History dependence without unstable steady state

Hartl, R.F.; Kort, P.M.

Published in:
Journal of Mathematical Economics

Publication date:
2003

Citation for published version (APA):
Hartl, R. F., & Kort, P. M. (2003). History dependence without unstable steady state: A non-differentiable framework. *Journal of Mathematical Economics*, *39*(8), 891-900.

History dependence without unstable steady state: a non-differentiable framework.

Richard F. Hartl (corresponding author)

Institute of Management, University of Vienna,

Brünnerstraße 8, A-1210 Vienna, Austria

Richard.Hartl@univie.ac.at

Tel. +43-1-4277-38091, Fax. +43-1-4277-38094

Peter M. Kort

Department of Econometrics and CentER, Tilburg University,

P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Kort@kub.nl, and

Department of Economics, UFSIA University of Antwerp,

Antwerp, Belgium

revised:

February 23, 2005

Abstract

In this paper, a non-differentiability is identified as a new mechanism that creates history dependence in dynamic economic models. A unique characteristic is that this occurs although no unstable steady state exists. This is shown by studying a capital accumulation model in which the revenue function exhibits a kink. It can be expected that history dependence will occur in other models with non-differentiabilities as well.

JEL #: C62, D92

Key Words: history dependence, dynamic economics, Skiba threshold

1 Introduction

Recent work on dynamic economics has stressed the fact that economic outcomes may be history-dependent. The term history-dependence has been made popular by the work of W.B. Arthur describing economic paths of new technologies that exhibit increasing returns and positive feedbacks (see, e.g., Arthur 1989, 1994). Positive feedbacks may arise locally, and can already be found in early models in development economics using convex - concave production functions. From the onset it has been recognized that such convex-concave production functions may imply multiple steady states - usually, two saddle-points and one unstable middle steady-state in between. As a consequence, the long-term behavior of the economy will be history-dependent. According to the conditions prevailing in the first phases of development, the economy will converge to either one or the other saddle-point. Consequently, there exists a threshold where the dynamics leading to these two different long-term solutions separate. Following the pioneering articles of Skiba (1978) and Dechert and Nishimura (1983), such thresholds have been occasionally called 'Skiba points' in the economic literature (see, e.g., Brock and Malliaris, 1989, and Feichtinger and Hartl, 1986). Recently, these thresholds have often been called DNS-points because of the contributions of Dechert, Nishimura and Skiba to the discovery of this phenomenon of thresholds.

History-dependence means that, depending on the initial conditions, the optimal solutions of the dynamic system considered converge towards two or more distinct attractors. These attractors can be saddle-points or boundary equilibria such as the origin (if the feasible state-space is bounded by non-negativity conditions). In higher-dimensional models, limit cycles may also occur as attractors. The existence of two or more steady-states usually implies the existence of at least one unstable steady-state. Furthermore, the existence of optimal paths converging to different attractors implies the existence of a threshold, i.e. of a DNS-point, on which the decision-maker is indifferent between choosing one of these optimal paths.

A common denominator of most existing contributions in this area is that a non-concavity is introduced to explain history-dependent outcomes. Although there are several economic characteristics leading to 'convexities' with respect to a state, 'increasing returns' dominates by far. The hypothesis of increasing returns is picked up also in other areas, e.g. in regulatory economics and in endogenous growth theory pioneered by Romer (1986); see also Barro and Sala-I-Martin (1995), Ciccone and Matsuyama (1996), or Santos (1999). Of course, increasing returns are capable of generating history-dependent outcomes too, even for the efficient, i.e. 'planned', economy; see Matsuyama (1991) and Ladron-de-Guevara et al. (1999).

In this paper we identify a yet unknown mechanism that causes the occurrence of history dependence. In particular we show that a non-differentiability with respect to the state variable in a dynamic economic problem can also be a source for multiple long run equilibria. A remarkable characteristic of the resulting solution is that this history dependence appears without simultaneous occurrence of an unstable steady state. To the best of our knowledge this is a completely new feature. We apply the well known capital accumulation framework to make our point but our analysis is not restricted to this application area.

The paper is organized as follows. Sections 2 contains the general analysis of a capital accumulation model. The existence of the DNS-point is verified in a linear quadratic framework in Section 3, and Section 4 concludes.

