}\sigma^2) \tau}{\sigma \sqrt{\tau}} \right), \]  
(35)
where \( T \) denotes the time to reach the threshold and \( \Phi \) is the standard normal cumulative density function. As already pointed out in Sarkar \[19\], the derivative \( \frac{\partial P(T < \tau)}{\partial \sigma} \) does not have an unambiguous sign and it can thus be shown
\(^{17}\)For a derivation of the probability distribution of the first passage time see Harrison \[9\] for a formal exposition and Dixit \[3\] for a heuristic approach.
\(^{18}\)Increasing \( \sigma^2 \) beyond \( 2\alpha \) implies that the probabilities of surviving without reaching the threshold before a given time do not fall sufficiently fast for longer hitting times (nevertheless, the probability that the process will reach the barrier in infinity is still positive). Since the expectation is the sum of the product of the first passage times and their probabilities, an insufficient decay in the survival probabilities (without reaching the threshold) results in the divergence of the expectation.
\(^{19}\)For a discussion of the capital budgeting process at a corporate level see Kaplan and Atkinson \[12\], Ch. 14 and Bower \[1\], Ch. 1-3.
that, in general, uncertainty can affect the probability of reaching the threshold within a given time in both directions.

First, we illustrate the relationship between the first passage time, volatility and related probabilities for the follower threshold since this threshold is unaffected by the strategic considerations. Subsequently, we present results of simulations related to the threshold of the leader. In this part we use the model of Section 3-4. The results for the decision to start production are qualitatively similar and are not reported.

Figure 3

Figure 3. The cumulative probability of reaching the optimal follower replacement threshold as a function of demand uncertainty (left) and time (right) for the set of parameter values: $A_t = 4$, $r = 0.05$, $\alpha = 0.015$, $K = 3$, $k = 0$ and $I = 60$.

From Figure 3 (left) it can be concluded that the form of relationship between uncertainty and the probability of reaching the threshold depends on the time to reach. For high values of the time to reach, the cumulative probability of reaching the threshold decreases in volatility since the probability mass of the first passage time density function moves to the right (cf. (34)). For low values of $\tau$ the probability of reaching the threshold first increases due to a spread in the probability mass. However, for high volatilities the effect of the probability mass shifting to the right dominates, so that the cumulative probability of reaching the threshold becomes smaller again.

Figure 3 (right) indicates that the probability of reaching the follower threshold always increases with the time interval which is of course trivial. The relevant observation is that this relationship is more pronounced for low levels of market uncertainty. This results from the fact that in the absence of uncertainty the optimal investment trigger is reached at a specified point in time with probability 1 and the corresponding cumulative density function is a heaviside step function. Increasing volatility spreads the probability mass around the point corresponding to the deterministic case. This leads to an increased cumulative chance of reaching the trigger at points in time situated to the left of this specified point in time, while the reverse is true for the point situated to the right. This influences the shape of the cumulative distribution function whose slope decreases in uncertainty.
Figure 4.

The derivative with respect to market uncertainty of the cumulative probability of reaching the optimal follower threshold as a function of uncertainty itself (left) and time (right) for the set of parameter values: $A_t = 4$, $r = 0.05$, $\alpha = 0.015$, $K = 3$, $k = 0$ and $I = 60$.

Figure 4 allows for a closer inspection of the relationship between the timing of asset replacement and uncertainty. From the left window it can be concluded that, irrespective from the time to reach, there exists a level of uncertainty beyond which a further increase in uncertainty reduces the probability of the optimal asset replacement. This relationship is inverse, i.e. the longer the time interval, the lower level of uncertainty for which its further growth reduces the probability of the optimal replacement. For example, using the parameters from Figure 4 we can conclude that for $\tau = 5$ this critical value of uncertainty, $\sigma$, is 0.234, for $\tau = 10$ it is only 0.118, whereas for $\tau = 20$ increased uncertainty always reduces the cumulative probability of optimal investment.

The right window of Figure 4 indicates that the probability of the optimal replacement increases in uncertainty for a sufficiently small time to reach and decreases for a sufficiently large time. Moreover, the derivative of the probability of reaching the optimal threshold changes its sign only once. Finally, Figure 4 (right) allows for the conclusion that the time to reach beyond which uncertainty negatively affects the probability of optimal replacement is, again, negatively related to its level. For $\sigma = 0.1$ the time to reach which separates the areas of a positive and a negative relationship reaches the value of 11.46 years, for $\sigma = 0.2$ it equals 5.87 years, while for $\sigma = 0.3$ it drops to 4.06 years.

