Modified Normal Demand Distributions in (R,S)-Inventory Models

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Modified normal demand distributions in (R, S)-inventory control

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To model demand, the normal distribution is by far the most popular; the disadvantage that it takes negative values is taken for granted. This paper proposes two modifications of the normal distribution, both taking non-negative values only. Safety factors and order-up-to-levels for the familiar (R, S)-control system are derived and compared with the standard values corresponding with the original normal distribution.

1 Introduction

In all textbooks and much literature on inventory control, the normal distribution plays an important part in modelling demand. In the celebrated textbook Silver et al. (1998), normal demand is discussed in numerous places; from more recent literature we mention six examples. *Only* normal demand is considered in Artto & Pylkkänen (1999), Chen & Chuang (2000), Geunes & Zeng (2001) and Alstrøm (2001), in quite different settings. Zeng & Hayya (1999) made comparisons for four families of demand distributions; they state: 'the normal distribution always enjoys wide applications in both research and practice' (p. 149). Bartezzaghi et *al.* (1999) considered the numerical impact of six specific distributions, among which a normal.

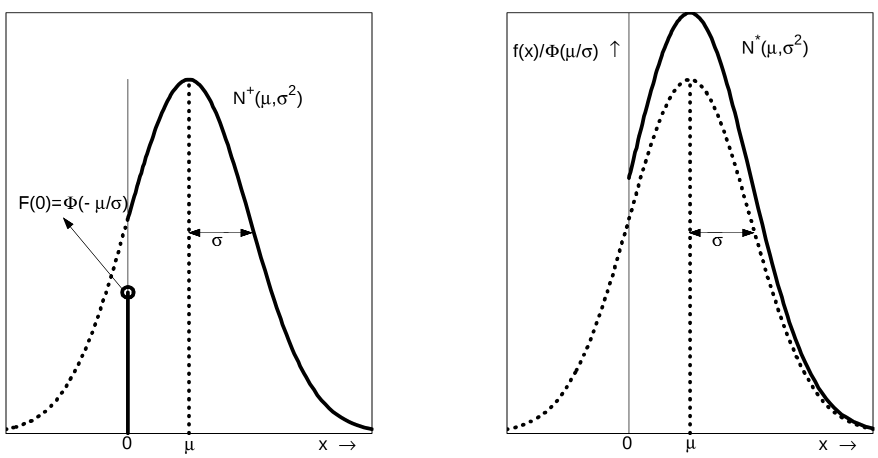
An obvious argument for this choice of the demand distribution are the nice analytical properties of the normal distribution, enabling the derivation of exact expressions for important control parameters like safety factors.

Nevertheless, normal demand distributions have two clear disadvantages. The first is their symmetry, which does not reflect the fact that in practice demand distributions generally are skewed to the right. Of even more importance is the occurrence of negative values, particularly for higher values of the coefficient of variation. This problem is either neglected, or negative demand is interpreted as deliveries being sent back. So, implicit additional assumptions are that customers are allowed to return delivered goods - even if delivery is rather long ago - and that this phenomenon may occur rather frequently.

In case of constant order-up-to levels, even more additional assumptions are necessary; if total demand during a replenishment cycle is negative, it has to be assumed that the returned stock is sent back to the original supplier.

To avoid these akward constructions, we here suggest two modifications of the normal distributions; both only take non-negative values and are asymmetrical, while allowing theoretical derivations. The first is obtained by replacing all negative values by 0, creating a probability mass in the origin. The second is the normal distribution, one-sidedly truncated at 0. Starting from the normal distribution $N(\mu, \sigma^2)$, they will be denoted as $N^+(\mu, \sigma^2)$ and $N^*(\mu, \sigma^2)$, respectively. Figure 1 shows these distributions; $f(F)$ denotes the density (distribution function) of $N(\mu, \sigma^2), \varphi(\Phi)$ of the standard normal.

Figure 1.1. The modified normal distributions $N^+(\mu, \sigma^2)$ and $N^*(\mu, \sigma^2)$.



It is worth mentioning that normal distributions with a probability mass in 0 feature in Geunes & Zeng (2001). The two distributions will be studied within the (R, S)-inventory control system with lead time zero. This means that stock is measured whenever a review period R has passed, and immediately replenished to the order-up-to level S. Two criteria will be considered:

- the fraction of review periods in which total demand can be delivered from stock should be equal to a given level P_1,

- the fraction of total demand that can be delivered from stock should equal a given level P_2 (the fill rate).

They are called the P_1- and P_2- service criterion, respectively.

