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**EXISTENCE OF EQUILIBRIUM AND PRICE
ADJUSTMENTS IN A FINANCE ECONOMY
WITH INCOMPLETE MARKETS**

By A.J.J. Talman and J.J.J. Thijssen

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Existence of Equilibrium and Price Adjustments in a Finance Economy with Incomplete Markets

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Abstract

In this paper the standard two-period general equilibrium model with incomplete financial markets is considered. First, existence of equilibrium is proved using a stationary point argument on the set of no-arbitrage prices. Prices are normalized with respect to the market portfolio. The proof does not use the commonly applied normalization on the unit sphere or truncation of the set of prices. It is shown that there exists a connected set from an arbitrary price vector to an equilibrium. The path can be followed by a simplicial algorithm for stationary point problems on polytopes. It is argued that this algorithm can be interpreted as originating from a market-maker maximizing the value of excess demand.

Keywords: General Equilibrium, Incomplete markets, Price adjustments.

JEL-codes: C62, C63, C68, D52, D58.

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1 Introduction

The main focus of this paper is to describe a price-adjustment process in an economy with incomplete financial markets, that converges to an equilibrium price vector. It turns out that the simplicial algorithm for calculating stationary points of a continuous function on a polytope as developed by Talman and Yamamoto (1989) has a nice interpretation that can be used to give an explanation for price formation on financial markets.

In this paper the simplest general equilibrium model with incomplete markets as is explained in e.g. Magill and Quinzii (1996) is considered. There are two periods of time (present and future), a finite number of possible future states, one consumption good and a number of financial securities that can be used to transfer income from the present to the future. For the consumption good there are spot markets, so at present one cannot trade the consumption good for the future. The financial markets are incomplete if not all possible income streams for present and future can be attained by trading on the existing financial markets.

Existence of equilibrium in a two-period general equilibrium model with multiple consumption goods and (possibly) incomplete markets is proved in Geanakoplos and Polemarchakis (1986). They prove existence on the set of no-arbitrage prices. These are prices such that it is impossible to create a portfolio of assets which generates a non-negative income stream in the future and has non-positive costs at present. The proof uses a fixed point argument for functions on compact sets. For that, since the set of no-arbitrage prices can be unbounded, the proof of Geanakoplos and Polemarchakis (1986) uses a compact truncation of this set. In this paper we present an existence proof for the one consumption good model that uses a stationary point argument without truncating the set of no-arbitrage prices. Other existence proofs use some transformation of the underlying model. Hens (1991) for example, introduces an artificial asset to translate the present to the future. The approach taken by Hirsch et al. (1990) shows existence of equilibrium in a model with only state prices. Then it is argued that each equilibrium in the original model corresponds one-to-one to an equilibrium in state prices.

Given that an equilibrium exists the question arises how to compute one. There is a homotopy method introduced in Herings and Kubler (2002) that requires differentiability assumptions on the utility functions. In this paper we show that one can use the simplicial approach developed by Talman and Yamamoto (1989), which does not require additional assumptions to the ones needed to prove existence. Note that the Talman and Yamamoto (1989) algorithm is defined for functions on polytopes. The set of no-arbitrage prices for the model can, however, be an unbounded poly-

hedron. Therefore, we extend the algorithm of Talman and Yamamoto (1989) to unbounded polyhedra. Note, however, that there are simplicial algorithms for functions on possibly unbounded polyhedra, notably by Dai et al. (1991) and Dai and Talman (1993). These algorithms cannot be applied, however, since they assume pointedness of the polyhedron and affine functions, respectively.

Since a general equilibrium model can (and generally will) have multiple equilibria, the method used to calculate an equilibrium should ideally also be an equilibrium selection device. The algorithm assumes a market maker that maximises the market's turn-over on an expanding set of prices. It does so by, given a starting vector of asset prices, collecting all demand and supply orders and by relatively increasing the price of the asset that has the greatest excess demand. This continues until another asset has the greatest excess demand. Then the price of this asset will also be relatively increased, and so on. This procedure eventually converges to an equilibrium. The demand and supply orders of the agents represent the agents' valuations of their portfolios in the different states of the future.

The static two period model can be given a more dynamic interpretation where the present represents a point in time just after a fundamental change has taken place in the economy that influences for example utility functions, endowments or future payoffs to assets. The starting vector of asset prices for the market maker is then the equilibrium price vector from just before the shock. Such an interpretation implicitly assumes agents to be boundedly rational since the agents are just one-period forward looking.

The paper is organised as follows. In Section 2 the economic model is described. In Section 3 we prove the existence of equilibrium and in Section 4 we interpret the simplicial algorithm of Talman and Yamamoto (1989) as describing a boundedly rational path of asset prices. In Section 5 the algorithm is presented in some detail and illustrated by means of a numerical example.

2 The Finance Economy

The General Equilibrium model with Incomplete markets (GEI) explicitly includes incomplete financial markets in a general equilibrium framework. In this paper the simplest version is used. It consists of two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. At $t = 0$ the state of nature is known to be $s = 0$. The state of nature at $t = 1$ is unknown and denoted by $s \in \{1, 2, \dots, S\}$. In the economy there are $I \in \mathbb{N}$ consumers, indexed by $i = 1, \dots, I$. There is one consumption good that can be interpreted as income. A consumption plan for consumer $i \in \{1, \dots, I\}$ is a vector $x^i \in \mathbb{R}_+^{S+1}$, where x_s^i gives the consumption level

in state $s \in \{0, 1, \dots, S\}$.¹

Each consumer $i = 1, \dots, I$, is characterised by a vector of initial endowments, $\omega^i \in \mathbb{R}_+^{S+1}$, and a utility function $u^i : \mathbb{R}_+^{S+1} \rightarrow \mathbb{R}$. Denote aggregate initial endowments by $\omega = \sum_{i=1}^I \omega^i$. Regarding the initial endowments and utility functions we make the following assumptions.

Assumption 1 *The vector of aggregate initial endowments is strictly positive, i.e.*

$$\omega \in \mathbb{R}_{++}^{S+1}.$$

Assumption 2 *For each agent $i = 1, \dots, I$, the utility function u^i satisfies:*

1. *continuity on \mathbb{R}_+^{S+1} ;*
2. *strict monotonicity on \mathbb{R}_+^{S+1} ;*
3. *strict quasi-concavity on \mathbb{R}_+^{S+1} .*

Assumption 1 ensures that in each period and in each state of nature there is at least one agent who has a positive amount of the consumption good. Assumption 2 ensures that the consumer's demand is a continuous function.

It is assumed that the market for the consumption good is a spot market. The consumers can smoothen consumption by trading on the asset market. At the asset market, $J \in \mathbb{N}$ financial contracts are traded, indexed by $j = 1, \dots, J$. The future payoffs of the assets are put together in a matrix

$$V = (V^1, \dots, V^J) \in \mathbb{R}^{S \times J},$$

where V_s^j is the payoff of one unit of asset j in state s . The following assumption is made with respect to V .

Assumption 3 *There are no redundant assets, i.e. $\text{rank}(V) = J$.*

Actually, Assumption 3 can be made without loss of generality; if there are redundant assets then a no-arbitrage argument guarantees that its price is uniquely determined by the other assets. Let the marketed subspace be denoted by $\langle V \rangle = \text{Span}(V)$. That is, the marketed subspace consists of those income streams that can be generated by trading on the asset market. If $S = J$, the marketed subspace consists of all possible income streams, i.e. markets are complete. If $J < S$ there is idiosyncratic risk and markets are incomplete.