Despite the presence of strategic effects, the probability of asset replacement of the leader responds to changes in uncertainty and time to reach in a similar way as the corresponding probabilities of the follower. For low $\sigma$’s the probability of investing increases more rapidly with the length of the time interval than for high $\sigma$’s. Moreover, for high $\tau$’s the probability of replacing the existing asset is always decreasing in uncertainty, while for low $\tau$’s the probability behaves in a non-monotonic way.

These latter observations are confirmed by the following proposition.

**Proposition 4** Define

$$
\tau^* = \frac{1}{\alpha} \ln \frac{A^*}{A_t},
$$

(36)
as the point in time at which the replacement threshold $A^*$ is reached in the deterministic case. Then it holds that for $\tau < \tau^*$ the probability of reaching the investment threshold $A^*$ before $\tau$ increases in uncertainty at a relatively low level of uncertainty and decreases for a relatively high level, whereas for $\tau > \tau^*$ the probability of reaching the optimal threshold before $\tau$ always decreases in uncertainty.

**Proof.** See Appendix. ■

The results of simulations concerning the relationship between uncertainty, first passage time and probabilities of reaching the leader threshold are presented in Table 2 below.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\tau = 1$</th>
<th>$\tau = 2$</th>
<th>$\tau = 5$</th>
<th>$\tau = 10$</th>
<th>$\tau = 15$</th>
<th>$\tau = 20$</th>
</tr>
</thead>
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<td>2.306</td>
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<td>54.32</td>
<td>71.17</td>
<td>80.97</td>
</tr>
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<td>0.061</td>
<td>5.935</td>
<td>26.79</td>
<td>47.94</td>
<td>59.70</td>
<td>67.24</td>
</tr>
<tr>
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<td>21.00</td>
<td>36.47</td>
<td>45.10</td>
<td>50.72</td>
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<td>0.063</td>
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<td>16.50</td>
<td>28.66</td>
<td>35.31</td>
<td>39.55</td>
</tr>
<tr>
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<td>0.063</td>
<td>3.301</td>
<td>13.57</td>
<td>23.22</td>
<td>28.29</td>
<td>31.39</td>
</tr>
<tr>
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<td>0.063</td>
<td>2.932</td>
<td>11.55</td>
<td>19.23</td>
<td>23.02</td>
<td>25.21</td>
</tr>
</tbody>
</table>

Table 2. The cumulative probability (in percentages) of reaching the optimal leader replacement threshold as a function of demand uncertainty for the set parameter values: $A_t = 2, \tau = 0.05, \alpha = 0.015, k = 0, K = 3$ and $I = 60$. The optimal timing of replacement in the deterministic case equals $\tau^* = 9.36$.

We conclude that, while under increased uncertainty the threshold increases, the probability that the firm optimally replaces its production facility within a given amount of time decreases when this amount of time is sufficiently large. However, when this amount of time is sufficiently low there are two contradictory effects. On the one hand, this probability goes up because higher volatility enhances the chance of reaching a particular threshold early. On the other hand, this probability eventually goes down with uncertainty because then the effect of the probability mass shifting to the right begins to dominate.

## 8 Conclusions

The purpose of this paper is to analyze the firm’s decision to replace an existing production facility with a technologically superior one. In order to capture the effect of strategic interactions among the firms operating in an imperfectly competitive and uncertain environment we model the product market as a Cournot duopoly with a stochastic demand parameter. Such a formulation results in the payoff functions being convex in the stochastic demand parameter.
We determine the type of equilibria of the real option game played by the firms. We show that it is optimal for the firms to replace their production facilities sequentially when the associated cost is relatively low and simultaneously otherwise.

Furthermore, we find that the direct effect of uncertainty (related to the waiting option) on the replacement threshold of the leader is always larger than the indirect effect (strategic option) resulting from the delay in the follower decision to replace its production facility. Consequently, irrespective of the type of equilibrium, increasing uncertainty always raises the level of demand triggering the optimal replacement. This result also holds in case of the decision to start production rather than the decision to replace.

Moreover, it can be concluded that the expected timing of replacement increases with uncertainty. This result supports the view that uncertainty delays the implementation of the new technology, even in the presence of strategic interactions combined with a convex profit function.