Let X refer to demand during a review period and assume that μ and σ^2 are known. We will calculate the order-up-to levels S_i^+ and S_i^* for criterion P_i $(i = 1, 2)$ and - by standardizing - the corresponding safety factors c_i^+ and c_i^*. They will be compared with the well-known safety factors c_i, relating to the normal distribution itself:

$$\begin{cases} c_1 = & \Phi^{-1}(P_1) \\ c_2 = & G^{-1}[(1-P_2)/\nu] \end{cases} \tag{1.1}$$

where $\nu = \sigma/\mu$ denotes the coefficient of variation and G is given by

$$G(k) = \int_k^\infty (z-k)\varphi(z)dz = \varphi(k) - k\Phi(-k) \tag{1.2}$$

Compare Silver et al. (1998) e.g. or Strijbosch & Moors (1998). Table 1.1. gives numerical values of the $c_{i.}$

Table 1.1. Values of c_i

	P_i / ν	0.9	0.925	0.95	0.975
c_1	-	1.2816	1.4395	1.6449	1.9600
c_2	**0.25**	-0.0021	0.2165	0.4929	0.9023
	0.50	0.4929	0.6711	0.9023	1.2556
	0.75	0.7405	0.9023	1.1146	1.4430
	1.00	0.9023	1.0546	1.2556	1.5689
	1.25	1.0212	1.1671	1.3602	1.6631
	1.50	1.1146	1.2556	1.4430	1.7379
	1.75	1.1910	1.3283	1.5111	1.7997
	2.00	1.2556	1.3898	1.5689	1.8523

In Section 2 the modified normal demand distribution $N^+(\mu, \sigma^2)$ will be covered, in Section 3 the truncated normal distribution $N^*(\mu, \sigma^2)$. Section 4 presents a more refined comparison with the normal distribution. In the final Section 5 the results are discussed.

2 Demand distribution $N^+(\mu, \sigma^2)$

The distribution $N^+(\mu, \sigma^2)$ has a point mass $F(0) = \Phi(-1/\nu)$ in 0 and the normal density $f(x)$ for $x > 0$. Hence, its distribution function F^+ is given by

$$F^+(x) = \begin{cases} 0, & x < 0 \\ F(0), & x = 0 \\ F(x), & x > 0 \end{cases} \tag{2.1}$$

The mean μ^+ follows from (1.2):

$$\mu^+ = \int_0^\infty x f(x) dx = \sigma \int_{-1/\nu}^\infty (z + 1/\nu)\varphi(z) dz = \sigma G(-1/\nu) \tag{2.2}$$

The second moment μ_2^+ is found similarly:

$$\begin{aligned} \mu_2^+ &= \int_0^\infty x^2 f(x) dx = \sigma^2 \int_{-1/\nu}^\infty (z + 1/\nu)^2 \varphi(z) dz \\ &= \sigma^2 \left[\int_{-1/\nu}^\infty z^2 \varphi(z) dz + \frac{2}{\nu} \int_{-1/\nu}^\infty (z + 1/\nu)\varphi(z) dz - \frac{1}{\nu^2} \int_{-1/\nu}^\infty \varphi(z) dz \right] \\ &= \sigma^2[\varphi(1/\nu)/\nu + (1 + 1/\nu^2)\Phi(1/\nu)] \end{aligned}$$

using partial integration and (1.2) once more. Writing

$$H(k) = k\varphi(k) + (k^2 + 1)\Phi(k) \tag{2.3}$$

for brevity, the variance ${\sigma^+}^2$ is given by

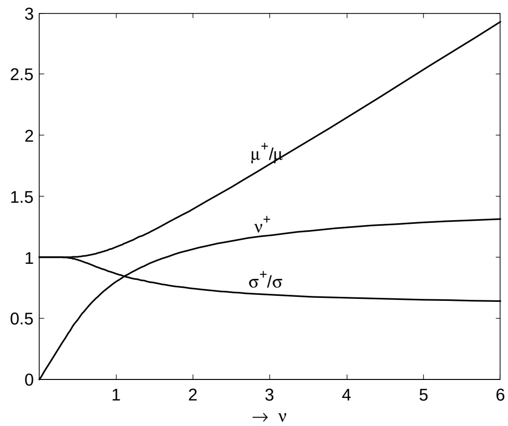
$${\sigma^+}^2 = \sigma^2[H\left(1/\nu\right) - G^2(-1/\nu\big)] \tag{2.4}$$

so that the coefficient of variation ν^+ equals

$$\nu^+ = \sqrt{H(1/\nu)/G^2(-1/\nu) - 1} \tag{2.5}$$

Figure 2.1. shows these parameters of $N^+(\mu, \sigma^2)$ as function of ν; some theoretical properties are derived in Appendix A.