¹In general we denote for a vector $x \in \mathbb{R}^{S+1}$, $x = (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^S$ to separate x_0 in period $t = 0$ and $x_1 = (x_1, \dots, x_S)$ in period $t = 1$.

A *finance economy* is defined as a tuple $\mathcal{E} = \left((u^i, \omega^i)_{i=1, \dots, I}, V \right)$. Given a finance economy \mathcal{E} , agent i can trade assets by buying a portfolio $z^i \in \mathbb{R}^J$ given the (row)vector of prices $q = (q_0, q_1) \in \mathbb{R}^{J+1}$, where q_0 is the price for consumption in period $t = 0$ and $q_1 = (q_1, \dots, q_J)$ is the vector of security prices with q_j the price of security j , $j = 1, \dots, J$. Given a vector of prices $q = (q_0, q_1) \in \mathbb{R}^{J+1}$, the budget set for agent $i = 1, \dots, I$ is given by

$$B^i(q) = \left\{ x \in \mathbb{R}_+^{S+1} \mid \exists z \in \mathbb{R}^J : q_0(x_0 - \omega_0^i) \leq -q_1 z, x_1 - \omega_1^i = Vz \right\}. \quad (1)$$

Given the asset payoff matrix V we will restrict attention to asset prices that generate no arbitrage opportunities, i.e. asset prices q such that there is no portfolio generating a semi-positive income stream. In other words, we only consider asset prices that exclude the possibility of "free lunches". The importance of restricting ourselves to no-arbitrage prices becomes clear from the following well-known theorem (cf. Magill and Quinzii (1996)).

Theorem 1 *Let \mathcal{E} be a finance economy satisfying Assumption 2. Then the following conditions are equivalent:*

1. $q \in \mathbb{R}^{J+1}$ permits no arbitrage opportunities;
2. $\forall_{i=1, \dots, I} : \arg \max \{u^i(x^i) \mid x^i \in B^i(q)\} \neq \emptyset$;
3. $\exists_{\pi \in \mathbb{R}_{++}^S} : q_1 = \pi V$;
4. $B^i(q)$ is compact for all $i = 1, \dots, I$.

The vector $\pi \in \mathbb{R}_{++}^S$ can be interpreted as a vector of state prices. Condition 3 therefore states that a no-arbitrage price for security j equals the present value of security j given the vector of state prices π . As a consequence of this theorem, in the remainder we restrict ourselves to the set of no-arbitrage prices

$$Q = \{q \in \mathbb{R}^{J+1} \mid q_0 > 0, \exists_{\pi \in \mathbb{R}_{++}^S} : q_1 = \pi V\}. \quad (2)$$

Under Assumption 2, Theorem 1 shows that the demand function $x^i(q)$, maximising agent i 's utility function $u^i(x)$ on $B^i(q)$, is well-defined for all $i = 1, \dots, I$, and all $q \in Q$. Since the budget correspondence $B^i : Q \rightarrow \mathbb{R}_+^{S+1}$ is upper- and lower-semicontinuous, Berge's maximum theorem gives that $x^i(q)$ is continuous on Q . Because the mapping $z^i \mapsto Vz^i + \omega^i$ is continuous, one-to-one and onto, the security demand function $z^i(q)$, determined by $Vz^i(q) = x_1^i(q) - \omega_1^i$, is a continuous function on Q .

Define the *excess demand* function $f : Q \rightarrow \mathbb{R}^{J+1}$ by

$$f(q) = (f_0(q), f_1(q)) = \left(\sum_{i=1}^I (x_0^i(q) - \omega_0^i), \sum_{i=1}^I z^i(q) \right).$$

Note that since there are no initial endowments of asset j , $j = 1, \dots, J$, excess demand is given by $\sum_{i=1}^I z_j^i(q)$. With respect to the excess demand function we can derive the following result.

Lemma 1 *Under Assumptions 1–3 the excess demand function $f : Q \rightarrow \mathbb{R}^J$ satisfies the following properties:*

1. *continuity on Q ;*
2. *homogeneity of degree 0;*
3. *$(f_0(q), V f_1(q)) \geq -\omega$ for all $q \in Q$;*
4. *for all $q \in Q$, $qf(q) = 0$ (Walras' law).*

The proof of this lemma is elementary and therefore omitted.

A *financial market equilibrium* (FME) for a finance economy \mathcal{E} is a tuple $((\bar{x}^i, \bar{z}^i)_{i=1, \dots, I}, \bar{q})$ with $\bar{q} \in Q$ such that:

1. $\bar{x}^i \in \arg \max\{u^i(x^i) | x^i \in B^i(\bar{q})\}$ for all $i = 1, \dots, I$;
2. $V \bar{z}^i = \bar{x}_1^i - \omega_1^i$ for all $i = 1, \dots, I$;
3. $\sum_{i=1}^I \bar{z}^i = 0$.

Note that the market-clearing conditions for the financial markets imply that the goods market also clears, since there is only one consumption good.

3 Existence of Equilibrium

Existence of equilibrium is proved on the space of asset prices \bar{Q} , where

$$\bar{Q} = \{q \in \mathbb{R}^{J+1} | q_0 \geq 0, \exists \pi \in \mathbb{R}_+^S : q_1 = \pi V\}.$$

Before proving a general existence theorem we present the following lemmata.

Lemma 2 *Under Assumption 3 it holds that $\bar{Q} = cl(Q)$.*

Proof. Since \bar{Q} is a finitely generated cone it is a closed set (cf. Rockafellar (1970, Theorem 19.1)) and hence $cl(Q) \subset \bar{Q}$.

Let $\bar{q} \in \bar{Q}$. Then $\bar{q}_0 \geq 0$ and there exists a $\bar{\pi} \in \mathbb{R}_+^S$ satisfying $\bar{q}_1 = \bar{\pi}V$. Take any $(\pi^\nu)_{\nu \in \mathbb{N}}$ in \mathbb{R}_{++}^S converging to $\bar{\pi}$. Such a sequence exists, since $\bar{\pi} \in \mathbb{R}_+^S$. Moreover, take $q_0^\nu = \max\{\bar{q}_0, \frac{1}{\nu}\}$ for all $\nu \in \mathbb{N}$. Define $q^\nu = (q_0^\nu, q_1^\nu)$, where $q_1^\nu = \pi^\nu V$. Clearly, $q^\nu \in Q$ for all $\nu \in \mathbb{N}$ and $q^\nu \rightarrow \bar{q}$ since $q_0^\nu \rightarrow \bar{q}_0$ and $q_1^\nu = \pi^\nu V \rightarrow \bar{\pi}V = \bar{q}_1$. Consequently, $(q^\nu)_{\nu \in \mathbb{N}}$ is a sequence in Q converging to \bar{q} . Hence, $\bar{Q} \subset cl(Q)$. \square

An important result needed to prove existence of an FME is the existence of a convergent sequence of state prices to the boundary.

Lemma 3 *Let $(q^\nu)_{\nu \in \mathbb{N}}$ be a sequence in Q converging to $\bar{q} \in \partial\bar{Q} \setminus \{0\}$. Then under Assumption 3 there exists a sequence of state prices $(\pi^\nu)_{\nu \in \mathbb{N}}$ in \mathbb{R}_{++}^S converging to some $\bar{\pi} \in \mathbb{R}_+^S$ satisfying $\bar{q}_1 = \bar{\pi}V$. Moreover, if $\bar{q}_0 > 0$, it holds that $\bar{\pi} \in \partial\mathbb{R}_+^S$.*

Proof. Define

$$\bar{Q}_1 = \{q_1 \in \mathbb{R}^J \mid \exists \pi_1 \in \mathbb{R}_+^S : q_1 = \pi_1 V\}.$$

Since \bar{Q}_1 is a finitely generated cone it consists of all nonnegative linear combinations of finitely many directions $\{q^1, \dots, q^m\} \subset \bar{Q}_1$. Hence, there exist corresponding vectors $\pi^1, \dots, \pi^m \in \mathbb{R}_+^S$ such that for all $k = 1, \dots, m$, $q^k = \pi^k V$.