We also look at the probability of replacing the production facility within a certain time interval. Here, the point in time at which replacement is made optimally in the deterministic case plays a crucial role. For an interval that contains this point in time, the probability of optimal replacement within this time interval decreases with uncertainty. However, if this time interval is that short that the optimal replacement time in the deterministic case lies outside this interval, then the replacement probability goes up with uncertainty when uncertainty is low while it goes down otherwise.

9 Appendix

Proof of Proposition 1. First, let us define

$$
\zeta(A_t) = V_S(A_t) - V_L(A_t).
$$

(37)

After substitution we get

$$
\zeta(A_t) = -\frac{4}{9} K A_t^r - \alpha + I + \left( A_S^\beta \frac{4}{9} K \alpha - I \right) A_t^\beta - \frac{1}{2} \left( I + \frac{4}{9} K \alpha \right) A_t^\beta + \frac{1}{2} \left( I + \frac{4}{9} K \alpha \right) A_t^\beta.
$$

(38)

for $A_t \leq A_F$. From (19) it follows that if on the interval $[A_0, A_F]$ the minimum of $\zeta(A_t)$ is smaller than zero, a preemptive equilibrium occurs. Otherwise, the firms replace their production facilities simultaneously.\(^\text{20}\) The existence of a negative minimum of $\zeta(A_t)$ depends, as mentioned above, on the value of the input parameters. The minimum of $\zeta(A_t)$ occurs for

$$
A^{**} = \left( \frac{\frac{4}{9} K \alpha - I}{\left( \frac{1}{2} \left( I + \frac{4}{9} K \alpha \right) + \frac{1}{2} \left( I + \frac{4}{9} K \alpha \right) \right) (A_F^\beta A_S^\beta)} \right)^{\frac{1}{\beta - 1}}.
$$

(39)

\(^{20}\)Strictly speaking, the equilibria with sequential entry still exists in this case but is Pareto-dominated by the simultaneous entry equilibrium (cf. Fudenberg and Tirole [6]).
It is sufficient to show that
\[
\frac{d \zeta (A_t)}{d I} \bigg|_{A_t = A^{**}} = \frac{\partial \zeta (A_t)}{\partial I} + \frac{\partial \zeta (A_t)}{\partial A_t} \bigg|_{A_t = A^{**}} \frac{d A^{**}}{d I} > 0. \tag{40}
\]
From (38) we derive that
\[
\frac{d \zeta (A_t)}{d I} = 1 - \left( \frac{A_t}{A^*} \right)^\beta - \beta \frac{1}{2} I + \frac{K^2}{3 r} \left( \frac{A_t}{A^*} \right)^\beta. \tag{41}
\]
Subsequently, we substitute for \( A_t \) in (41) the expression (39) for \( A^{**} \). Complexity of (41) yields the necessity to use a numerical procedure. A geometric grid search indicates that \( \frac{d \zeta (A_t)}{d I} \bigg|_{A_t = A^{**}} \) is positive for \( \beta \in [1, \infty), \alpha \in \mathbb{R}, r \in (\alpha, \infty), K \in \mathbb{R}_{++} \) and \( I \in \left( \frac{4}{3} \frac{K^2}{r}, \infty \right). \)