2 General Analysis

It is convenient to adopt the capital accumulation framework for our model. Let the state k denote the capital stock and the control u be the investment rate. The revenue function is given by $r(k)$ while the investment costs are $c(u)$. The discount rate is ρ while μ denotes the depreciation rate. This leads to the following model:

$$\max_u \int_0^\infty e^{-\rho t} \left[r(k) - c(u) \right] dt, \tag{1}$$

$$\dot{k} = u - \mu k, \quad k(0) = k_0, \tag{2}$$

where labor is assumed to be proportional to capital stock so that it does not need to be explicitly included. The function $c(u)$ includes the costs of acquisition and adjustment costs. It is convex and increasing in u, i.e., $c(0) = 0$, $c' > 0$, $c'' > 0$. The function $r(k)$, with $r(0) = 0$, is increasing, $r' > 0$, while the second order derivative is negative, $r'' < 0$. So far everything is standard; see, e.g., Lucas (1967) and Gould (1968).

The new element we introduce is that at one point, $k = \kappa$, the function r is not differentiable in such a way that the first order derivative jumps upwards: $r'(\kappa^+) > r'(\kappa^-)$. The interpretation, for instance[1], can be that the firm has the possibility to produce using some old software or technology that the firm owns, which gives revenue $r_1(k)$.

[1] An alternative interpretation would be that when capital stock and, in turn, production is low, only the home market is served. If the production rate is larger it pays for the firm to serve a second market but then a fixed cost per unit of time is incurred because of, e,g, new outlets, new personnel, etc.

In addition, the firm has the option to license or subscribe to a new software or technology which makes capital stock more productive, in the sense that it leads to a higher revenue $r_2(k)$ as well as a higher marginal revenue, $r_2'(k) > r_1'(k)$. It is only cost effective to do so if the capital stock is sufficiently large, because of the existence of a fixed license or subscription fee, g. Then, it holds that $r_1(\kappa) = r_2(\kappa) - g$ and that $r'(\kappa^+) = r_2'(\kappa) > r_1'(\kappa) = r'(\kappa^-)$ so that

$$\begin{aligned} r(k) &= r_1(k) \quad \text{for} \quad k \leq \kappa \quad \text{and} \\ r(k) &= r_2(k) - g \quad \text{for} \quad k > \kappa. \end{aligned}$$

Pontryagin's maximum principle is used to obtain the necessary optimality conditions. The Hamiltonian is

$$H = r(k) - c(u) + q[u - \mu k]$$

which leads to [see Clarke (1983)]

$$\partial H / \partial u = 0 \qquad \text{i.e.} \qquad c'(u) = q. \tag{3}$$

and

$$\dot{q} = \rho q - \partial H / \partial k = (\rho + \mu) q - r'(k). \tag{4}$$

Since $H_{uu} < 0$, maximization of the Hamiltonian yields a unique solution. It follows that u is continuous over time (see Feichtinger and Hartl 1986, Corollary 6.2).

As usual, we can obtain from the previous two equations that

$$\dot{u} = \frac{1}{c''(u)}\left[(\rho + \mu)c'(u) - r'(k)\right]. \tag{5}$$

The above differential equations (4) and (5) are not valid for $k = \kappa$. There, the Hamiltonian is non-differentiable and Clarke's (1983) maximum principle prescribes that the adjoint equation is replaced by the differential inclusion

$$\dot{q} \in \left[(\rho + \mu)q - r_2'(\kappa), (\rho + \mu)q - r_1'(\kappa)\right]. \tag{6}$$

This implies that for $k = \kappa$, equation (5) must be replaced by

$$\dot{u} \in \left[\frac{1}{c''(u)}\left[(\rho + \mu)c'(u) - r_2'(k)\right], \frac{1}{c''(u)}\left[(\rho + \mu)c'(u) - r_1'(k)\right]\right]. \tag{7}$$

We continue by applying the phase diagram analysis in the state-control space. The dynamic system is given by (2) and (5). The $\dot{k} = 0$ isocline, $u = \mu k$, is upward sloping. The $\dot{u} = 0$ isocline is decreasing almost everywhere:

$$\left.\frac{\partial u}{\partial k}\right|_{\dot{u}=0} = \frac{r''}{(\rho + \mu)c''} < 0 \quad \text{for} \quad k \neq \kappa.$$

while it exhibits an upward jump at $k = \kappa$.