Figure 2.1. Behaviour of $\mu^+/\mu, \sigma^+/\sigma$ and ν^+.



Since μ is positive, $F(0) \leq 0.5$ holds, so that in general P_1 exceeds $F(0)$. Then the order-up-to level S_1^+ corresponding to the P_1-criterion remains unchanged:

$$S_1^+ = S_1$$

and standardizing with the parameters of $N^+(\mu, \sigma^2)$ gives

$$c_1^+ = \frac{S_1^+ - \mu^+}{\sigma^+} = \frac{1/\nu - G(-1/\nu) + \Phi^{-1}(P_1)}{\sqrt{H(1/\nu) - G^2(-1/\nu)}} \tag{2.6}$$

The level S_2^+ has to satisfy

$$\int_{S_2^+}^{\infty} (x - S_2^+) f(x) dx = (1 - P_2)\mu^+$$

The left-hand side equals $\sigma G[(S_2^+ - \mu)/\sigma]$, so that

$$S_2^+ = \mu + \sigma G^{-1}[(1 - P_2)G(-1/\nu)]$$

and finally

$$c_2^+ = \frac{1/\nu - G(-1/\nu) + G^{-1}[(1 - P_2)G(-1/\nu)]}{\sqrt{H(1/\nu) - G^2(-1/\nu)}} \tag{2.7}$$

Table 2.1 presents numerical results for the same values of ν and P as in Tabel 1.1.

Table 2.1 Values of c_i^+.

P_i		c_1^+				c_2^+			
ν	ν^+	0.9	0.925	0.95	0.975	0.9	0.925	0.95	0.975
0.25	**0.2500**	1.2816	1.4396	1.6449	1.9600	-0.0021	0.2165	0.4929	0.9024
0.50	**0.4879**	1.2992	1.4604	1.6699	1.9915	0.4916	0.6736	0.9098	1.2706
0.75	**0.6703**	1.3437	1.5150	1.7376	2.0793	0.7372	0.9139	1.1455	1.5035
1.00	**0.8000**	1.3826	1.5649	1.8018	2.1654	0.8943	1.0729	1.3082	1.6744
1.25	**0.8945**	1.4109	1.6028	1.8523	2.2351	1.0048	1.1868	1.4273	1.8031
1.50	**0.9659**	1.4311	1.6311	1.8911	2.2900	1.0870	1.2724	1.5179	1.9025
1.75	**1.0216**	1.4458	1.6526	1.9212	2.3336	1.1506	1.3391	1.5890	1.9814
2.00	**1.0661**	1.4568	1.6691	1.9451	2.3687	1.2013	1.3924	1.6462	2.0452

Comparison with Table 1.1. shows that c_1^+ exceeds c_1 throughout; the difference is increasing in ν and P_1. The picture for c_2^+ is quite similar, but c_2^+ is smaller than c_2 for P relatively low. The general conclusion is that to attain high performance, safety factors should be increased if the demand model $N^+(\mu, \sigma^2)$ is preferred to $N(\mu, \sigma^2)$. For the tabled values, the increase is up to 20% for c_1 and up to 10% for c_2.

3 Demand distribution $N^*(\mu, \sigma^2)$

For positive x, the distribution function of the truncated normal distribution is given by $[F(x) - \Phi(-1/\nu]/\Phi(1/\nu)$. Immediate results therefore are

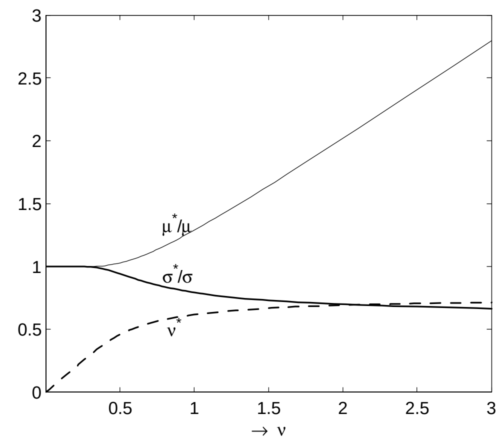
$$\mu^* = \mu^+/\Phi(1/\nu), \quad \mu_2^* = \mu_2^+/\Phi(1/\nu)$$

and consequently

$$\begin{cases} \mu^* = & \sigma G(-1/\nu)/\Phi(1/\nu) \\ \sigma^{*2} = & \dfrac{\sigma^2}{\Phi^2(1/\nu)}[\Phi(1/\nu)H(1/\nu) - G^2(-1/\nu)] \\ \nu^* = & \sqrt{\Phi(1/\nu)H(1/\nu)/G^2(-1/\nu) - 1} \end{cases} \tag{3.1}$$

Figure 3.1 shows the parameters σ^* and ν^* as function of ν.