**Proof of Proposition 2.** Differentiating (16) yields
\[
\frac{d \xi (A_t)}{d \beta} = \left( \frac{\beta}{\gamma - 1} \left( 3 I + \frac{2 K^2}{3} \right) - \frac{K^2}{3 r} - I \right) \ln \left( \frac{A_t}{A^*} \right) - \frac{1}{\beta - 1} \left( \frac{1}{2} I + \frac{1}{3} \frac{K^2}{r} \right). \tag{42}
\]
Since the threshold of the leader is equal to \( A^P \), and \( A^P \) is the smallest root of the concave function \( \xi (A_t) \), we know that
\[
\frac{\partial \xi (A_t)}{\partial A_t} \bigg|_{A_t = A^P} > 0. \tag{43}
\]
Consequently, from the envelope theorem we conclude that it is sufficient to show that
\[
\frac{d \xi (A_t)}{d \beta} \bigg|_{A_t = A^P} > 0 \tag{44}
\]
and conclude that the replacement threshold of the leader is increasing in uncertainty (decreasing in \( \beta \)). Moreover, we know from (42) that \( \frac{d \xi (A_t)}{d \beta} \) changes its sign only once and the corresponding realization of \( A_t \) to the zero value of the derivative is
\[
A^* = A^P e^{-\left( \frac{1}{\beta - 1} \left( 3 I + \frac{2 K^2}{3} \right) - \frac{K^2}{3 r} - I \right)}. \tag{45}
\]
Therefore
\[
\frac{d \xi (A_t)}{d \beta} \bigg|_{A_t = A^P} > 0 \quad \text{iff} \quad A_t < A^*. \tag{46}
\]
Consequently, \( \xi (A^*) > 0 \) would imply that \( A^* > A^P \) and \( \frac{d \xi (A_t)}{d \beta} \bigg|_{A_t = A^P} > 0 \). In order to prove that \( \xi (A^*) > 0 \), we plug (45) into (16) to obtain
\[
\xi (A^*) = \frac{\beta}{\beta - 1} \left( 3 I + \frac{2 K^2}{3} \right) e^{-\left\{ \beta \left( \frac{1}{\beta - 1} \left( 3 I + \frac{2 K^2}{3} \right) - \frac{K^2}{3 r} - I \right) \right\}} - \left( \frac{\beta}{\beta - 1} \left( 3 I + \frac{2 K^2}{3} \right) - \frac{K^2}{3 r} - I \right) e^{-\left\{ \beta \left( \frac{1}{\beta - 1} \left( 3 I + \frac{2 K^2}{3} \right) - \frac{K^2}{3 r} - I \right) \right\}}. \tag{47}
\]
An analytical proof is again not possible but numerically it can be shown that \( \xi(A^*) \) is positive for \( \beta \in [1, \infty), \alpha \in \mathbb{R}, r \in (\alpha, \infty), K \in \mathbb{R}^+ \) and \( I \in \left( \frac{3}{2} \frac{K^2}{r}, \infty \right) \).

**Limiting value of the optimal threshold to start production (follower).**

We are interested in the following limit

\[
\lim_{\sigma^2 \to r-2\alpha} 3 \sqrt{\frac{\beta}{\beta - 2} I (r - 2\alpha - \sigma^2)} = 3 \sqrt{I} \lim_{\sigma^2 \to r-2\alpha} \frac{\beta}{\beta - 2} (r - 2\alpha - \sigma^2). \tag{48}
\]

Furthermore, we have

\[
\lim_{\sigma^2 \to r-2\alpha} \frac{\beta}{\beta - 2} (r - 2\alpha - \sigma^2) = \lim_{\sigma^2 \to r-2\alpha} \frac{\frac{\beta}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{\beta}{2}\right)^2 + \frac{2r}{\sigma^2}}}{r - 2\alpha - \sigma^2}
\]

\[
= \frac{2 (r - 2\alpha)}{\frac{3}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}
\]

Applying l’Hôpital yields

\[
2 (r - 2\alpha) \lim_{\sigma^2 \to r-2\alpha} \frac{-1}{\frac{3}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}
\]

\[
= 2 (r - 2\alpha) \frac{-1}{\frac{3}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}(r - 2\alpha)\right)^2 + 2r(r - 2\alpha)}}
\]

\[
= \frac{-3 \sqrt{(\alpha - \frac{1}{2}(r - 2\alpha))^2 + 2r(r - 2\alpha)} - \alpha + \frac{1}{2}(r - 2\alpha) + 2r}{-3 \left| 2\alpha - \frac{3}{2}r \right| - \alpha + \frac{1}{2}(r - 2\alpha) + 2r}
\]

Since \( 2\alpha < r \), this is equal to

\[
\frac{4 (r - 2\alpha) \left( 2\alpha - \frac{3}{2}r \right)}{3 \left( 2\alpha - \frac{3}{2}r \right) - \alpha + \frac{1}{2}(r - 2\alpha) + 2r}
\]

\[
= \frac{4 (r - 2\alpha) \left( 2\alpha - \frac{3}{2}r \right)}{- (r - 2\alpha)}
\]

\[
= 3r - 4\alpha. \tag{49}
\]
Substituting (49) into (48) yields the desired result.

