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van Velzen, S.

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By Bas van Velzen

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Bas van Velzen

CentER and Department of Econometrics and Operations Research,
Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.
E-mail: S.vanVelzen@uvt.nl

Abstract

In this paper we study cooperative cost games arising from domination problems on graphs. We introduce three games to model the cost allocation problem and we derive a necessary and sufficient condition for the balancedness of all three games. Furthermore we study concavity of these games.

Keywords: TU game, dominating set, cost allocation

1 Introduction

In this paper we consider cooperative cost games that arise from domination problems on graphs. A domination problem consists of a given graph $G = (V, E)$, a positive integer $k \in \mathbb{N}$, and a nonnegative function $w : V \rightarrow \mathbb{R}_+$ that assigns a fixed cost to each vertex. A $k$-dominating set is a set $D \subseteq V$ such that the distance between any vertex in $V$ and at least one vertex in $D$ is at most $k$. A $k$-domination problem is the problem of finding a so-called minimum weighted $k$-dominating set of $G$, i.e. a $k$-dominating set that minimizes the total cost of its vertices.

Domination problems are widely studied in graph theory. Meir and Moon (1975) investigate domination problems where the underlying graph is a tree. Some results of Meir and Moon (1975) are extended to larger classes of graphs in Farber (1981). In Haynes, Hedetniemi, Slater (1998) an overview of literature on domination problems is given.

An illustration of a domination problem is the following example. Consider a number of regions in which certain facilities are going to be placed. There is a fixed cost for the placement of a facility in a certain region. The problem is to select the regions in which to place facilities at minimum costs, such that each region is served by a facility in it or by a facility in a neighbouring region. The problem of placing the facilities at minimum costs can be regarded as a domination problem. Let $G = (V, E)$ be the graph where regions correspond to vertices and edges represent pairs of regions that are neighbours. The fixed cost can be described by a map $w : V \rightarrow \mathbb{R}_+$. The problem of placing the facilities at minimum cost is equivalent to finding a minimum weighted $1$-dominating set on $G$.

A natural question that now arises is how to allocate the total costs of placing the facilities among the participating regions. In this paper we use cooperative game theory to study this problem. We introduce three cooperative cost games that model the cost allocation problem.

The three dominating set games have in common that the cost of the grand coalition $N$ equals the minimum weighted $k$-domination number. However, the cost of coalitions may take different values in each of the three games. This is due to the fact that coalitions have different possibilities of placing the facilities in each of the three games. In the relaxed dominating set game coalitions are allowed to use every vertex and every edge of the graph. These games belong to the class of combinatorial optimization games introduced in Deng, Ibaraki, Nagamochi (1999).
In some situations it makes sense to assume that coalitions will place facilities in their own region. For instance because regions outside the coalition do not allow the placement of a facility in their region. Therefore we introduce the intermediate dominating set game. In this game coalitions are only allowed to place facilities in their own regions. However, coalitions are still allowed to use every edge of the graph. Consider for example the situation where the facilities are libraries. Then coalitions will place the libraries in their own regions but inhabitants of one region may freely travel through regions outside the coalition in order to reach a library.

Now consider a situation where the facilities are power stations. Then coalitions will place the power stations in their own regions. However, cooperation between non-adjacent members of a coalition can be obstructed by regions outside the coalition if these regions do not allow transportation of electricity through their region. For this reason we define the rigid dominating set game. In this game coalitions are only allowed to use vertices and edges of the induced subgraph corresponding to the coalition.

In spite of the differences between these games, we will obtain a common necessary and sufficient condition for non-emptiness of the cores of all three dominating set games. In particular, if one of the dominating set games possesses a core element, then the other two dominating set games possess core elements as well. We also derive relations between the cores of the dominating set games. Furthermore we present a class of graphs for which the corresponding dominating set games have a non-empty core for all cost functions \( w : V \to \mathbb{R}_+ \) and all \( k \in \mathbb{N} \). Finally we study concavity of the dominating set games.