Figure 3.1. Behaviour of $\mu^*/\mu, \sigma^*/\sigma$ and ν^*.



Now, S_1^* has to satisfy

$$P_1\Phi(1/\nu) = F(S_1^*) - \Phi(-1/\nu) = \Phi[(S_1^* - \mu)/\sigma] - \Phi(-1/\nu)$$

so that

$$S_1^* = \mu + \sigma\Phi^{-1}[P_1\Phi(1/\nu) + \Phi(-1/\nu)]$$

and

$$c_1^* = \frac{\Phi(1/\nu)/\nu - G(-1/\nu) + \Phi^{-1}[P_1\Phi(1/\nu) + \Phi(-1/\nu)]\Phi(1/\nu)}{\sqrt{\Phi(1/\nu)H(1/\nu) - G^2(-1/\nu)}} \tag{3.2}$$

The level S_2^* follows from

$$(1 - P_2)\mu^*\Phi(1/\nu) = \int_{S_2^*}^{\infty}(x - S_2^*)f(x)dx = \sigma G[(S_2^* - \mu)/\sigma]$$

and equals

$$S_2^* = \mu + \sigma G^{-1}[(1 - P_2)G(-1/\nu)]$$

which is identical to S_2. Hence,

$$c_2^* = \frac{\Phi(1/\nu)/\nu - G(-1/\nu) + G^{-1}[(1 - P_2)G(-1/\nu)]\Phi(1/\nu)}{\sqrt{\Phi(1/\nu)H(1/\nu) - G^2(-1/\nu)}} \tag{3.3}$$

Values of safety factors for the truncated normal distribution are given in Table 3.1.

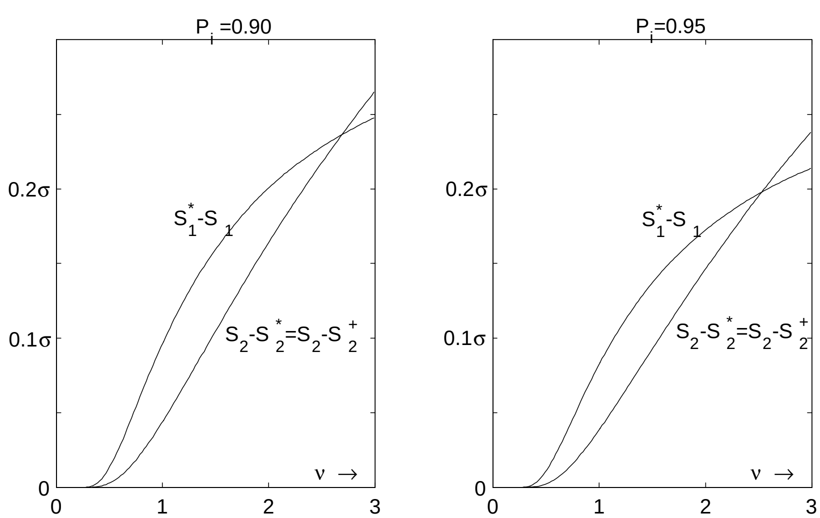
Table 3.1 Values of c_i^*.

P_i		c_1^*				c_2^*			
ν	ν^+	0.9	0.925	0.95	0.975	0.9	0.925	0.95	0.975
0.25	0.2499	1.2818	1.4398	1.6452	1.9604	-0.0022	0.2164	0.4929	0.9025
0.50	0.4581	1.3164	1.4832	1.7002	2.0335	0.4619	0.6514	0.8973	1.2727
0.75	0.5632	1.3547	1.5356	1.7716	2.1349	0.6355	0.8267	1.0772	1.4644
1.00	0.6163	1.3738	1.5647	1.8142	2.1994	0.7193	0.9143	1.1713	1.5712
1.25	0.6471	1.3838	1.5813	1.8398	2.2399	0.7673	0.9656	1.2276	1.6369
1.50	0.6670	1.3895	1.5915	1.8564	2.2671	0.7981	0.9988	1.2645	1.6809
1.75	0.6808	1.3931	1.5984	1.8679	2.2863	0.8195	1.0220	1.2906	1.7122
2.00	0.6909	1.3955	1.6033	1.8763	2.3007	0.8352	1.0391	1.3099	1.7356

Comparison with Table 1.1 shows that c_1^* exceeds c_1 throughout. Comparing with Tabel 2.1 reveals that c_1^* is relatively close to c_1^+; note that c_1^* takes both lower and higher values than c_1^+. As to c_2^*, most values are below c_2, although exceptions occur; even $c_2^* > c_2^+$ holds occasionally.