**Limiting value of the optimal threshold to start production (leader).**

To obtain the leader’s limiting threshold, we are interested in the form of function $\xi^N$ when $\sigma^2$ tends to $r - 2\alpha$. For any $A_t \in (0, A^{FN})$ we have (cf. (28), (29) and $\xi^N = V^{LN} - V^{FN}$)

$$
\lim_{\sigma^2 \to r - 2\alpha} \frac{1}{4} \frac{A_t^2}{r - 2\alpha - \sigma^2} - I - \left(\frac{1}{4} \frac{(A^{FN})^2}{r - 2\alpha - \sigma^2} - I\right) \left(\frac{A_t}{A^{FN}}\right) - I
$$

$$
= \lim_{\sigma^2 \to r - 2\alpha} \frac{1}{4} \frac{A_t^2}{r - 2\alpha - \sigma^2} - I - \left(\frac{1}{4} \frac{(3r - 4\alpha) I}{r - 2\alpha - \sigma^2} - I\right) \left(\frac{A_t}{3 \sqrt{(3r - 4\alpha) I}}\right)^\beta,
$$

by using the limit of (25). Consequently, we rearrange (50) to get

$$
\lim_{\sigma^2 \to r - 2\alpha} \frac{1}{4} \frac{A_t^2}{r - 2\alpha - \sigma^2} - I + \frac{A_t^2}{9(3r - 4\alpha)} - \frac{1}{4} \frac{(3r - 4\alpha) I}{r - 2\alpha - \sigma^2} \left(\frac{A_t}{3 \sqrt{(3r - 4\alpha) I}}\right)^\beta
$$

$$
= \frac{A_t^2}{9(3r - 4\alpha)} - I + \frac{A_t^2}{4} \lim_{\sigma^2 \to r - 2\alpha} \frac{1 - \left(\frac{A_t}{3 \sqrt{(3r - 4\alpha) I}}\right)^\beta}{r - 2\alpha - \sigma^2}. \quad (51)
$$

The limit in the last component can be calculated as follows

$$
\lim_{\sigma^2 \to r - 2\alpha} \frac{1 - \left(\frac{A_t}{3 \sqrt{(3r - 4\alpha) I}}\right)^\beta}{r - 2\alpha - \sigma^2} = \lim_{\sigma^2 \to r - 2\alpha} \frac{1 - \left(\frac{A_t}{3 \sqrt{(3r - 4\alpha) I}}\right)^\beta}{r - 2\alpha - \sigma^2}
$$

Application of l’Hôpital rule yields

$^2$We do so by observing that $\lim_{\sigma^2 \to r - 2\alpha} \beta = 2$. 

23
\[
\lim_{\sigma^2 \to r-2\alpha} \frac{\partial}{\partial (\sigma^2)} \left( \frac{A_t}{3\sqrt{(3r-4\alpha)I}} \right)^{-\frac{3}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{\alpha r}{\sigma^2}}} \quad (53)
\]

\[
= \lim_{\sigma^2 \to r-2\alpha} \ln \left( \frac{A_t}{3\sqrt{(3r-4\alpha)I}} \right) \left( \frac{A_t}{3\sqrt{(3r-4\alpha)I}} \right)^{-\frac{3}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{\alpha r}{\sigma^2}}}
\times \left( \frac{\alpha - \frac{\alpha r}{\sigma^2} \left( \frac{\alpha}{\sigma^2} - \frac{1}{2} \right) + \frac{r}{\sigma^2}}{\sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{\alpha r}{\sigma^2}}} \right)
\]

\[
= \ln \left( \frac{A_t}{3\sqrt{(3r-4\alpha)I}} \right) \left( \frac{\alpha}{(r-2\alpha)^2} - \frac{\alpha - \frac{\alpha r}{\sigma^2} \left( \frac{\alpha}{\sigma^2} - \frac{1}{2} \right) + \frac{r}{\sigma^2}}{\frac{3}{2}r - 2\alpha} \right)
\]

\[
= \ln \left( \frac{A_t}{3\sqrt{(3r-4\alpha)I}} \right) \left( -\frac{1}{\frac{3}{2}r - 2\alpha} \right). \quad (54)
\]

Consequently, after substituting (54) into (51), we obtain the formula for the limiting case of \( \xi^N \)

\[
\left( \left( \frac{A_t}{A^{PN}} \right)^2 - 1 \right) I - \frac{A_t^2}{2(3r-4\alpha)} \ln \left( \frac{A_t}{A^{PN}} \right). \quad (55)
\]

**Proof of Proposition 3.** First we substitute the functional forms of the value functions into (30) and rewrite the derivative (31) as

\[
\frac{d\xi^N (A_t)}{d(\sigma^2)} = \frac{1}{4} \frac{A_t^2}{(r-2\alpha-\sigma^2)^2} - \frac{5I}{\beta - 2} \left( \frac{A_t}{A^{PN}} \right) \beta
\times \left[ \frac{\partial \beta}{\partial (\sigma^2)} + \left( \beta + \frac{8}{5} \right) \left( \frac{\beta}{2(r-2\alpha-\sigma^2)} + \ln \left( \frac{A_t}{A^{PN}} \right) \frac{\partial \beta}{\partial (\sigma^2)} \right) \right]. \quad (56)
\]