Other game theoretical approaches to location problems include facility location games (Kolen, Tamir (1990), Tamir (1992)) and minimum spanning forest games (Granot, Granot (1992)). In Kolen, Tamir (1990) and Tamir (1992) a general class of facility location problems is studied. They mainly focus on tree graphs, and for these graphs non-emptiness of the core is established. In Granot, Granot (1992) no restrictions are made on the proximity of the facilities. If the underlying graph is a tree, non-emptiness of the core is shown.

This paper is organized as follows. In Section 2 we recall notions from cooperative game theory, graph theory and matrix theory. In Section 3 we introduce three cooperative cost games that model the cost allocation problem arising from domination problems on graphs. In Section 4 we focus on the core of the dominating set games. Section 5 is dedicated to concavity and in Section 6 some final remarks are made.

2 Preliminaries

In this section we recall some notions from cooperative game theory and introduce some notation. Also some graph and matrix theoretical concepts are discussed.

2.1 Game theory

A cooperative \( TU \) cost game is a pair \((N, c)\) where \( N = \{1, \ldots, n\} \) is a finite player set, and \( c \), the characteristic function, is a map \( c : 2^N \to \mathbb{R} \) with \( c(\emptyset) = 0 \). The map \( c \) assigns to each coalition \( S \subset N \) a real number \( c(S) \) called the cost of \( S \). The core of a game \((N, c)\) is the set

\[
C(c) = \{ x \in \mathbb{R}^N | \ x(S) \leq c(S) \text{ for every } S \subset N \text{ and } x(N) = c(N) \},
\]

where \( x(S) \) denotes \( \sum_{j \in S} x_j \). Intuitively, the core of a game is the set of cost allocation vectors, for which no coalition has an incentive to leave the grand coalition \( N \). Note that the core of a game can be empty. If the core of a game is nonempty, then the game is called balanced. A monotonic game is a game \((N, c)\) satisfying \( c(S) \leq c(T) \) for every \( S \subset T \subset N \). If \((N, c)\) is a balanced monotonic game and \( x \in C(c) \), then it holds that \( x_i = c(N) - \sum_{j \in N \setminus \{i\}} x_j \geq c(N) - c(N \setminus \{i\}) \geq 0 \) for every
that \( c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \).

Equivalently, a game is concave if and only if for all \( i, j \in N \), with \( i \neq j \), and \( S \subset N \setminus \{i, j\} \) it holds that

\[
c(S \cup \{i\}) - c(S) \geq c(S \cup \{i, j\}) - c(S \cup \{j\}).
\]

Hence, for concave games the marginal contribution of a player to any coalition is at most his marginal contribution to a smaller coalition. Shapley (1971) showed that the core of a concave game is nonempty.

Let \((N, c)\) be a cooperative cost game, and let \( \sigma : N \to \{1, \ldots, n\} \) be a bijection. The marginal vector \( m^\sigma(c) \) is defined as \( m^\sigma(c) = c([i, \sigma]) - c((i, \sigma)) \), where \([i, \sigma] = \{ j \in N : \sigma(j) \leq \sigma(i) \}\) is the set of predecessors of \( i \) with respect to \( \sigma \) including \( i \), and \((i, \sigma) = \{ j \in N : \sigma(j) < \sigma(i) \}\) is the set of predecessors of \( i \) with respect to \( \sigma \) excluding \( i \). Shapley (1971) and Ichishii (1981) showed that the set of marginal vectors coincides with the set of extreme points of the core if and only if the game is concave.

### 2.2 Graph theory and matrix theory

A graph \( G \) is a pair \((V, E)\) where \( V = \{1, \ldots, n\} \) is a finite set of vertices, and \( E \) is the set of edges, i.e. a set of unordered pairs of \( V \). If \( (v, w) \in E \) for all distinct \( v, w \in V \), then \( G \) is called complete.

The subgraph induced by \( V' \subset V \) is the graph \( G_{V'} = (V', E_{V'}) \), where \( E_{V'} \) is the set of edges having both endpoints in \( V' \).

Two distinct vertices \( v, w \in V \) are called adjacent if \( (v, w) \in E \). For \( v, w \in V \), a \( v \to w \) path of length \( m \) is a chain \( (v, v_1, \ldots, v_{m-1}, w) \) of pairwise disjoint vertices, where each subsequent pair of vertices is adjacent, i.e. \( (v, v_1) \in E \), \((v_i, v_{i+1}) \in E \) for all \( i \in \{1, \ldots, m-2\} \) and \((v_{m-1}, w) \in E \).