From the above results on the shape of the isoclines it follows that one or two positive equilibria can exist. These equilibria are always saddle point stable. This follows from the general observation that an equilibrium is a saddle point if and only if the derivative of the $\dot{k} = 0$-isocline is larger than the derivative of the $\dot{u} = 0$-isocline there:

$$\det \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial u} \\ \frac{\partial \dot{u}}{\partial k} & \frac{\partial \dot{u}}{\partial u} \end{bmatrix} < 0 \text{ iff } \left.\frac{du}{dk}\right|_{\dot{k}=0} = -\frac{\frac{\partial \dot{k}}{\partial k}}{\frac{\partial \dot{k}}{\partial u}} > -\frac{\frac{\partial \dot{u}}{\partial k}}{\frac{\partial \dot{u}}{\partial u}} = \left.\frac{du}{dk}\right|_{\dot{u}=0},$$

because $\partial \dot{k}/\partial u = 1 > 0$ and $\partial \dot{u}/\partial u = \rho + \mu > 0$. Since we already found that the $\dot{k} = 0$-isocline is increasing and the $\dot{u} = 0$-isocline is decreasing, both steady states are saddle points. Usually, between two saddle points one would expect an unstable steady state. Here, this is not the case and we will demonstrate that the kink in the revenue function at $k = \kappa$ plays the same role as an unstable steady state.

We are interested in the case where two steady states exist. Proposition 1 identifies the scenario in which this happens.

Proposition 1. *Two equilibria exist iff*

$$\frac{r_1'(\kappa)}{\rho + \mu} < c'(\mu\kappa) < \frac{r_2'(\kappa)}{\rho + \mu}. \tag{8}$$

The proof directly follows from (2) and (7). The corresponding phase diagram is depicted in Figure 1.

Insert Figure 1 about here

At $k = \kappa$ the jump in the $\dot{u} = 0$ isocline occurs and thus the trajectories exhibit a kink there. The bold lines represent the optimal policy functions while the dotted lines are candidates which are not optimal. The thin solid lines are the unstable branches or other trajectories of the dynamical system and the dash-dotted lines are the isoclines. As can be inferred from Figure 1, there are two candidate long run equilibria, $\bar{k}_L$ and

$\bar{k}_H$. For initial values of capital stock lower than k_{LB} it is always optimal to converge to $\bar{k}_L$, while convergence to $\bar{k}_H$ always occurs for $k_0 > k_{UB}$. In the "overlap region" between k_{LB} and k_{UB} there are two solution candidates, because either convergence to $\bar{k}_L$ or convergence to $\bar{k}_H$ is possible.

Using the same arguments as in Dechert (1983), in the "overlap region" between k_{LB} and k_{UB} a unique threshold k_{DNS} can be identified where the firm is indifferent between the two candidate policies. Then, for initial values of capital stock lower than k_{DNS} it is always optimal to converge to $\bar{k}_L$, while convergence to $\bar{k}_H$ always occurs for $k_0 > k_{DNS}$. Note that – contrary to Dechert (1983) – this threshold exists in the absence of an unstable steady state.

Figure 1 covers the case that the overlap region does not contain one of the two steady states, i.e., $\bar{k}_L < k_{LB} < \kappa < k_{UB} < \bar{k}_H$. The following proposition explains that in the remaining cases where two steady states exist, only one of them is a long run optimal equilibrium.

Proposition 2. *If a steady state is contained in the overlap region, convergence to that particular steady state can never be optimal.*

Proof. It is known from Dechert (1983) that the minimum of the Hamiltonian H for a fixed value of k is reached at the $\dot{k} = 0$ isocline. Furthermore the value of the objective functional evaluated at any point of a bounded solution candidate equals H/ρ; see Michel (1982). Now consider e.g. the situation where at $\bar{k}_L$ a solution candidate converging to $\bar{k}_H$ exists. Then from the above we can conclude that this solution candidate offers a higher value of the objective than the other solution candidate which is staying at $\bar{k}_L$.□

In the next section we treat an example in which the above results are illustrated while most values can be computed analytically.