Denote the smallest solution of \( \xi^N (A_t) = 0 \) by \( A^{PN} \). Since \( A^{PN} \) cannot be explicitly derived, we proceed as follows. First, we consider a particular point \( \overline{A} > A^{PN} \). Second, we show that \( \frac{d\xi^N (A_t)}{d(\sigma^2)} \) is negative for all \( A_t \in (\underline{A}, \overline{A}) \), where \( \underline{A} \) is a realization of \( A_t \) such that \( \underline{A} < A^{PN} \). Let us define

\[
\overline{A} = 2 \sqrt{\beta - 2} I (r-2\alpha-\sigma^2). \quad (57)
\]

First, we show that \( \xi^N (\overline{A}) > 0 \) which would imply that \( \overline{A} > A^{PN} \). After substituting (57) into (30) we obtain

\[
\xi^N (\overline{A}, \beta) = \frac{2I}{\beta - 2} \left( 1 - \left( \frac{5}{8} \beta + 1 \right) \left( \frac{2}{3} \right) \right) = \frac{2I}{\beta - 2} \phi (\beta). \quad (58)
\]

24
Since $\beta > 2$ (recall that for $\beta \leq 2$ no firm is willing to enter), we know that $\frac{\partial^2 I}{\partial \beta^2}$ is always positive. Therefore we are interested only in the sign of $\phi (\beta)$. For $\beta \downarrow 2$ we obtain

$$\lim_{\beta \downarrow 2} \phi(\beta) = 0.$$  \hspace{1cm} (59)$$

Then we establish that

$$\frac{\partial \phi (\beta)}{\partial \beta} = - \left( \frac{2}{3} \right)^{\beta} \left( \frac{5}{8} \beta + \frac{5}{8} + 1 \ln \left( \frac{2}{3} \right) \right) > 0$$  \hspace{1cm} (60)$$

for $\beta \in (2, \infty)$. This implies that $\xi^N (\overline{A})$ is positive which implies that $\overline{A} > A^{pN}$. Furthermore, we prove that (56) changes signs twice, i.e. it is positive for $A_t \in (0, \overline{A}) \cup (A^{pN}, A^{\infty})$, where $\overline{A}$ is some realization of $A_t$ such that $\overline{A} > A$, and negative otherwise. First, we express (56) as

$$\frac{d\xi^N (A_t)}{d(\sigma^2)} = A^2_t \left[ K A_t^{\beta - 2} + L A_t^{\beta - 2} \ln \left( \frac{A_t}{A^{pN}} \right) + M \right],$$  \hspace{1cm} (61)$$

where

$$K = - \frac{5}{3} I (A^{pN})^{-3} \beta \left( \frac{\partial \beta}{\partial (\sigma^2)} + \frac{\beta (\beta + \frac{8}{5})}{2 (r - 2 \alpha - \sigma^2)} \right),$$  \hspace{1cm} (62)$$

$$L = - \frac{5}{3} I (A^{pN})^{-3} \left( \beta + \frac{8}{5} \right) \frac{\partial \beta}{\partial (\sigma^2)} > 0,$$  \hspace{1cm} (63)$$

$$M = \frac{1}{4} (r - 2 \alpha - \sigma^2)^2 > 0.$$  \hspace{1cm} (64)$$

From (61) - (64) we know that\textsuperscript{22}

$$\lim_{A_t \downarrow 0} K A_t^{\beta - 2} + L A_t^{\beta - 2} \ln \left( \frac{A_t}{A^{pN}} \right) + M = M, \quad \text{and}$$  \hspace{1cm} (65)$$

$$\lim_{A_t \rightarrow \infty} K A_t^{\beta - 2} + L A_t^{\beta - 2} \ln \left( \frac{A_t}{A^{pN}} \right) + M = \infty.$$  \hspace{1cm} (66)$$

Moreover

$$\frac{\partial}{\partial A_t} \left( K A_t^{\beta - 2} + L A_t^{\beta - 2} \ln \left( \frac{A_t}{A^{pN}} \right) + M \right) = A_t^{\beta - 3} \left( (\beta - 2) K + (\beta - 2) L \ln \left( \frac{A_t}{A^{pN}} \right) + L \right),$$

which implies that there exists only one optimum of $\frac{d\xi^N (A_t)}{d(\sigma^2)}$ that is different from zero. This result, combined with (65) and (66), implies that $\frac{d\xi^N (A_t)}{d(\sigma^2)}$ is

\textsuperscript{22}The result (65) has been derived using the l'Hôpital and observing that $A_t^{2 - \beta}$ explodes in the neighborhood of zero faster than $\ln A_t$. 