From Carathéodory's theorem (cf. Rockafellar (1970, Theorem 17.1)) we know that for every $\nu \in \mathbb{N}$, the vector q_1^ν can be written as

$$q_1^\nu = \sum_{k \in K_\nu} \lambda_k^\nu q^k,$$

where $\lambda_k^\nu \geq 0$ for all $k \in K_\nu$ and K_ν is such that $\{q^k \mid k \in K_\nu\}$ is a set of linearly independent vectors. Take

$$\pi^\nu = \sum_{k \in K_\nu} \lambda_k^\nu \pi^k,$$

then $\pi^\nu \in \mathbb{R}_+^S$ and $\pi^\nu V = q_1^\nu$, for all $\nu \in \mathbb{N}$.

Since \bar{Q}_1 is finitely generated, there exists a subsequence of $(q_1^\nu)_{\nu \in \mathbb{N}}$ generated from the same subset K of linearly independent elements of $\{q^1, \dots, q^m\}$. Without loss of generality we take it to be the sequence itself. Since $(q_1^\nu)_{\nu \in \mathbb{N}}$ is convergent and hence bounded, and $\{q^k \mid k \in K\}$ is a set of linearly independent vectors, we have that λ_k^ν , $k \in K$, $\nu \in \mathbb{N}$, is unique and thus bounded. Therefore, the sequence $(\pi^\nu)_{\nu \in \mathbb{N}}$ is bounded.

Since $q^\nu \in Q$ for all $\nu \in \mathbb{N}$, there exists a $\tilde{\pi}^\nu \in \mathbb{R}_{++}^S$ such that $q_1^\nu = \tilde{\pi}^\nu V$. Note that the sequence $(\tilde{\pi}^\nu)_{\nu \in \mathbb{N}}$ might not be bounded. Since $(\pi^\nu)_{\nu \in \mathbb{N}}$ is bounded (in any given norm) by, say, $M > 0$, for all $\nu \in \mathbb{N}$, there exists a convex combination $\hat{\pi}^\nu$ of $\tilde{\pi}^\nu$ and π^ν that is bounded by $2M$, such that $\hat{\pi}^\nu \in \mathbb{R}_{++}^S$. Since $(\hat{\pi}^\nu)_{\nu \in \mathbb{N}}$ is

bounded there exists a convergent subsequence with limit, say, $\bar{\pi}$, that without loss of generality we take to be the sequence itself. Clearly, $\bar{q} = \bar{\pi}V$ and $\bar{\pi} \in \mathbb{R}_+^S$.

Furthermore, when $\bar{q}_0 > 0$, $\bar{\pi} \in \partial\mathbb{R}_+^S$, since if $\bar{\pi} \in \mathbb{R}_{++}^S$, there would be a full-dimensional ball around $\bar{\pi}$ in the interior of \mathbb{R}_+^S . This ball would be mapped in a ball of full dimension around \bar{q}_1 , which contradicts $\bar{q} \in \partial\bar{Q}$ when $\bar{q}_0 > 0$. \square

The following lemma concerns the boundary behaviour of the excess demand function.

Lemma 4 *Let $(q^\nu)_{\nu \in \mathbb{N}}$ be a sequence in Q with $\lim_{\nu \rightarrow \infty} q^\nu = \bar{q} \in \partial\bar{Q} \setminus \{0\}$. Under Assumptions 1–3 it holds that*

$$f_0(q^\nu) + e^\top V f_1(q^\nu) \rightarrow \infty.$$

Proof. Suppose not. Then

$$\exists M > 0 \forall \nu \in \mathbb{N} : f_0(q^\nu) + e^\top V f_1(q^\nu) \leq M.$$

Since $(f_0, V f_1)$ is bounded from below, this implies that $(f_0(q^\nu), V f_1(q^\nu))_{\nu \in \mathbb{N}}$ is bounded and has a convergent sequence (w.l.o.g. we assume it is the sequence itself) with limit, say, $\bar{f} = (\bar{f}_0, V \bar{f}_1)$.

For all $\nu \in \mathbb{N}$, let $\pi^\nu \in \mathbb{R}_{++}^S$ be a supporting vector, i.e. $q_1^\nu = \pi^\nu V$. By Lemma 3 we can choose π^ν , $\nu \in \mathbb{N}$, such that $(\pi^\nu)_{\nu \in \mathbb{N}}$ has a convergent subsequence with $\lim_{\nu \rightarrow \infty} \pi^\nu = \bar{\pi} \in \mathbb{R}_+^S$, satisfying $\bar{q}_1 = \bar{\pi}V$.

We consider three cases. First, consider the case where $\bar{q}_0 > 0$ and $\bar{q}_1 \neq 0$. Let $\mathcal{S} = \{s | \bar{\pi}_s = 0\}$ and $\mathcal{S}^c = \{s | \bar{\pi}_s > 0\}$. Since $\bar{q}_1 \neq 0$ and since by Lemma 3, $\bar{\pi} \in \partial\mathbb{R}_+^S$, both sets are non-empty. Take $s^c \in \mathcal{S}^c$. Since $\omega_{s^c} > 0$, there exists an $i^c \in \{1, \dots, I\}$ with $\omega_{s^c}^{i^c} > 0$. For $q \in Q$, this consumer has excess demand defined by

$$f^{i^c}(q) = (x_0^{i^c}(q) - \omega_0^{i^c}, z^{i^c}(q)).$$

Since $(f_0(q^\nu), V f_1(q^\nu))_{\nu \in \mathbb{N}}$ is bounded from above and $(f_0^{i^c}(q^\nu), V f_1^{i^c}(q^\nu))_{\nu \in \mathbb{N}}$ is bounded from below by $-\omega^{i^c}$ the sequence $(f_0^{i^c}(q^\nu), V f_1^{i^c}(q^\nu))_{\nu \in \mathbb{N}}$ is bounded and therefore there is a convergent subsequence with limit $\bar{f}^{i^c} = (\bar{f}_0^{i^c}, V \bar{f}_1^{i^c})$. Let $\bar{x}^{i^c} = (\bar{f}_0^{i^c} + \omega_0^{i^c}, V \bar{f}_1^{i^c} + \omega_1^{i^c})$. Note that for all $q \in Q$ with supporting vector $\pi \in \mathbb{R}_{++}^S$ we have

$$B^{i^c}(q) = \{x \in \mathbb{R}_+^{S+1} | q_0 x_0 + \pi x_1 \leq q_0 \omega_0^{i^c} + \pi \omega_1^{i^c}\}.$$

Hence, $\bar{q}_0 \bar{x}_0^{i^c} + \bar{\pi} \bar{x}_1^{i^c} = \bar{q}_0 \omega_0^{i^c} + \bar{\pi} \omega_1^{i^c}$ since u^{i^c} is continuous and strictly monotonic. Consider the bundle $\tilde{x}^{i^c} = \bar{x}^{i^c} + e(s)$ for some $s \in \mathcal{S}$, where $e(s) \in \mathbb{R}^{S+1}$ is the s -th unit vector. Because of strict monotonicity it holds that $u^{i^c}(\tilde{x}^{i^c}) > u^{i^c}(\bar{x}^{i^c})$. However, since $\bar{\pi}_s = 0$ we have

$$\bar{q}_0 \tilde{x}_0^{i^c} + \bar{\pi} \tilde{x}_1^{i^c} = \bar{q}_0 \bar{x}_0^{i^c} + \bar{\pi} \bar{x}_1^{i^c} = \bar{q}_0 \omega_0^{i^c} + \bar{\pi} \omega_1^{i^c}.$$