3 Example

Following Barucci (1998) we assume quadratic revenue and cost functions. The differences are that the second order derivative of the revenue function is negative and we also assume that the revenue function has a convex kink:

$$r(k) = \max\left\{a_L k - bk^2,\ a_H k - bk^2 - g\right\},\ c(u) = cu + du^2.$$

We require all parameters a_L, a_H, b, c, d, g, μ, and ρ to be positive and $a_L < a_H$. The characteristic values of the capital stock, for which revenue $r(k)$ reaches its maximum, and where the kink occurs, respectively, are

$$k_{\max} = \frac{a_H}{2b}, \quad \kappa = \frac{g}{a_H - a_L}. \tag{9}$$

We restrict ourselves to the scenario where revenue is increasing in the relevant region $0 < k < k_{\max}$, implying that:

$$g < \frac{a_L\,(a_H - a_L)}{2b}. \tag{10}$$

In this case the $\dot{k} = 0$-isocline is the upward sloping straight line $u = \mu k$ and the $\dot{u} = 0$-isocline is given by the downward sloping straight line(s)

$$u = \frac{1}{2d(\rho+\mu)}[a_i - (\rho+\mu)c - 2bk], \tag{11}$$

where a_i equals a_L for $k < \kappa$, and a_H for $k > \kappa$, respectively. There is an upward jump of size $\frac{a_H - a_L}{2d(\rho+\mu)}$ at $k = \kappa$.

The steady states must satisfy:

$$\bar{k}_i = \frac{a_i - (\rho + \mu) c}{2\mu d (\rho + \mu) + 2b}. \tag{12}$$

for $i = L, H$. In order to have multiple steady states, we impose $a_L > (\rho + \mu) c$, because otherwise $\bar{k}_L$ becomes negative. It is easily obtained that $\bar{k}_L < \kappa < \bar{k}_H$ holds iff

$$\frac{(a_L - (\rho + \mu) c) (a_H - a_L)}{2\mu d (\rho + \mu) + 2b} < g < \frac{(a_H - (\rho + \mu) c) (a_H - a_L)}{2\mu d (\rho + \mu) + 2b}. \tag{13}$$

and that values of g in this interval automatically satisfy (10).

Apparently, in each of the two regions $k < \kappa$ and $k > \kappa$ the canonical system formed by (2) and the isocline $\dot{u} = 0$ is linear and can be solved analytically. Following Hartl and Kort (2000), we obtain that

$$k = \bar{k}_i + \left(k_0 - \bar{k}_i\right) e^{\lambda_1 t} \tag{14}$$

$$u = \mu \bar{k}_i + \left(k_0 - \bar{k}_i\right) (\mu + \lambda_1) e^{\lambda_1 t} \tag{15}$$

where λ_1 is the negative eigenvalue of the Jacobian of the dynamic system (2) and the isocline $\dot{u} = 0$:

$$\lambda_1 = \frac{\rho - \sqrt{(\rho + 2\mu)^2 + \frac{4b}{d}}}{2} \tag{16}$$

From (14) and (15), we also obtain analytical expressions for the candidate policy functions (saddle point paths), showing that the saddle point paths are decreasing straight lines:

$$u = \mu \bar{k}_i + \left(k - \bar{k}_i\right)(\mu + \lambda_1) \tag{17}$$

We now investigate the location of the DNS-threshold and establish the following result.

Proposition 3. *The location of the DNS-threshold satisfies*

$$\begin{aligned} \text{If } g &< \frac{1}{2}\left(\bar{k}_H + \bar{k}_L\right)(a_H - a_L)\text{, then } k_{DNS} < k^* < \kappa && \text{(18a)} \\ \text{if } g &= \frac{1}{2}\left(\bar{k}_H + \bar{k}_L\right)(a_H - a_L)\text{, then } k_{DNS} = k^* = \kappa\text{, and} && \text{(18b)} \\ \text{if } g &> \frac{1}{2}\left(\bar{k}_H + \bar{k}_L\right)(a_H - a_L)\text{, then } k_{DNS} > k^* > \kappa. && \text{(18c)} \end{aligned}$$

where

$$k^* = \frac{g - d\lambda_1^2\left(\bar{k}_H^2 - \bar{k}_L^2\right)}{(a_H - a_L) - 2d\lambda_1^2\left(\bar{k}_H - \bar{k}_L\right)} \tag{19}$$

Proof. Here we only give an outline of the proof, which can be found in full detail in Hartl and Kort (2002). First, we compute the objective values Π_L when starting from $k_0 < \kappa$ converging to equilibrium $\bar{k}_L$ and Π_H starting from $k_0 > \kappa$ converging to equilibrium $\bar{k}_H$. This yields

$$\begin{aligned} \rho\Pi_L(k) &= \left(a_L - c\mu - 2d\lambda_1^2\bar{k}_L\right)k + \left(d\lambda_1^2 - d\mu^2 - b\right)k^2 + d\bar{k}_L^2\lambda_1^2, && (20) \\ \rho\Pi_H(k) &= \left(a_H - c\mu - 2d\lambda_1^2\bar{k}_H\right)k + \left(d\lambda_1^2 - d\mu^2 - b\right)k^2 + d\bar{k}_H^2\lambda_1^2 - g. && (21) \end{aligned}$$