Comparison of the order-up-to-levels is easier. It is immediately clear that S_1^* always exceeds S_1; Appendix A shows that, on the other hand, S_2 always exceeds $S_2^* = S_2^+$. Figure 3.2 shows the behaviour of the differences $S_1^* - S_1$ and $S_2 - S_2^+ = S_2 - S_2^*$ (in units σ).

Figure 3.2. Behaviour of $S_1^* - S_1$ and $S_2 - S_2^* = S_2 - S_2^+$.



4 Standard safety factors using ν^+ and ν^*

If not $N(\mu, \sigma^2)$ but rather $N^+(\mu, \sigma^2)$ or $N^*(\mu, \sigma^2)$ is thought to be the appropriate demand model, this implies in fact two changes:

- normality does not hold anymore,
- mean and variance are replaced by $\mu^+(\mu^*)$ and $\sigma^+(\sigma^*)$, respectively.

It is interesting to separate these two effects by neglecting the first, while adopting the second. This means that the standard safety factors c_i are used, with for c_2, however, the correct coefficient of variation plugged in. Denoting the resulting safety factors by c_{is}, this gives more precisely:

$$\begin{cases} c_{1s}^* = c_{1s}^+ = c_1 = \Phi^{-1}(P_1), \\ c_{2s}^* = G^{-1}[(1-P_2)/\nu^*],\ c_{2s}^+ = G^{-1}[(1-P_2)/\nu^+] \end{cases} \tag{4.1}$$

The exact performance P_{is} of the corresponding (R, S)-rules can be calculated, e.g.

$$P_{2s}^+ = 1 - \frac{1}{\mu^+} \int_{S_{2s}^+}^{\infty} x f(x) dx = 1 - G[(S_{2s}^+ - \mu)/\sigma]/G(-1/\nu)$$

Denoting

$$\begin{cases} a(\nu) = \sigma^+/\sigma = \sqrt{H(1/\nu) - G^2(-1/\nu)} \\ b(\nu) = \sigma^*/\sigma = \sqrt{\Phi(1/\nu)H(1/\nu) - G^2(-1/\nu)}/\Phi(1/\nu) \end{cases}$$

this leads to the performances

$$\begin{cases} P_{1s}^+ = \Phi[G(-1/\nu) - 1/\nu + \Phi^{-1}(P_1)a(\nu)] \\ P_{2s}^+ = 1 - G[G(-1/\nu) - 1/\nu + G^{-1}[(1-P_2)/\nu^+]a(\nu)]/G(-1/\nu) \\ P_{1s}^* = \Phi[G(-1/\nu)/\Phi(1/\nu) - 1/\nu + \Phi^{-1}(P_1)b(\nu)]/\Phi(1/\nu) - \Phi(-1/\nu) \\ P_{2s}^* = 1 - G[G(-1/\nu)/\Phi(1/\nu) - 1/\nu + G^{-1}[(1-P_2)/\nu^+]b(\nu)]/G(-1/\nu) \end{cases} \tag{4.2}$$

The short Tables 4.1 and 4.2 shows the deviations 100 $(P_{is} - P_i)$ of the prescribed level P_i, due to neglecting non-normality.

Table 4.1. Deviations 100 $(P_{is}^+ - P_i)$.

		P_1				P_2			
		0.900	0.925	0.950	0.975	0.900	0.925	0.950	0.975
ν	ν^+								
0.5	0.4879	-0.31	-0.29	-0.26	-0.19	-0.22	-0.21	-0.19	-0.14
1.0	0.8000	-1.62	-1.66	-1.57	-1.24	-1.96	-1.85	-1.62	-1.20
1.5	0.9659	-2.23	-2.38	-2.35	-1.96	-3.47	-3.28	-2.91	-2.20
2.0	1.0661	-2.48	-2.73	-2.76	-2.39	-4.53	-4.29	-3.82	-2.93

Table 4.2. Deviations 100 $(P_{is}^* - P_i)$.