A graph is said to be connected if for any two vertices \( v, w \in V \) it contains a \( v \to w \) path. The maximal connected parts of a graph are called components.

A graph \( G = (V, E) \) is said to be a circuit if it is a connected graph containing \( n \geq 3 \) vertices such that each vertex is adjacent to precisely two other vertices. A circuit containing \( n \) vertices is denoted by \( C_n \).

The distance \( d_G(v, w) \) between \( v, w \) is the length of a shortest \( v \to w \) path. The diameter \( \Delta(G) \) is the maximum distance within the graph. For each \( v \in V \), let \( e_G(v) \) denote the eccentricity of \( v \), i.e. \( e_G(v) = \max\{d_G(v, x) : x \in V\} \). The radius \( r(G) \) is the minimum over all eccentricities, i.e. \( r(G) = \min\{e_G(v) : v \in V\} \).

For each vertex \( v \in V \), the \( k \)-neighbourhood of \( v \), denoted by \( N_k[v] \), consists of the vertices at distance at most \( k \) of \( v \), i.e. \( N_k[v] = \{ w \in V : d_G(v, w) \leq k \} \). A \( k \)-neighbourhood of \( v \) is also called a \( k \)-star at \( v \).

If \( T \subset N_k[v] \) contains \( v \), then \( T \) is called a \( k \)-substar at \( v \). The set of \( k \)-substars at \( j \in V \) is denoted by \( S_j \), i.e. \( S_j = \{ S \subset V : S \text{ is a } k-\text{substar at } j \} \). If \( T \subset N_k[v] \), \( T \neq \emptyset \), is such that \( d_G(v, x) \leq k \) for all \( x \in T \), then \( T \) is called a proper \( k \)-substar at \( v \). Note that if \( T \) is a proper \( k \)-substar at \( v \), then \( G_T \) is necessarily connected and contains \( v \). The set of proper \( k \)-substars at \( j \in V \) is denoted by \( P_j \), i.e. \( P_j = \{ S \subset V : S \text{ is a proper } k-\text{substar at } j \} \).

**Example 2.1** Let \( G \) be the graph depicted in Figure 1. Let \( k = 2 \). Then it holds that \( \{1, 2, 3\} \) is a 2-star at 1 and that \( \{1, 3\} \) is a 2-substar at 1. However, \( \{1, 3\} \) is not a proper 2-substar at 1, since \( G_{\{1,3\}} \) is a disconnected subgraph and hence \( d_G(1,3) = \infty \). Also note that \( \{1, 3\} \) is not a 1-substar at 2 because it does not contain 2 itself.
Algorithm 1: Construction of disjoint proper k-substars at \( j, j \in K \), that cover \( T \).

Step 1: Let \( U^0_l = \{ l \} \) for all \( l \in K \). Set \( S = T \setminus K \) and \( j = 0 \).

Step 2: If \( \bigcup_{l=1}^m U^0_l = T \), i.e. if \( S = \emptyset \), then stop. Else, \( j = j + 1, l = 1 \) and go to Step 3.

Step 3: \( U^j_l = U^{j-1}_l \cup \{ i \in S : d_{G_T}(i, l) = j \} \) and \( S = S \setminus U^j_l \). If \( l = m \), then return to Step 2. Else \( l = l + 1 \), and return to Step 3.

The following lemma shows that Algorithm 1 indeed produces disjoint proper k-substars at \( j, j \in K \).

**Lemma 2.1** The sets \( U_j \), \( j \in K \), produced by Algorithm 1, are disjoint proper k-substars at \( j \) that cover \( T \).

**Proof:** Because of the phrase \( S = S \setminus U^j_l \) in Step 3 of the algorithm it is guaranteed that \( U^j_i \cap U^j_l = \emptyset \) for all \( j \) and all \( i, l \in K \) with \( i \neq l \). Note that, because it holds that \( \bigcup_{l \in K} N^G_T[l] = T \), the algorithm stops after \( p \leq k \) loops. Therefore we only need to show that each \( U^p_l, l \in K \) is a proper k-substar.