Equating these, and solving for k, yields (19). We note that k^* is only an approximation of the DNS point, since Π_L and Π_H only hold on one side of κ. On the other side of κ, the extension of this solution candidate yields a higher value of the objective function since - because of the kink - revenue is, in fact, higher than assumed in Π_L and Π_H. Thus, we know that $k_{DNS} < k^* < \kappa$ if $k^* < \kappa$ and that $k_{DNS} > k^* > \kappa$ if $k^* > \kappa$. Only if k^* happens to coincide with κ, then $k_{DNS} = k^* = \kappa$. The inequality $k^* \lesseqgtr \kappa$ can be reformulated as (18a-c), which completes the proof. $\square$

Note that λ_1 in (16) and thus also $\bar{k}_H$ and $\bar{k}_L$ do not depend on the parameter g which can be chosen independently. Thus, for any given set of the other parameters, values of g small enough lead to a situation that the smaller steady state, although strictly positive, is no equilibrium since it is always optimal to converge to the larger steady state being the only long run optimal equilibrium here. On the other hand, a sufficiently large g will lead to the situation that $k^* > \bar{k}_H$, implying that it is never optimal to converge to the larger steady state. In the next subsection we provide a numerical example in which the DNS-point occurs.

3.1 Numerical Examples

We now identify different scenarios by choosing appropriate parameter values. Let us assume $a_L = 5$, $a_H = 10$, $b = 0.1$, $d = 1$, $\mu = 0.2$, $\rho = 0.1$, and $c = 1$. From (10) we need $g < 125$ to have the revenue function increasing for $k < \kappa$.

3.1.1 Two equilibria

To consider the case of two equilibria, we choose $g = 110$. This gives the characteristic values $k_{\max} = 50$, $\kappa = 22$, $\bar{k}_L = 14.688$, $\bar{k}_H = 30.313$, and $k^* = 20.234$.

In Figure 2 we plot the phase diagram. At $\kappa = 22$ the jump in the $\dot{u} = 0$ isocline occurs and thus the trajectories exhibits a kink there. Compared to Figure 1, we see that now isoclines, policy functions and also the unstable paths are straight lines. This is caused by the quadratic specifications for revenue and investment cost functions.

Insert Figure 2 about here

From Proposition 3 we can derive that $k_{DNS} < k^*$. We can now compute this DNS-point k_{DNS} by first obtaining the intersection of policy function u_H with $k = \kappa$. Next, we can determine the hyperbola like trajectory belonging to the region $k < \kappa$ and ending in the point $(k, u) = (\kappa, u_H(\kappa))$. This hyperbola like trajectory in question is vertical when $\dot{k} = 0$, which happens for $k_{LB} = 17.697$, while the corresponding investment rate is $u = 3.5393$. Capital stock k_{LB} is the lower boundary of the overlap region. Similarly, also the upper boundary $k_{UB} = 24.592$ is obtained.

After computing the real value function Π_H for $k_{LB} < k < \kappa$, and intersecting it with Π_L, we obtain that $k_{DNS} = 19.884$. More details of this derivation can be found in Hartl and Kort (2002). In Figure 3, we plot all relevant value function candidates.

Insert Figure 3 about here

Note that in the interval $[k_{LB}, \kappa]$ the value function candidate Π_H representing solutions converging to $\bar{k}_H$ is not given by (21) but by a slightly larger value. In Figure 3, this is represented by the bold dotted curve in $[k_{DNS}, \kappa]$ and the thin dotted curves in $[k_{LB}, k_{DNS}]$.

In the numerical example just presented, the DNS-point k_{DNS} was smaller than the kink

κ. From Proposition 3 we derive that for $g = 112.5$ DNS-point and kink coincide, where $\kappa = k^* = k_{DNS} = 22.5$. If g is decreased further, then the DNS-point k_{DNS} will be larger than the kink κ.

3.1.2 Only one equilibrium

For sufficiently low values of g, the kink κ is smaller than the k-value of the intersection point of the stable path belonging to k_H and the unstable path corresponding to k_L in Figure 3. Consequently, the policy function corresponding to k_H can be extended to the left beyond k_L. From Proposition 2 we now know that only the larger steady state is a long run equilibrium. This occurs e.g. for $g = 100$, where $\kappa = 20$.