Since $\bar{\pi}_{s^c} > 0$ and $\omega_{s^c}^{i^c} > 0$ it holds that $\bar{\pi}\omega_{\mathbf{1}}^{i^c} > 0$ and hence $\bar{q}_0\tilde{x}_0^{i^c} + \bar{\pi}\tilde{x}_{\mathbf{1}}^{i^c} > 0$. So there exists $s^* \in \mathcal{S}$ satisfying $\bar{q}_0\tilde{x}_0^{i^c} + \bar{\pi}_{s^*}\tilde{x}_{s^*}^{i^c} > 0$ and thus $\tilde{x}_0^{i^c} > 0$ or $\tilde{x}_{s^*}^{i^c} > 0$. Suppose first that $\tilde{x}_{s^*}^{i^c} > 0$. Since $u^{i^c}(\tilde{x}^{i^c}) > u^{i^c}(\bar{x}^{i^c})$ and u^{i^c} is continuous, it holds that

$$\exists_{\delta>0} : u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(\bar{x}^{i^c}).$$

However,

$$\begin{aligned} \bar{q}_0\tilde{x}_0^{i^c} + \bar{\pi}(\tilde{x}_{\mathbf{1}}^{i^c} - \delta e_{\mathbf{1}}(s^*)) &= \bar{q}_0\tilde{x}_0^{i^c} + \bar{\pi}\tilde{x}_{\mathbf{1}}^{i^c} - \bar{\pi}_{s^*}\delta \\ &= \bar{q}_0\omega_0^{i^c} + \bar{\pi}\omega_{\mathbf{1}}^{i^c} - \bar{\pi}_{s^*}\delta \\ &< \bar{q}_0\omega_0^{i^c} + \bar{\pi}\omega_{\mathbf{1}}^{i^c}. \end{aligned}$$

Since $q_0^\nu \rightarrow \bar{q}_0$ and $\pi^\nu \rightarrow \bar{\pi}$ we also have

$$\exists_{\nu_1 \in \mathbb{N} \forall \nu > \nu_1} : q_0^\nu \tilde{x}_0^{i^c} + \pi^\nu (\tilde{x}_{\mathbf{1}}^{i^c} - \delta e_{\mathbf{1}}(s^*)) \leq q_0^\nu \omega_0^{i^c} + \pi^\nu \omega_{\mathbf{1}}^{i^c}.$$

Moreover, since $x^{i^c}(q^\nu) \rightarrow \bar{x}^{i^c}$ and u^{i^c} is continuous,

$$\exists_{\nu_2 \in \mathbb{N} \forall \nu > \nu_2} : u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(x^{i^c}(q^\nu)).$$

So, for all $\nu > \max\{\nu_1, \nu_2\}$ we have $\tilde{x}^{i^c} - \delta e(s^*) \in B^{i^c}(q^\nu)$ and $u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(x^{i^c}(q^\nu))$, which contradicts $x^{i^c}(q^\nu)$ being a best element in $B^{i^c}(q^\nu)$. Suppose now that $\tilde{x}_0^{i^c} > 0$. Using a similar reasoning as above we can show that there exists a $\delta > 0$ such that

$$\exists_{\nu_1 \in \mathbb{N} \forall \nu > \nu_1} : q_0^\nu (\tilde{x}_0^{i^c} - \delta) + \pi^\nu \tilde{x}_{\mathbf{1}}^{i^c} \leq q_0^\nu \omega_0^{i^c} + \pi^\nu \omega_{\mathbf{1}}^{i^c},$$

and

$$\exists_{\nu_2 \in \mathbb{N} \forall \nu > \nu_2} : u^{i^c}(\tilde{x}^{i^c} - \delta e(0)) > u^{i^c}(x^{i^c}(q^\nu)).$$

So, for all $\nu > \max\{\nu_1, \nu_2\}$ we have $\tilde{x}^{i^c} - \delta e(0) \in B^{i^c}(q^\nu)$ and $u^{i^c}(\tilde{x}^{i^c} - \delta e(0)) > u^{i^c}(x^{i^c}(q^\nu))$, which contradicts $x^{i^c}(q^\nu)$ being a best element in $B^{i^c}(q^\nu)$.

If $\bar{q}_0 = 0$, there exists an $i^c \in \{1, \dots, I\}$ with $\omega_0^{i^c} > 0$. Now the proof follows along the same lines as above with some $s^* \in \mathcal{S}$ satisfying $\bar{\pi}_{s^*}\tilde{x}_{s^*}^{i^c} > 0$.

If $\bar{q}_{\mathbf{1}} = 0$, the set \mathcal{S} is empty and $\bar{q}_0\tilde{x}_0^{i^c} > 0$. We can now look at the proof of the first case and continue in a similar way as before. \square

Since $0 \in \partial\bar{Q}$ there is a tangent hyperplane at 0, i.e. there exists $\tilde{z} \in \mathbb{R}^{J+1} \setminus \{0\}$ such that $q\tilde{z} \geq 0$ for all $q \in \bar{Q}$. Since \bar{Q} is full-dimensional, it holds that $q\tilde{z} > 0$ for all $q \in Q$. We show existence of FME by normalising asset prices to $q\tilde{z} = 1$, i.e. on a hyperplane parallel to the tangent hyperplane in 0. One possible choice for \tilde{z} is the market portfolio z_M which is defined in the following way (cf. Herings and

Kubler (2000)). Decompose the vector of total initial endowments in $\omega = \omega_M + \omega_\perp$, where $\omega_M \in \langle V \rangle$ and $\omega_\perp \in \langle V \rangle^\perp$, the null-space of $\langle V \rangle$. The market portfolio z_M is defined to be the unique portfolio satisfying $Vz = \omega_M$. If $\omega_M \gg 0$ this implies $qz_M \geq 0$ for all no-arbitrage prices $q \in \bar{Q}$.

In the remainder, we fix $\tilde{z} \in \mathbb{R}^{J+1} \setminus \{0\}$ such that $q\tilde{z} \geq 0$ for all $q \in \bar{Q}$. Denote the set of normalised prices by \tilde{Q} , i.e.

$$\tilde{Q} = \{q \in \bar{Q} | q\tilde{z} = 1\}.$$

Note that \tilde{Q} can contain half-spaces and is hence not necessarily bounded. Based on the previous lemma, however, one can show that $f_0 + e^\top V f_1$ becomes arbitrarily large by moving to the boundary of \tilde{Q} or by taking $\|q\|_\infty$ large enough. In the following, let $d(q, A)$ denote the distance from $q \in \bar{Q}$ to $A \subset \bar{Q}$, i.e. $d(q, A) = \inf\{\delta | \exists a \in A : \|a - q\| = \delta\}$.

Lemma 5 *Under Assumptions 1–3 it holds that for all $M > 0$ there exists $\varepsilon > 0$ and $N > 0$ such that for all $q \in \tilde{Q}$,*

1. $d(q, \partial\tilde{Q}) \leq \varepsilon \Rightarrow f_0(q) + e^\top V f_1(q) > M$;
2. $\|q\|_\infty \geq N \Rightarrow f_0(q) + e^\top V f_1(q) > M$.