Suppose that \( U^p_l \) is not a proper k-substar. Then there is a \( j \in U^p_l \) with \( d_{G_{U^p_l}}(l, j) > k \). From \( p \leq k \) and Step 3 of Algorithm 1 it follows that \( d_{G_T}(l, j) \leq k \). Let \( (j, a_1, \ldots, a_h, l) \) be a shortest \( j - l \) path in \( G_T \). Because it holds that \( d_{G_{U^p_l}}(l, j) > k \geq d_{G_T}(l, j) \) it follows that there is a \( g \in \{1, \ldots, h\} \) with \( a_g \notin U^p_l \). In particular there is an \( i \in K \setminus \{l\} \) with \( a_g \in U^p_i \). Because \( a_g \) is assigned to \( i \) in the algorithm it either holds that \( d_{G_T}(a_g, i) < d_{G_T}(a_g, l) \) or that \( d_{G_T}(a_g, i) = d_{G_T}(a_g, l) \) and \( i < l \). It follows that it either holds that

\[
  d_{G_T}(j, i) \leq d_{G_T}(j, a_g) + d_{G_T}(a_g, i) < d_{G_T}(j, a_g) + d_{G_T}(a_g, l) = d_{G_T}(j, l)
\]

or that

\[
  d_{G_T}(j, i) \leq d_{G_T}(j, a_g) + d_{G_T}(a_g, i) = d_{G_T}(j, a_g) + d_{G_T}(a_g, l) = d_{G_T}(j, l) \text{ and } i < l.
\]
But this implies that \( j \) is assigned to \( i \) as well, i.e. \( j \in U_i^P \). This contradicts \( j \in U_i^D \). We conclude that \( U_i^D \) is a proper \( k \)-substar at \( l \).

Let \( G = (V, E) \) be a graph and let \( k \in \mathbb{N} \). A set \( D \subseteq V \) is called a \( k\)-dominating set if for all \( v \in V \setminus D \) it holds that there is a \( z \in D \) with \( d_G(v, z) \leq k \). The \( k\)-domination number \( \gamma_k(G) \) is the minimum number of vertices in a \( k\)-dominating set. A fractional \( k\)-domination is a vector of nonnegative weights on the vertices such that for each \( k\)-neighbourhood the weights sum up to at least one. The fractional \( k\)-domination number \( \gamma_k^*(G) \) is the minimum sum of the weights in a fractional \( k\)-domination. Let \( w : V \to \mathbb{R}_+ \) be a cost function on the vertices. The weighted \( k\)-domination number \( \gamma_k(G, w) \) is the minimum sum of the costs in a \( k\)-dominating set and the fractional weighted \( k\)-domination number \( \gamma_k^*(G, w) \) is the minimum sum of the costs in a fractional \( k\)-domination.

**Example 2.2** Let \( G \) be the graph depicted in Figure 2, let \( w_1 = (1,1,1,1) \) and let \( k = 1 \). The minimum number of vertices in a 1-dominating set is 2. Hence, \( \gamma_1(G, w_1) = 2 \). It holds that \( y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) is an optimal fractional 1-domination. It follows that \( \gamma_1(G, w_1) = 2 > \frac{4}{3} = \gamma_1^*(G, w_1) \).

Now let \( w_2 = (10,1,10,1) \). It holds that \((0,1,0,1)\) is an optimal 1-domination, as well as an optimal fractional 1-domination. Hence, \( \gamma_1(G, w_2) = 2 = \gamma_1^*(G, w_2) \).

![Figure 2](image-url)  
Figure 2: It holds that \( \gamma_1(G, w_1) = 2, \gamma_1^*(G, w_1) = \frac{4}{3} \) and \( \gamma_1(G, w_2) = \gamma_1^*(G, w_2) = 2 \).

