The maximum edge biclique problem is NP-complete

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Abstract

We prove that the maximum edge biclique problem in bipartite graphs is NP-complete.

Key words: Complexity, bipartite graphs, biclique
1 Introduction

Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$. A pair of two disjoint subsets $A$ and $B$ of $V$ is called a biclique if $\{a,b\} \in E$ for all $a \in A$ and $b \in B$. Thus the edges $\{a,b\}$ form a complete bipartite subgraph of $G$ (which is not necessarily an induced subgraph if $G$ is not bipartite). A biclique $\{A,B\}$ clearly has $|A| + |B|$ vertices and $|A| \times |B|$ edges. In this note we restrict ourselves to case when $G$ is bipartite. The two colour classes of $G$ will be denoted by $V_1$ and $V_2$, so $V = V_1 \cup V_2$.

Already in the book of Garey and Johnson [2, GT24] the complexity of deciding whether or not a bipartite graph contains a biclique of a certain size is discussed. If the requirement is that $|A| = |B| = K$ for some integer $K$ (this is called the balanced complete bipartite subgraph problem or balanced biclique problem), then the problem is NP-complete. If however the requirement is that $|A| + |B| \geq K$ (the maximum vertex biclique problem), the problem can be solved in polynomial time via the matching algorithm. The complexity of the maximum vertex biclique problem for general graphs depends on the precise definition of a biclique in this case. With the above definition the problem is solvable in polynomial time since there is a one to one correspondence between bicliques in the bipartite double\footnote{The bipartite double of a graph with adjacency matrix $A$ is the bipartite graph with adjacency matrix \[ \begin{bmatrix} O & A \\ A & O \end{bmatrix} \].} of the graph and bicliques in the graph itself (see also [4]). If one defines a biclique as an induced complete bipartite subgraph (so $A$ and $B$ are independent sets...
in $G$), then the maximum vertex biclique problem for general graphs is NP-complete (see [8]). A natural third variant is the so-called maximum edge biclique problem (MBP) where the requirement is that $|A| \times |B| \geq K$. Up to now the complexity of this problem was still undecided.

In various papers the complexity of MBP is mentioned and guessed to be NP-complete ([1, 4, 3, 7]. In [1] some applications of MBP are discussed and it is shown that the weighted version of MBP is NP-complete. Furthermore the authors show that four variants of MBP are NP-complete. Using different techniques Hochbaum [4], Haemers [3] and Pasechnik [7] derive upper bounds for the maximum number of edges in a biclique. Hochbaum [4] presents a 2-approximation algorithm for the minimum number of edges needed to be removed so that the remainder is a biclique based on an LP-relaxation. Inspired by the work of Lovász on the Shannon capacity of a graph ([6]), Haemers [3] and Pasechnik [7] derive similar inequalities for the maximum biclique problem. Pasechnik uses semidefinite programming techniques whereas Haemers uses eigenvalue techniques.

In the next section we prove that indeed MBP is NP-complete. The reduction used is inspired by the reduction that is used to prove the NP-completeness of the balanced biclique problem (see [5]). As a consequence MBP is also NP-complete for general graphs.

2 The reduction

We define MBP as follows:
Maximum edge biclique problem (MBP): Given a bipartite graph $G = (V_1 \cup V_2, E)$ and a positive integer $K$, does $G$ contain a biclique with at least $K$ edges?

**Theorem 1** MBP is NP-complete.

**Proof:** We shall reduce CLIQUE to MBP. This reduction is a modification of the reduction from CLIQUE to BALANCED COMPLETE BIPARTITE SUBGRAPH referred to in [2, GT24] and published in [5].

Let $G = (V, E)$ and $K$ provide an instance of CLIQUE. Without loss of generality we may assume that $K = \frac{1}{2}|V|$.

Now construct an instance $G' = (V_1 \cup V_2, E')$, $K'$ of MBP as follows: Let

$V_1 = V,$

$V_2 = E \cup W,$

where $W$ is a set of $\frac{1}{2}K^2 - K$ new elements.

$E' = \{\{v, e\} : v \in V; e \in E; v \notin e\} \cup \{\{v, w\} : v \in V; w \in W\}$

$K' = K^3 - \frac{3}{2}K^2$

This construction can clearly be performed in polynomial time. Suppose $G$ has a clique $C$ of size $K$. Take $A := V - C$ and $B := W \cup \{c, d\} : c, d \in C; c \neq d$. Then $\{A, B\}$ is a biclique in $G'$ with $|A \ast B| = K \ast (\frac{1}{2}K^2 - K + \frac{1}{2}K(K - 1)) = K^3 - \frac{3}{2}K^2$ edges. So if $G$ has a clique of size $K$ then $G'$ has a biclique with $K'$ edges.
Now suppose $G$ has no clique of size $K$. Let $\{A, B\}$ be a biclique of $G'$ with $A \subseteq V_1$ and $B \subseteq V_2$. We shall finish the proof by showing that $|A| \cdot |B| < K'$ in this case. Without loss of generality $W \subseteq B$. Let $a := |A|$ and $b := |B| - |W|$.

The $b$ elements of $B \cap E$ correspond with edges in $G$ whose endpoints are not in $A$. There are $2K - a$ vertices of $G$ that are not in $A$ so $b \leq \frac{1}{2}(2K - a)(2K - a - 1)$, with equality if and only if $V - A$ is a clique with edge set $B \cap E$.

We consider two cases:

1. Suppose $a > K$, so $|V - A| = K - c$ with $c := a - K$ (So $0 < c \leq K$).
   Then $b \leq \frac{1}{2}(K - c)(K - c - 1)$, so
   $$|A| \cdot |B| \leq (K + c) \cdot \left[\frac{1}{2}K^2 - K + \frac{1}{2}(K - c)(K - c - 1)\right]$$
   This reduces to
   $$|A| \cdot |B| - (K^3 - \frac{3}{2}K^2) \leq \frac{1}{2}c^2 - (K - 1)c - 2K$$
   Now $c^2 - (K - 1)c - 2K$ is negative for $0 \leq c \leq K$, so $|A| \cdot |B| < K'$ for $0 < c \leq K$.

2. Suppose $a \leq K$, so $|V - A| = K + c$ with $c := K - a$ (So $0 \leq c \leq K$).
   Since $G$ has no cliques with $K$ vertices, the number of edges in the subgraph of $G$ induced by $V - A$, and consequently $b$, is strictly less than $\frac{1}{2}(K + c)(K + c - 1) - c$. This leads to
   $$|A| \cdot |B| < (K - c) \cdot \left[\frac{1}{2}K^2 - K + \frac{1}{2}(K + c)(K + c - 1) - c\right]$$
which reduces to

\[ |A| \ast |B| - (K^3 - \frac{3}{2}K^2) < \frac{1}{2}c^2(-c + 3 - K) \]

Since we may assume that \( K \geq 4 \), the right hand side is negative for \( 1 \leq c \leq K \) and zero for \( c = 0 \). So \( |A| \ast |B| < K' \). 

\[ \square \]

References