Proof. Suppose that 1) does not hold. Therefore, there exists an $\bar{M} > 0$ such that for all $\varepsilon > 0$ there exists a $q^\varepsilon \in \tilde{Q}$ with $d(q^\varepsilon, \partial\tilde{Q}) \leq \varepsilon$ such that $f_0(q^\varepsilon) + e^\top V f_1(q^\varepsilon) \leq \bar{M}$. Let $(q^\nu)_{\nu \in \mathbb{N}}$ be the sequence in \tilde{Q} such that $\varepsilon = \frac{1}{\nu}$ and $q^\nu = q^\varepsilon$ for all $\nu \in \mathbb{N}$.

In case $(q^\nu)_{\nu \in \mathbb{N}}$ is bounded it has a convergent subsequence. Without loss of generality, $\lim_{\nu \rightarrow \infty} q^\nu = \tilde{q}$. Since $d(q^\nu, \partial\tilde{Q}) \leq \frac{1}{\nu}$ and $0 \notin \partial\tilde{Q}$ we have that $\tilde{q} \in \partial\tilde{Q} \setminus \{0\}$. From Lemma 4 it follows that $f_0(q^\nu) + e^\top V f_1(q^\nu) \rightarrow \infty$. This implies that there exists an $\nu_{\bar{M}} \in \mathbb{N}$ such that for all $\nu > \nu_{\bar{M}}$ it holds that $f_0(q^\nu) + e^\top V f_1(q^\nu) > \bar{M}$, which gives a contradiction.

In case $(q^\nu)_{\nu \in \mathbb{N}}$ is unbounded we are in case 2) with $\|q^\nu\|_\infty \rightarrow \infty$. So, suppose $f_0(q^\nu) + e^\top V f_1(q^\nu) \leq M$ for all $\nu \in \mathbb{N}$ with $\|q^\nu\|_\infty \rightarrow \infty$. For all $\nu \in \mathbb{N}$ define $\tilde{q}^\nu = \frac{q^\nu}{\|q^\nu\|_\infty}$. Then

$$\tilde{q}^\nu \tilde{z} = \frac{q^\nu \tilde{z}}{\|q^\nu\|_\infty} = \frac{1}{\|q^\nu\|_\infty} \rightarrow 0.$$

Moreover, for all $\nu \in \mathbb{N}$ it holds that $\|\tilde{q}^\nu\|_\infty = 1$. Hence, $(\tilde{q}^\nu)_{\nu \in \mathbb{N}}$ is bounded and therefore has a convergent subsequence with limit, say, \tilde{q} . Then $\tilde{q}\tilde{z} = \lim_{\nu \rightarrow \infty} \tilde{q}^\nu \tilde{z} = 0$, i.e. $\tilde{q} \in \partial\tilde{Q}$. Furthermore, $\|\tilde{q}\|_\infty = 1$ and hence $\tilde{q} \neq 0$. From Lemma 4 we know that $f_0(\tilde{q}^\nu) + e^\top V f_1(\tilde{q}^\nu) \rightarrow \infty$. Since the budget correspondence is homogeneous of degree 0, we also get $f_0(q^\nu) + e^\top V f_1(q^\nu) \rightarrow \infty$, which gives a contradiction. \square

With these lemmas in place, existence of an FME can be proved by using a direct approach as opposed to the indirect proof of e.g. Magill and Quinzii (1996).

Theorem 2 *Let \mathcal{E} be a finance economy satisfying Assumptions 1–3. Then there exists an FME with asset price vector $\bar{q} \in Q$.*

Proof. A vector of prices $\bar{q} \in Q$ gives rise to an FME if and only if $f(\bar{q}) = 0$. Take $M > 0$. According to Lemma 5 there exists $\varepsilon > 0$ and $N > 0$ such that $d(q, \partial\tilde{Q}) \leq \varepsilon \Rightarrow f_0(q) + e^\top V f_1(q) > M$ and $\|q\|_\infty \geq N \Rightarrow f_0(q) + e^\top V f_1(q) > M$. Define the set of asset prices

$$\tilde{Q}' = \text{conv}\left(\{q \in \tilde{Q} \mid d(q, \partial\tilde{Q}) \geq \varepsilon, \|q\|_\infty \leq N\}\right),$$

where $\text{conv}(\cdot)$ denotes the convex hull. Obviously, $\tilde{Q}' \subset Q$ is compact and convex. Since the excess demand function f is continuous on \tilde{Q}' there exists a stationary point, i.e.

$$\exists_{\bar{q} \in \tilde{Q}'} \forall_{q \in \tilde{Q}'} : qf(\bar{q}) \leq \bar{q}f(\bar{q}).$$

Note that for all $q \in \tilde{Q}'$ one obtains

$$qf(\bar{q}) \leq \bar{q}f(\bar{q}) = 0,$$

because of Walras' law.

It is easy to see that $\bar{q} \in \tilde{Q}' \setminus \partial\tilde{Q}'$. For suppose $\bar{q} \in \partial\tilde{Q}'$ and take $(q_0, q_1) = \frac{(1, e^\top V)}{\tilde{z}_0 + e^\top V \tilde{z}_1} \in \tilde{Q}'$.² Then Lemma 5 shows that

$$q_0 f_0(\bar{q}) + q_1 f_1(\bar{q}) = \frac{f_0(\bar{q}) + e^\top V f_1(\bar{q})}{\tilde{z}_0 + e^\top V \tilde{z}_1} \geq \frac{M}{\tilde{z}_0 + e^\top V \tilde{z}_1} > 0,$$

which contradicts \bar{q} being a stationary point. Hence, $\bar{q} \in \text{int}(\tilde{Q}') \subset Q$.

Since \bar{q} is a stationary point it solves the linear programming problem $\max\{qf(\bar{q})\}$ such that q satisfies $q\tilde{z} = 1$. The dual of this problem is $\min\{\lambda\}$ such that λ satisfies $\lambda\tilde{z} = f(\bar{q})$. Using Walras' law we obtain,

$$\begin{aligned} 0 &= \bar{q}f(\bar{q}) = \lambda\bar{q}\tilde{z} = \lambda \\ &\iff \lambda = 0. \end{aligned}$$

Hence, $f(\bar{q}) = 0$. □

4 Price Adjustments Towards Equilibrium

In this section we present a path of points in \tilde{Q} from an arbitrary starting point in \tilde{Q} to an FME. First, we prove the existence of such a path. Note that the set of

²We can take ε and N such that $\frac{(1, e^\top V)}{\tilde{z}_0 + e^\top V \tilde{z}_1} \in \tilde{Q}'$.

prices \tilde{Q} is a (possibly unbounded) polyhedron generated by vertices $\{v^1, \dots, v^n\}$ and directions $\{q^1, \dots, q^m\}$. The recession cone of \tilde{Q} is given by

$$re(\tilde{Q}) = \left\{ q \in \mathbb{R}^{J+1} \mid q = \sum_{k=1}^m \mu_k q^k, \mu \geq 0 \right\}.$$

Let $q^0 \in \tilde{Q}$ be an arbitrary starting point in \tilde{Q} and denote

$$\tilde{Q}_1 = conv(\{v^1, \dots, v^n\}),$$

i.e. \tilde{Q}_1 is the convex hull of the vertices of \tilde{Q} . It is assumed without loss of generality that the directions $\{q^1, \dots, q^m\}$ are such that $q^0 + q^k \notin \tilde{Q}_1$, for all $k = 1, \dots, m$. Define the polytope

$$\tilde{Q}(1) = \left\{ q \in \tilde{Q} \mid q = \sum_{h=1}^n \mu_h (v^h - q^0) + \sum_{k=1}^m \mu_{n+k} q^k + q^0, \mu_k \geq 0, \sum_{k=1}^{n+m} \mu_k \leq 1 \right\},$$

and the set

$$K = \left\{ q \in re(\tilde{Q}) \mid q = \sum_{k=1}^m \mu_k q^k, \mu_k \geq 0, \sum_{k=1}^m \mu_k \leq 1 \right\}.$$

For simplicity we assume $q^0 \in int(\tilde{Q}(1))$. We can now define the expanding set $\tilde{Q}(\lambda)$,

$$\tilde{Q}(\lambda) = \begin{cases} (1 - \lambda)\{q^0\} + \lambda\tilde{Q}(1) & \text{if } 0 \leq \lambda \leq 1; \\ \tilde{Q}(1) + (\lambda - 1)K & \text{if } \lambda \geq 1. \end{cases}$$

Note that for all $\lambda \geq 0$ the set $\tilde{Q}(\lambda)$ is a polytope and that $\lim_{\lambda \rightarrow \infty} \tilde{Q}(\lambda) = \tilde{Q}$. In Figure 1 some of these sets are depicted.