25
negative at most in only one interval. Substituting $\overline{A}$ into (56) yields

$$\frac{d\xi^N(A_t)}{d(\sigma^2)} \bigg|_{A_t=\overline{A}} = \frac{2L}{r - 2\alpha - \sigma^2} - \frac{3L}{\beta - 2} \left( \frac{2}{3} \right)^{\beta}$$

(67)

$$\times \left[ \frac{\partial \beta}{\partial (\sigma^2)} + \left( \beta + \frac{8}{5} \right) \left( \beta + \frac{2}{5} (r - 2\alpha - \sigma^2) + \ln \left( \frac{2}{3} \right) \frac{\partial \beta}{\partial (\sigma^2)} \right) \right]$$

Numerically it can be shown that $\frac{d\xi^N(A_t)}{d(\sigma^2)} \bigg|_{A_t=\overline{A}}$ is negative for $\beta \in [2, \infty)$, $\alpha \in \mathbb{R}$, $r \in (\alpha, \infty)$ and $I \in (0, \infty)$. Therefore the only remaining part of the proof is to show that $\underline{A} < A^{PN}$ for any vector of input parameters. Since the explicit analytical forms of $\underline{A}$ and $A^{PN}$ do not exist, we use a numerical procedure. For any given vector of input parameters (from the domains as in the preceding part of the proof), we calculate the difference $A^{PN} - \underline{A}$ and it turns out that it is always positive. Given that $\frac{d\xi^N(A_t)}{d(\sigma^2)} \bigg|_{A_t=\underline{A}} < 0$, $A^{PN} \in (\underline{A}, \overline{A})$ and $\xi^N(\overline{A}) > 0$, we conclude that $\frac{dA^{PN}}{d(\sigma^2)} > 0$, i.e. the investment threshold of the leader increases in uncertainty. 

**Proof of Proposition 4.** First, we show that $\tau^*$ is the time to reach the replacement threshold $A^*$ in the deterministic case. After observing that $x = \alpha t$ is the solution to $dx = \alpha dt$ with initial condition $x_0 = 0$, and substituting $x^* = \ln A^*$, we obtain

$$\ln \frac{A^*}{A_t} = \alpha \tau^*,$$

(68)

so $\tau^*$ in (36) is the time to reach the threshold $A^*$. Now, we consider the density function $\varphi(\tau; \mu(\sigma), \sigma^2)$. For the moment we assume that $\mu = \tau^*$ irrespective from $\sigma$. Then increasing $\sigma$ is equivalent to performing a mean preserving spread. Consequently in such a case

$$\frac{\partial}{\partial \sigma} \left( \int_0^\tau \varphi(s) \, ds \right) (\tau - \tau^*) < 0.$$ 

(69)

When $\varphi(\tau; \mu(\sigma), \sigma^2)$ is the density function of the first passage time for a geometric Brownian motion, $E[\tau]$ is increasing in $\sigma$ (cf. (34)) when $A^*$ is increasing in $\sigma$, too. Therefore, there is another effect contributing to the sign of the derivative $\frac{\partial}{\partial \sigma} \left( \int_0^\tau \varphi(s) \, ds \right)$. For $\tau > \tau^*$, an increase in uncertainty not only reduces the probability mass to the left of $\tau$ via the mean preserving spread but also because of the mean moving to the right itself. Therefore the effect of uncertainty on the probability of the replacement decision is unambiguous in this region and negative. For $\sigma \to \infty$ the probability of investing before $\tau$ decreases to zero. The latter conclusion is true since from (35) it is obtained
\[
\lim_{\sigma \to \infty} P(T < \tau) = \lim_{\sigma \to \infty} \Phi \left( \frac{-\ln \frac{A^*}{A} + (\alpha - \frac{1}{2}\sigma^2)}{\sigma \sqrt{\tau}} \right) \\
+ \lim_{\sigma \to \infty} \left[ \Phi \left( \frac{-\ln \frac{A^*}{A} - (\alpha - \frac{1}{2}\sigma^2)}{\sigma \sqrt{\tau}} \right) \left( \frac{A^*}{A} \right)^{2\alpha - 1} \right] \\
= \lim_{\sigma \to \infty} \left[ \Phi \left( \frac{-\ln \frac{A^*}{A} - (\alpha - \frac{1}{2}\sigma^2)}{\sigma \sqrt{\tau}} \right) \left( \frac{A^*}{A} \right) \right] = 0 \quad (70)
\]

for
\[
\lim_{\sigma \to \infty} A^* = \infty. \quad (71)
\]