The \( k\)\,-neighbourhood matrix of \( G = (V, E) \) is the \( n \times n \)-matrix \( A_k(G) \), defined by \((A_k(G))_{ij} = 1\) if \( j \in N_k[i] \) and \((A_k(G))_{ij} = 0\) if \( j \notin N_k[i] \). The \( k\)\,-th power of \( G = (V, E) \) is the graph \( G^k = (V, E^k) \), where \((v, w) \in E^k\) if and only if \( d_G(v, w) \leq k \). Note that it holds that \( A_k(G) = A_1(G^k) \). It holds that

\[
\gamma_k(G, w) = \min\{yw : yA_k(G) \geq 1, y \in \{0,1\}^n\} \geq \min\{yw : yA_k(G) \geq 1, y \geq 0\} = \gamma_k^*(G, w).
\]

Let \( A \) be a \{0,1\}-matrix. Then \( A \) is called ideal if each extreme point of the polyhedron \( P = \{x \in \mathbb{R}^n : xA \geq 1, x \geq 0\} \) is integer. It holds that \( P \) has only integer extreme points if and only if \( \min\{yw : yA \geq 1, y \geq 0\} = \min\{yw : yA \geq 1, y \in \{0,1\}^n\} \) for all \( w : V \to \mathbb{R}_+ \) (cf. Lehman (1990)). A \{0,1\}-matrix is called balanced (cf. Berge (1972)) if it does not contain an odd-sized square submatrix with exactly two nonzero entries in each row and each column. If \( A \) is a balanced matrix, then \( A \) is ideal (cf. Fulkerson et al. (1974)). Note that if \( A_k(G) \) is ideal, then it holds that \( \gamma_k(G, w) = \gamma_k^*(G, w) \) for every \( w : V \to \mathbb{R}_+ \).

### 3 The dominating set games

In this section we introduce three cooperative dominating set games that model the cost allocation problem arising from domination problems on graphs.
Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \rightarrow \mathbb{R}_+$. Define the corresponding **rigid dominating set game** $(N, c_k^w)$ by $N = V$, and $c_k^w(S) = \gamma_k(G[S], w_S) = \min\{yw : yA_k(G) \geq e(S), y_i = 0 \text{ if } i \not\in S, y \in \{0, 1\}^N\}$, where $w_S$ is $w$ restricted to $S$. In this game, the cost of a coalition equals the minimum weighted $k$-domination number of the subgraph induced by this coalition. Obviously, in the rigid dominating set game, coalitions cannot place facilities in vertices corresponding to non-members. Furthermore, a coalition can only make use of an edge if both endpoints of this edge are member of the coalition.

The rigid dominating set game is restrictive in the sense that coalitions cannot make use of edges that are not present in the induced subgraph corresponding to its coalitions. In the **intermediate dominating set game** $(N, ce^w_k)$ this requirement is dropped. Formally, $(N, ce^w_k)$ is defined by $N = V$, and $ce^w_k(S) = \min\{yw : yA_k(G) \geq e(S), y_i = 0 \text{ if } i \not\in S, y \in \{0, 1\}^N\}$, where $e(S)$ denotes the vector with each entry corresponding to $S$ equal to one, and zero otherwise.

The **relaxed dominating set game** is obtained by dropping the requirement that coalitions are only allowed to use vertices corresponding to members of the coalition. That is, in the relaxed dominating set game a coalition may place a facility in any vertex present in the graph. Formally, the relaxed dominating set game is defined by $N = V$ and $cv^w_k(S) = \min\{yw : yA_k(G) \geq e(S), y \in \{0, 1\}^N\}$. For $(N, cv^w_k)$ it holds that $cv^w_k(S) \leq cv^w_k(T)$ for all $S \subset T$. Hence, $(N, cv^w_k)$ is a monotonic game. The relaxed dominating set game is included in the class of combinatorial optimization games, introduced in Deng et al. (1999).

**Example 3.1** Let $G$ be the graph depicted in Figure 1, let $w = (3, 1, 2)$ and let $k = 2$. It holds that $c_2^w(\{1, 3\}) = 5$, because coalition $\{1, 3\}$ cannot make use of the edges present in the graph. Because in the intermediate dominating set game coalition $\{1, 3\}$ can make use of the edges $(1, 2)$ and $(2, 3)$, it follows that $ce^w_2(\{1, 3\}) = 2$. Finally, $cv^w_2(\{1, 3\}) = 1$, because coalition $\{1, 3\}$ can make use of the vertex of player 2.

**Remark:** Throughout the paper we assume that graphs are connected. For disconnected graphs the cost allocation problem can be analyzed for each of its components.

Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \rightarrow \mathbb{R}_+$. Let $(N, c_k^w)$, $(N, ce^w_k)$ and $(N, cv^w_k)$ be the corresponding dominating set games. Obviously, coalitions have more possibilities of placing the facilities in the intermediate dominating set game than in the rigid dominating set game. A similar observation holds for $(N, ce^w_k)$ and $(N, cv^w_k)$, i.e. coalitions have more possibilities of placing the facilities in the relaxed dominating set game than in the intermediate dominating set game. Hence, for all $S \subset N$ it holds that $c_k^w(S) \geq ce^w_k(S) \geq cv^w_k(S)$. The grand coalition $N$ has the same possibilities in each of the three games. Therefore we have that $c_k^w(N) = ce^w_k(N) = cv^w_k(N) = \gamma_k(G, w)$.

Because making use of edges with endpoints outside the coalition makes no sense if $k = 1$, it follows that $c_1^w(S) = ce^w_1(S)$ for all $S \subset N$.

### 4 The cores of dominating set games

In this section we study the cores of the three dominating set games. We derive a relation between the cores, and we provide efficient descriptions of these sets. Furthermore we derive one necessary and sufficient condition for the non-emptiness of the cores of the three dominating set games. Hence, if one of the dominating set games is balanced, then the other two games are balanced as well. Finally, we obtain graphs with the property that the induced dominating set games are balanced for all cost functions $w : V \rightarrow \mathbb{R}_+$. 


Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. In the previous section we already concluded that for every $S \subset N$ it holds that $c^w_k(S) \geq c^w_k(S) \geq c^w_k(S)$ and that $c^w_k(N) = c^w_k(N) = c^w_k(N)$. From this we derive that $C(c^w_k) \subset C(c^w_k) \subset C(c^w_k)$. Moreover, the core of $(N, c^w_k)$ coincides with the nonnegative part of the core of $(N, c^w_k)$, as well as with the nonnegative part of the core of $(N, c^w_k)$.

**Theorem 4.1** Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let $(N, c^w_k)$, $(N, c^w_k)$ and $(N, c^w_k)$ be the corresponding dominating set games. It holds that $C(c^w_k) = C(c^w_k) \cap \mathbb{R}_+^n$, and $C(c^w_k) = C(c^w_k) \cap \mathbb{R}_+^n$. 

**Proof:** We only show that $C(c^w_k) = C(c^w_k) \cap \mathbb{R}_+^n$. The proof of $C(c^w_k) = C(c^w_k) \cap \mathbb{R}_+^n$ runs similar.

First we show that $C(c^w_k) \subset C(c^w_k) \cap \mathbb{R}_+^n$. As noted before, it holds that $C(c^w_k) \subset C(c^w_k)$. Because $(N, c^w_k)$ is a monotonic game, it holds that $x \geq 0$ for all $x \in C(c^w_k)$. Hence, $C(c^w_k) \subset C(c^w_k) \cap \mathbb{R}_+^n$.

Now we show that $C(c^w_k) \cap \mathbb{R}_+^n \subset C(c^w_k)$. Let $x \in C(c^w_k) \cap \mathbb{R}_+^n$. Let $T \subset N$ be such that $c^w_k(T) > c^w_k(T)$. Obviously, coalition $T$ can save costs by using vertices of $N\{T\}$. Let $K \subset N$ be such that $T \subset \bigcup_{j \in K} N_k[j]$ and $\sum_{j \in K} w_j = c^w_k(T)$. Let $T = \bigcup_{j \in K} N_k[j]$. It follows that $c^w_k(T) = c^w_k(T)$. Therefore, $x(T) \leq x(T) \leq c^w_k(T) = c^w_k(T)$, where the first inequality holds because $x \geq 0$ and the second inequality holds because $x \in C(c^w_k)$. We conclude that it holds that $x(T) \leq c^w_k(T)$ for every $T \subset N$, and $x(N) = c^w_k(N) = c^w_k(N)$. This implies that $x \in C(c^w_k)$. □

Now we provide descriptions of the cores of the three dominating set games. Showing that a cost allocation vector is a core element, generally boils down to showing that the cost allocated to a coalition is at most its cost in the corresponding cost game. However, for dominating set games more efficient core descriptions exist. The following proposition provides a description of the core of $(N, c^w_k)$ in terms of coalitions corresponding to $k$-stars. For the proof of this proposition we refer to Lemma 2 in Deng et al. (1999).