By Lemma 5 we know that for all $M > 0$ there exists an $N > 0$ such that

$$\|q\|_\infty \geq N \Rightarrow f_0(q) + e^\top V f_1(q) > M.$$

This implies that there exists a $\lambda^0 > 0$ such that for all stationary points \bar{q} of f on \tilde{Q} it holds that $\bar{q} \in \tilde{Q}(\lambda^0)$. Recall that all stationary points of f on \tilde{Q} are FMEs. Let $\lambda^* = \max\{1, \lambda^0\}$ and define the homotopy $h : [0, \lambda^*] \times \tilde{Q}(\lambda^*) \rightarrow \tilde{Q}(\lambda^*)$ by

$$h(\lambda, q) = \begin{cases} proj_{\tilde{Q}(\lambda)}(q + f(q)) & \text{if } q \in \tilde{Q}(\lambda); \\ proj_{\tilde{Q}(\lambda)}(proj_{\tilde{Q}(\lambda)}(q) + f(proj_{\tilde{Q}(\lambda)}(q))) & \text{if } q \notin \tilde{Q}(\lambda), \end{cases}$$

where $proj_A(q)$ is the projection of q in $\|\cdot\|_2$ on the set A . An important property of this homotopy is stated in the following lemma, where $(\lambda, q) \in [0, \lambda^*] \times \tilde{Q}(\lambda^*)$ is a fixed point of h if $h(\lambda, q) = q$.

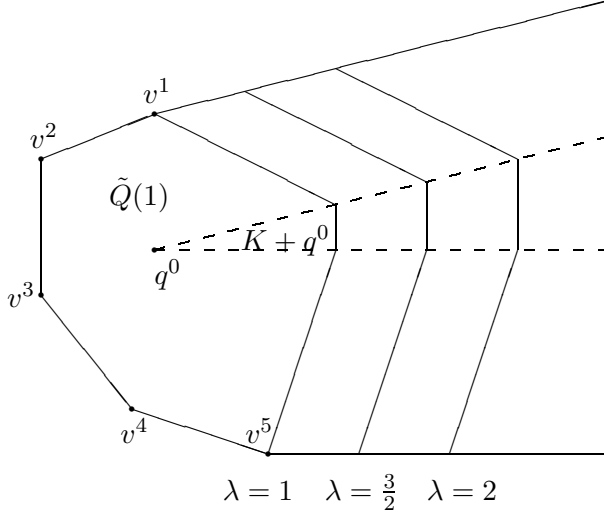


Figure 1: The expanding set $\tilde{Q}(\lambda)$.

Lemma 6 *Let $\lambda \in [0, \lambda^*]$. If (λ, q) is a fixed point of the homotopy h , then q is a stationary point of f on $\tilde{Q}(\lambda)$ and q is an FME if $q \notin \partial\tilde{Q}(\lambda)$. Moreover, if (λ^*, q) is a fixed point of h , then $q \notin \partial\tilde{Q}(\lambda^*)$, i.e. q is an FME.*

Proof. Note that if $q \notin \tilde{Q}(\lambda)$, q cannot be a fixed point of h . We therefore consider two cases. Firstly, if $q \in \tilde{Q}(\lambda)$ and $q + f(q) \in \tilde{Q}(\lambda)$, then $f(q) = 0$, i.e. q is an FME and a stationary point.

Secondly, if $q \in \tilde{Q}(\lambda)$ and $q + f(q) \notin \tilde{Q}(\lambda)$, then $q \in \partial\tilde{Q}(\lambda)$ and the orthogonal projection

$$\min_{q' \in \tilde{Q}(\lambda)} (q' - q - f(q))^\top (q' - q - f(q)),$$

is solved by q since q is a fixed point of h . Hence, for all $q' \in \tilde{Q}(\lambda)$ we have that

$$(q' - q - f(q))^\top (q' - q - f(q)) \geq f(q)^\top f(q),$$

and so

$$(q' - q)^\top (q' - q) \geq 2(q' - q)^\top f(q).$$

Take $\hat{q} = \mu q^0 + (1 - \mu)q$ for any μ , $0 < \mu \leq 1$. Since $\tilde{Q}(\lambda)$ is convex we have $\hat{q} \in \tilde{Q}(\lambda)$ and so for all $0 < \mu \leq 1$ it holds that

$$\mu^2 (q' - q)^\top (q' - q) \geq 2\mu (q' - q)^\top f(q),$$

and so

$$\frac{1}{2}\mu (q' - q)^\top (q' - q) \geq (q' - q)^\top f(q).$$

Let $\mu \downarrow 0$. Then $0 \geq (q' - q)f(q)$, i.e.

$$q'f(q) \leq qf(q).$$

So, q is a stationary point of f on $\tilde{Q}(\lambda)$. Furthermore, if $q \notin \partial\tilde{Q}(\lambda)$ Walras' law implies that $f(q) = 0$ and, hence, q is an FME. \square

By Browder's fixed point theorem (see Browder (1960)) it now follows that there is a connected set or path, C , of fixed points of h such that $C \cap \{0\} \times \tilde{Q}(\lambda^*) \neq \emptyset$ and $C \cap \{\lambda^*\} \times \tilde{Q}(\lambda^*) \neq \emptyset$. Lemma 6 then implies that there is a path in $\tilde{Q}(\lambda^*)$ of stationary points of f connecting q^0 with an FME \bar{q} .

The path starts in the price vector q^0 that can be interpreted as the current financial market equilibrium. Suppose that a structural change takes place in the economy, e.g. preferences, initial endowments or asset payoffs change. The resulting environment is the situation where $\lambda = 0$ and the set of possible prices equals $\tilde{Q}(0) = \{q^0\}$. If one interprets $t = 0$ as the start of a trading day and $t = \lambda^*$ to be the end, then the price vector q^0 can also be interpreted as the starting prices of assets on the stock exchange, like for example the outcome of the call auction on the NYSE that takes place at the start of each trading day.

In many financial markets, the market maker takes a position in trade and is hence also dealer. In some financial markets, the market maker buys from the sellers and sells to the buyers. In that case he wants to maximise the value of excess demand, since this determines his profit. One way to achieve this is for the market maker to start quoting prices in the direction that maximises the value of excess demand. In response the investors give asks and bids to the market maker. This continues until the asks and bids are such that prices in another direction give the highest value of excess demand.