For \( \tau < \tau^* \), the two effects work in the opposite direction. As in the previous case, the mean \( E[\tau] \) is increasing in uncertainty. Without a change in the volatility, an increase in the mean would then decrease the probability of replacing the existing production facility. However, increasing uncertainty results in a greater probability mass being present in the left tail of \( \varphi(\tau) \). Therefore, the total effect of increasing uncertainty is ambiguous in this region. However, we are able to conclude that the probability of investing at a given \( \tau \) behaves in a certain non-monotonic way. For \( \sigma = 0 \), there is no probability mass on the interval \([0, \tau]\). Therefore an increase in uncertainty initially leads to an increased probability of investment. For relatively large \( \sigma \) the effect of moving the mean of the distribution to the right starts to dominate and the probability of asset replacement falls. For \( \sigma \to \infty \) the probability of replacing the existing asset before a given time \( \tau \) decreases to zero.

Finally, we show that all the thresholds increase in uncertainty monotonically and unboundedly. We already know (from Proposition 2 and 3) that the optimal replacement thresholds increase in uncertainty monotonically. So now we only have to prove that the thresholds grow in uncertainty unboundedly. For the thresholds of the follower and in case of simultaneous replacement it is easy to observe that \( \frac{\beta}{\gamma} \) tends to infinity when \( \gamma \to \infty \). The replacement threshold of the leader requires slightly more attention. We already know that the leader replaces its asset as soon as the stochastic variable reaches the smallest root of the following equation (cf. (16))

\[
0 = \frac{2}{3} K_A \left( r + \frac{K^2}{3r} - I - \left( \frac{2}{3} K A^F \left( \frac{K^2}{3r} - I \right) \left( \frac{A^*}{A} \right)^{\beta} \right) \right). \quad (72)
\]

After substituting expression of \( A^F \) (see Table 1) into (72) and rearranging, we

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23For new market model the similar conclusion can be drawn after the substitution of parameters in the original geometric Brownian motion.

24The unboundedness of the leader threshold in the new market entry can be proven in a similar way as in the presented case of technology adoption.
obtain

\[
0 = \left( 1 - \left( \frac{\frac{4}{3} KA_t}{\beta - 1} (I + \frac{4K^2}{9r})(r - \alpha) \right)^{\beta - 1} + \frac{\left( \frac{K^2}{3r} + I \right) A_t^{\beta-1}}{\frac{2}{3} r - \frac{K^2}{3r} - I} \right) \times \frac{2}{3} r - \alpha - \frac{K^2}{3r} - I.
\]

(73)

It holds that

\[
\lim_{\beta \downarrow 1} \left( 1 - \left( \frac{\frac{4}{3} KA_t}{\beta - 1} (I + \frac{4K^2}{9r})(r - \alpha) \right)^{\beta - 1} + \frac{\left( \frac{K^2}{3r} + I \right) A_t^{\beta-1}}{\frac{2}{3} r - \frac{K^2}{3r} - I} \right) = 0.
\]

(74)

Since

\[
\lim_{\beta \downarrow 1} \frac{\partial}{\partial \beta} \left( \frac{\frac{4}{3} KA_t}{\beta - 1} (I + \frac{4K^2}{9r})(r - \alpha) \right)^{\beta - 1} < 0,
\]

(75)

the LHS of (74) approaches zero from above. Now, we are looking for the solution of

\[
0 = m(x) x - n, \quad \forall x \in \mathbb{R}^{++} \quad m(x) > 0,
\]

(76)

such that \( m(x) \) is tending to zero from above (this is guaranteed by (75)) \( \forall x \in \mathbb{R}^{++} \) when the uncertainty is increasing. Consequently, any solution (so the smallest one as well) of (76) is tending to infinity. This is equivalent to

\[
\lim_{\beta \downarrow 1} A^P = \infty,
\]

(77)

which completes the proof. □

References