Analogously, for sufficiently high values of g, the kink κ is larger than the k-value of the intersection point of the stable path belonging to k_L and the unstable path corresponding to k_H in Figure 3. Then, the policy function corresponding to k_L can be extended to the right beyond k_H so that only the smaller steady state is a long run equilibrium. This occurs e.g. for $g = 122$, where $\kappa = 24.4$.

4 Conclusions and Extensions

Exploiting the framework of a capital accumulation model, it was established that non-differentiability can be a source for multiple equilibria. Remarkable was that this occurs while an unstable steady state does not exist. Within a numerical example we were able to determine the exact value of the threshold, or DNS point, at which the decision maker is indifferent between choosing one of the optimal paths. As a general result we established that a necessary condition for a stable steady state to be a long run equilibrium is that this steady state should not be part of an overlap region.

The result that a non-differentiability can lead to DNS points is clearly not restricted to a capital accumulation framework. More generally, this phenomenon can occur in any dynamic economic model due to a jump in the costate isocline caused by a non-smoothness of some model function.

References

[1] Arthur, W.B., 1989, Competing technologies, increasing returns, and lock-in by historical events, The Economic Journal 99, 116-131.

[2] Arthur, W.B., 1994, Increasing Returns and Path Dependence in the Economy (University of Michigan Press, Ann Arbor).

[3] Barro, R.J. and X. Sala-I-Martin, 1995, Economic Growth (McGraw-Hill, New York).

[4] Barucci, E., 1998, Optimal investments with increasing returns to scale, International Economic Review 39, 789-808.

[5] Brock, W.A. and A.G. Malliaris, 1989, Differential Equations, Stability and Chaos in Dynamical Economics (North Holland, Amsterdam).

[6] Ciccone, A. and K. Matsuyama, 1996, Start-up costs and pecuniary externalities as barriers to economic development, Journal of Development Economics 49, 33-59.

[7] Clarke, F., 1983, Optimization and Nonsmooth Analysis (Wiley-Interscience, New York).

[8] Dechert, W.D., 1983, Increasing returns to scale and the reverse flexible accelerator, Economic Letters 13, 69-75.

[9] Dechert, W.D. and K. Nishimura, 1983, A complete characterization of optimal growth paths in an aggregated model with a non-concave production function, Journal of Economic Theory 31, 332-354.

[10] Feichtinger, G. and R.F. Hartl, 1986, Optimale Kontrolle Ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften (de Gruyter, Berlin).

[11] Gould, J.P., 1968, Adjustment costs in the theory of investment of the firm, Review of Economic Studies 35, 47-56.

[12] Hartl, R.F. and P.M. Kort, 2000, Optimal investments with increasing returns to scale: a further analysis, in: E.J. Dockner et al., eds., Optimization, Dynamics, and Economic Analysis: Essays in Honor of Gustav Feichtinger (Springer-Physica, Berlin) 226-238 .

[13] Hartl, R.F. and P.M. Kort, 2002, Skiba without unstable equilibrium in a linear quadratic framework, POM Working Paper 7/2002, University of Vienna (http://www.bwl.univie.ac.at/bwl/prod/papers/pom-wp-07-2002.pdf).

[14] Ladron-de-Guevara, A., S. Ortigueira, and M.S. Santos, 1999, A two sector model of endogenous growth with leisure, Review of Economic Studies 66, 609-631.

[15] Lucas, R.E. jr., 1967, Optimal investment policy and the flexible accelerator, International Economic Review 8, 78-85.

[16] Matsuyama, K., 1991, Increasing returns, industrialization, and indeterminacy of equilibrium, Quarterly Journal of Economics 106, 617-650.

[17] Michel, P., 1982, On the transversality condistion in infinite horizon optimal problems, Econometrica 50, 975-985.

[18] Romer, P.M., 1986, Increasing returns and long-run growth, Journal of Political Economy 94, 1002-1037.

[19] Santos, M.S., 1999, On non-existence of continuous markov equilibria in competitive-market economies, presented at the 1999 meeting of the Society of Economic Dynamics in Alghero, Sardinia, to appear in: Journal of Economic Theory.

[20] Skiba, A.K., 1978, Optimal growth with a convex-concave production function, Econometrica 46, 527-539.

List of Figure Captions

Figure 1. The phase diagram in case of two steady states.

Figure 2. The value function and the DNS-point in the quadratic example.

Figure 3. The phase diagram with two steady states in the quadratic example.