Since the market maker is unaware of the preferences of the agents it cannot simply quote the equilibrium prices. Therefore, he takes the myopic approach to quote prices that maximise excess demand in some restricted set, $Q(\lambda)$. Along the adjustment path, the market maker quotes prices from a growing set of possible prices while λ moves from 0 towards λ^* . It can happen, however, that prices have been moving in the wrong direction. As a reaction set $Q(\lambda)$ shrinks by decreasing λ and the path starts moving in another direction. In this way, the set of prices expands, shrinks sometimes and expands again until an equilibrium price vector has been found.

5 The Algorithm

There are simplicial algorithms to follow approximately the path of stationary points of an excess demand function f from an arbitrary starting point q^0 to an FME. The algorithm of Talman and Yamamoto (1989) finds a path to an equilibrium on a polytope. The algorithm should first be applied to $\tilde{Q}(1)$. If the algorithm terminates in $\bar{q} \in \tilde{Q}(1) \setminus \partial\tilde{Q}(1)$, an FME has been found. If it terminates at $q \in \partial\tilde{Q}(1)$, we extend the algorithm to $\tilde{Q}(\lambda)$, $\lambda \geq 1$. This procedure is repeated until an approximate FME has been found.

The algorithm generates a path of stationary points of a piecewise linear approximation to the excess demand function f . The set $\tilde{Q}(1)$ is a J -dimensional polytope and can be written as

$$\tilde{Q}(1) = \{q \in \mathbb{R}^{J+1} | q\tilde{z} = 1, a^i q \leq b_i, i = 1, \dots, l\},$$

for some $a^i \in \mathbb{R}^{J+1} \setminus \{0\}$ and $b_i \in \mathbb{R}$, $i = 1, \dots, l$.

Let $I \subset \{1, \dots, l\}$. Then $F(I)$ is defined by

$$F(I) = \{q \in \tilde{Q}(1) | a^i q = b_i, i \in I\}.$$

The set $\mathcal{I} = \{I \subset \{1, \dots, m\} | F(I) \neq \emptyset\}$ is the set of all index sets I for which $F(I)$ is a $(J - |I|)$ -dimensional face of $\tilde{Q}(1)$. Let $q^0 \in \text{int}(\tilde{Q}(1))$ be the starting point. For any $I \in \mathcal{I}$ define

$$vF(I) = \text{conv}(\{q^0\}, F(I)).$$

Now \tilde{Q} is triangulated into simplices with finite mesh size in such a way that every $vF(I)$ is triangulated into $(J - |I| + 1)$ -dimensional simplices.

Suppose that the algorithm is in $q^* \in vF(I)$, then q^* lies in some t -dimensional simplex $\sigma(q^1, \dots, q^{t+1})$, with vertices the affinely independent points q^1, \dots, q^{t+1} , where $t = J - |I| + 1$ and $q^i \in vF(I)$ for all $i = 1, \dots, t + 1$. There exist unique $\lambda_1^*, \dots, \lambda_{t+1}^* \geq 0$, with $\sum_{i=1}^{t+1} \lambda_i^* = 1$, such that $q^* = \sum_{j=1}^{t+1} \lambda_j^* q^j$. The piecewise linear approximation of $f(\cdot)$ at q^* is then given by

$$\bar{f}(q^*) = \sum_{j=1}^{t+1} \lambda_j^* f(q^j).$$

Let λ , $0 < \lambda \leq 1$, be such that $q^* \in \partial\tilde{Q}(\lambda)$. Then $q^* = (1 - \lambda)q^0 + \lambda q'$, for some $q' \in F(I)$. For all $i = 1, \dots, m$, define $b_i(\lambda) = (1 - \lambda)a^i q^0 + \lambda b_i$. The point q^* is such that it is a stationary point of \bar{f} on $\tilde{Q}(\lambda)$, i.e. q^* is a solution to the linear program

$$\max\{q\bar{f}(q^*) | a^i q \leq b_i(\lambda), i = 1, \dots, m, q\tilde{z} = 1\}.$$

The dual problem is given by

$$\min \left\{ \sum_{i=1}^m \mu_i b_i(\lambda) + \beta \left| \sum_{i=1}^m \mu_i a^i + \beta \tilde{z} = \bar{f}(q^*), \mu \geq 0, \beta \in \mathbb{R} \right. \right\}.$$

This gives a solution μ^* , β^* . Using the complementary slackness condition and assuming non-degeneracy we get the following:

$$\begin{aligned} I &:= \{i | a^i q^* = b_i(\lambda)\} \\ &= \{i | a^i q' = b_i\} \\ &= \{i | \mu_i^* > 0\}. \end{aligned}$$

Hence,

$$\sum_{j=1}^{t+1} \lambda_j^* f(q^j) = \sum_{i \in I} \mu_i^* a^i + \beta^* \tilde{z},$$

$\sum_{j=1}^{t+1} \lambda_j^* = 1$, and $\mu_i^* \geq 0$, for all $i \in I$. In vector notation this system reads

$$\sum_{j=1}^{t+1} \lambda_j^* \begin{bmatrix} -f(q^j) \\ 1 \end{bmatrix} + \sum_{i \in I} \mu_i^* \begin{bmatrix} a^i \\ 0 \end{bmatrix} + \beta^* \begin{bmatrix} \tilde{z} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (3)$$

This linear system has $J + 2$ equations and $J + 3$ variables. The value β^* is a measure for how much the solution for the piecewise linear approximation deviates from Walras' law.

In each step of the algorithm one variable leaves and one new variable enters the basis of the linear system. This is achieved by making a linear programming pivot step in (3). Given that due to the pivot step a variable leaves the basis, the question is how to determine which variable enters the basis. First, suppose that some λ_k leaves the basis. This implies that q^* can be written as

$$q^* = \sum_{j=1, j \neq k}^{t+1} \lambda_j q^j.$$

Assuming again non-degeneracy, q^* then lies in the interior of the facet τ of the simplex $\sigma(q^1, \dots, q^{t+1})$ opposite to the vertex q^k . Now there are three possibilities. First, suppose that $\tau \in \partial vF(I)$ and $\tau \notin \partial \tilde{Q}(1)$. According to Lemma 4 this happens if and only if $\tau \subset vF(I \cup \{i\})$ for some $i \notin I$. Then we increase the dual dimension with one and μ_i enters the basis via a pivot step in (3). The second case comprises $\tau \subset \partial \tilde{Q}(1)$. Then the algorithm continues in $\tilde{Q} \setminus \tilde{Q}(1)$. The set τ is a facet of exactly one t -simplex σ' in the extension of $v(F(I))$ in $\tilde{Q}(2)$. The vertex opposite to τ of σ' is, say, q^k . The corresponding variable λ_k then enters the basis. Finally, it can be that $\tau \notin \partial vF(I)$. Then there is a unique simplex σ' in $vF(I)$ with vertex, say q^k , opposite to the facet τ . The corresponding variable λ_k then enters the basis.

The second possibility is that μ_i leaves the basis for some $i \in I$. Then the dual dimension is decreased with one, i.e. the set I becomes $I \setminus \{i\}$. Now there are two possibilities. If $I \setminus \{i\} = \emptyset$ then $\bar{f}(q^*) = \beta^* \bar{z}$ and the algorithm terminates. The vector q^* is an approximate equilibrium and the algorithm can be restarted at q^* with a smaller mesh for the triangulation in order to improve the accuracy of the approximation. Otherwise, if $I \setminus \{i\} \neq \emptyset$, then define $I' = I \setminus \{i\}$. Since the primal dimension is increased with one there exists a unique simplex σ' in $vF(I')$ having σ as a facet. The vertex opposite to σ is, say, q^k . The algorithm continues with entering λ_k in the basis by means of a pivot step in (3).