		P_1				P_2			
		0.900	0.925	0.950	0.975	0.900	0.925	0.950	0.975
ν	ν^+								
0.5	0.4879	-0.59	-0.60	-0.56	-0.43	-0.39	-0.40	-0.38	-0.31
1.0	0.8000	-1.41	-1.58	-1.61	-1.38	-1.44	-1.49	-1.43	-1.18
1.5	0.9659	-1.57	-1.83	-1.93	-1.73	-1.90	-1.96	-1.90	-1.59
2.0	1.0661	-1.62	-1.92	-2.07	-1.89	-2.12	-2.20	-2.14	-1.80

If ν^+ is used but non-normality neglected, the loss in performance may be as high as 4.5 percentage points (for $P_2 = 0.9$, $\nu = 2$). Note that this means that 15.5% in stead of the desired 10% of total demand can **not** be delivered from stock: this is an 55% increase.

The use of $N^*(\mu, \sigma^2)$ and hence ν^* in standard safety factors gives a similar, but somewhat less dramatic picture. However, for $P_1 = 0.975$ and $\nu = 2$, the percentage of review periods in which not all demand can be met increases with 75% (from 2.5 to 4.39%).

5 Discussion

The two families of distributions featuring in this paper both are nonnegative and skewed to the right. Both were obtained by modifying normal distributions; for both modifications we derived order-up-to-levels and safety factors.

The two families are proposed here as alternative demand models in case the demand does not show too much spread: our models fall short if demand populations have coefficients of variation exceeding the upperbound $\nu^+ \leq 1.463$ or $\nu^* \leq 0.7555$.

The choice between the two alternative models heavily depends on the occurrence of review periods with zero demand: if zero demand is a relatively frequent phenomenon, our distributions $N^+(\mu, \sigma^2)$ may give a useful model; otherwise, the model $N^*(\mu, \sigma^2)$ may be more appropriate.

An even richer family of distributions, showing all essential desirable properties, is the family of gamma distributions. Among many others, we propagated its use in the recent past (compare Moors en Strijbosch, 2002) and will continue to do so. We thought it useful, however, to stipulate the disadvantages of the much-used normal distribution and to present simple ameliorations.

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Appendix A

Here we present some interesting properties of G, H and related functions and mention the consequences for the parameters of $N^+(\mu, \sigma^2)$.

First note that

$$G(0) = 1\sqrt{2\pi}, \quad H(0) = 1/2.$$

so that $\mu^+/\mu \approx \nu/\sqrt{2\pi}$ for large ν. The properties

$$G'(k) = -\Phi(-k)$$

and $G(\infty) = 0$ imply $G(k) > 0$ for all $k \in \mathbb{R}$. Hence it follows

$$\begin{cases} H'(k) = 2G(-k) > 0, \\ [H(k) - G^2(-k)]' = 2\Phi(-k)G(-k) > 0 \end{cases}$$

Consequently, σ^+/σ is decreasing in ν, with limit $\sqrt{0.5 - 1/(2\pi)} = 0.5838$.

Further,

$$[\Phi(k)H(k) - G^2(-k)]' = \varphi(k)H(k) > 0$$

so that $\Phi(k)H(k) - G^2(-k) > 0$ holds. As a result,

$$\left[\frac{H(k) - G^2(-k)}{G^2(-k)}\right]' \propto G^2(-k) - \Phi(k)H(k) < 0$$

implying that ν^+ is increasing in ν, with limit $\sqrt{\pi - 1} = 1.4634$ voor $\nu \to \infty$.

Similar results can be obtained for $N^*(\mu, \sigma^2)$; we only mention here

$$\left.\begin{array}{l} \mu^*/\sigma \to \sqrt{2/\pi} = 0.7979 \\ \sigma^*/\sigma \to \sqrt{1 - 2/\pi} = 0.6028 \\ \nu^* \to \sqrt{\pi/2 - 1} = 0.7555 \end{array}\right\} \text{for } \nu \to \infty$$

Finally, note the relation with the so-called Mills' ratio $\Phi(-k)/\varphi(k)$. The well-known inequality

$$\frac{\Phi(-k)}{\varphi(k)} < \frac{1}{k}$$

(Feller, 1950 or Moors, 1985) implies

$$G(-k) > k$$

which gives

$$G^{-1}[(1 - P_2)/\nu] \geq G^{-1}[(1 - P_2)G(-1/\nu)]$$

and hence $S_2 \geq S_2^*$.