The first step of the algorithm consists of solving the linear program

$$\max\{qf(q^0) \mid a^i q \leq b_i, i = 1, \dots, m, q\bar{z} = 1\}.$$

Its dual program is given by

$$\min \left\{ \sum_{i=1}^m \mu_i b_i + \beta \mid \sum_{i=1}^m \mu_i a^i + \beta \bar{z} = f(q^0) \right\}.$$

This gives as solution μ^0 and β^0 . The set $F(I_0)$ is a vertex of $\tilde{Q}(1)$, where $I_0 = \{i \in \{1, \dots, m\} \mid \mu_i^0 > 0\}$. There is a unique one-dimensional simplex $\sigma(w^1, w^2)$ in $vF(I_0)$ with vertices $w^1 = q^0$ and $w^2 \neq w^1$. Then λ_2 enters the basis by means of a pivot step in system (3).

As an example of this procedure consider the finance economy $\mathcal{E}(u, \omega, V)$ with two consumers, two assets and three states of nature. The utility functions are given by

$$u^1(x^1) = (x_0^1)^3 x_1^1 x_2^1 x_3^1$$

and

$$u^2(x^2) = x_0^2 x_1^2 x_2^2 (x_3^2)^2,$$

and the initial endowments equal $\omega^1 = (1, 3, 3, 3)$ and $\omega^2 = (4, 1, 1, 1)$, respectively. On the financial markets, two assets are traded, namely a riskless bond and a contingent contract for state 3. That is,

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is easy to see that the set of no-arbitrage prices, Q , is given by

$$Q = \{(q_0, q_1, q_2) \mid q_0 > 0, q_2 > 0, q_1 > q_2\}.$$

Taking $\tilde{z} = (1, 1, 1)$, we get that

$$\tilde{Q} = \{q \in \mathbb{R}^3 \mid a^i q \leq 0, i = 1, 2, 3, q\tilde{z} = 1\},$$

where $a^1 = (-1, 0, 0)$, $a^2 = (0, 0, -1)$, and $a^3 = (0, -1, 1)$. Since \tilde{Q} is a polytope one can set $\tilde{Q}(1) = \tilde{Q}$. The set \tilde{Q} is the convex hull of the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 1/2, 1/2)$.

We start the algorithm at the price vector $q^0 = (\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$. The grid size of the simplicial subdivision is taken to be $\frac{1}{8}$. In the first step of the algorithm we solve the linear program

$$\min\{\beta \mid \mu_1 a^1, \mu_2 a^2, \mu_3 a^3, \beta \tilde{z} = f(q^0), \mu_i \geq 0, i = 1, 2, 3\},$$

where $f(q^0) = (-3.02, 8.4667, -1.8333)$. This gives as solution $\mu^0 = (11.4867, 10.3, 0)$ and $\beta^0 = 8.4667$. This implies that $I^0 = \{1, 2\}$. The basic variables are λ_1, μ_1, μ_2 , and β . The coefficient matrix corresponding to (3) equals

$$B = \begin{bmatrix} 3.02 & -1 & 0 & 1 \\ -8.4667 & 0 & 0 & 1 \\ 1.8333 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The first one-dimensional simplex that is generated is the simplex $\sigma(w^1, w^2) \in vF(I^0)$, where $w^1 = q^0$ and $w^2 = \frac{1}{64}(35, 22, 7)$. The algorithm proceeds by letting λ_2 enter the basis by means of a linear programming pivot step of $(-f(w^2), 1)$ into the matrix B^{-1} . This means, the algorithm leaves q^0 into the direction $vF(I^0) - q^0$ towards $vF(I^0) = (0, 1, 0)$. By doing so one finds that μ_2 leaves the basis. This implies that the dimension of the dual space is reduced and a two-dimensional simplex is generated in $vF(\{1\})$, namely $\sigma(w^1, w^2, w^3)$, where $w^3 = \frac{1}{64}(35, 20, 9)$. One proceeds by entering λ_3 into the basis by performing a pivot step in B^{-1} . In this way one obtains a sequence of two-dimensional adjacent simplices in $vF(\{1\})$ until the algorithm terminates when μ_1 leaves the basis. This happens after, in total, 12 iterations. The path of the algorithm is depicted in Figure 2.

The basic variables in the final simplex are $\lambda_2, \lambda_3, \lambda_1$, and β . The corresponding simplex is given by $\sigma(w^1, w^2, w^3)$, where $w^1 = \frac{1}{16}(5, 8, 3)$, $w^2 = \frac{1}{64}(15, 34, 15)$, and $w^3 = \frac{1}{64}(15, 36, 11)$. The corresponding values for λ are given by $\lambda_1 = 0.1223$, $\lambda_2 = 0.8460$, and $\lambda_3 = 0.0316$. This yields as an approximate FME the price vector

$$\bar{q} = \sum_{i=1}^3 \lambda_i w^i = (0.2439, 0.5284, 0.2267).$$

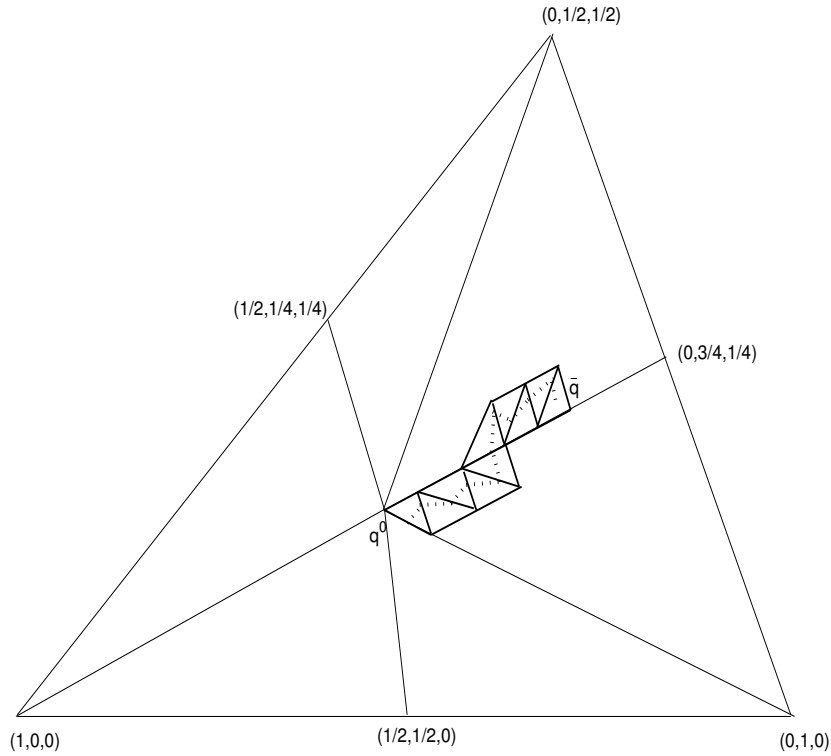


Figure 2: The path of prices (dotted line) generated by the algorithm.

The value of the excess demand function in \bar{q} is given by $f(\bar{q}) = (-0.0174, 0.0145, -0.0151)$. The approximate equilibrium values for consumption at $t = 0$ and the demand for assets are given by $\bar{x}_0^1 = 3.7494$, $\bar{x}_0^2 = 1.2332$, $\bar{z}^1 = (-0.9794, -0.6756)$, and $\bar{z}^2 = (0.9939, 0.6605)$, respectively. The accuracy of approximation can be improved by restarting the algorithm in \bar{q} and taking a smaller mesh size for the simplicial subdivision of \tilde{Q} .

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