**Proposition 4.1** Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let $(N, c^w_k)$ be the corresponding relaxed dominating set game. It holds that $x \in C(C^w_k)$ if and only if $x \geq 0$, $x(N_k[j]) \leq w_j$ for each $j \in V$ and $x(N) = c^w_k(N)$.

Proposition 4.1 provided a description of the core of relaxed dominating set games in terms of $k$-stars. Similarly, a core description can be obtained for intermediate dominating set games. However, this description deals with $k$-substars instead of $k$-stars. In other words, an efficient cost allocation vector is a core element of an intermediate dominating set game if and only if no coalition corresponding to a $k$-substar has an incentive to leave the grand coalition.

**Proposition 4.2** Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let $(N, c^w_k)$ be the corresponding intermediate dominating set game. It holds that $x \in C(c^w_k)$ if and only if for all $j \in V$ and all $S \subset S_j$ it holds that $x(S) \leq w_j$, and $x(N) = c^w_k(N)$. 

**Proof:** First we show the "only if" part. Let $x \in C(c^w_k)$, $j \in V$ and $S \subset S_j$. By definition of $k$-substars, it holds that $j \in S$. Hence, $S$ is a coalition with cost at most $w_j$. It follows that $x(S) \leq c^w_k(S) \leq w_j$. Trivially it holds that $x(N) = c^w_k(N)$.

Now we show the "if" part. Let $x \in \mathbb{R}_n$ be such that for all $j \in V$ and all $S \subset S_j$ it holds that $x(S) \leq w_j$, and $x(N) = c^w_k(N)$. Let $T \subset N$. We need to show that $x(T) \leq c^w_k(T)$.

Let $K \subset T$ be an optimal weighted $k$-dominating set of $T$, i.e. $T \subset \bigcup_{j \in K} N_k[j]$ and $\sum_{j \in K} w_j = \cdots$
ce_k^w(T). There exist disjoint k-substars S_j ∈ S_j, for j ∈ K, such that \( \cup_{j \in K} S_j = T \). It follows that

\[ x(T) = \sum_{j \in K} x(S_j) \leq \sum_{j \in K} w_j = ce_k^w(T). \]

Therefore we have that \( x \in C(ce_k^w) \).

Finally we consider the core of rigid dominating set games. In Proposition 4.3 we provide a description of the core of this set similar to the descriptions of the cores of the relaxed and the intermediate dominating set games. It states that an efficient cost allocation vector is in the core if and only if no coalition corresponding to a proper -substar has an incentive to leave the grand coalition.

**Proposition 4.3** Let \( G = (V,E) \) be a graph, \( k \in \mathbb{N} \) and \( w : V \to \mathbb{R}_+ \). Let \((N,ce_k^w)\) be the corresponding rigid dominating set game. It holds that \( x \in C(ce_k^w) \) if and only if for all \( j \in V \) and all \( S \in \mathcal{P}_j \) it holds that \( x(S) \leq w_j \), and \( x(N) = ce_k^w(N) \).

**Proof:** First we show the ”only if” part. Let \( x \in C(ce_k^w) \), \( j \in V \) and \( S \in \mathcal{P}_j \). It holds that \( S \) is a coalition with cost at most \( w_j \). It follows that \( x(S) \leq ce_k^w(S) \leq w_j \). Trivially it holds that \( x(N) = ce_k^w(N) \).

Now we show the ”if” part. Let \( x \in \mathbb{R}^n \) be such that for every \( j \in V \) and \( S \in \mathcal{P}_j \) it holds that \( x(S) \leq w_j \), and \( x(N) = ce_k^w(N) \). Let \( T \subset N \). We need to show that \( x(T) \leq ce_k^w(T) \).

Let \( G_T \) be the subgraph induced by \( T \) and let \( K \subset T \) be an optimal k-dominating set of \( G_T \). Hence, \( ce_k^w(T) = \sum_{j \in K} w_j \) and \( \cup_{j \in K} N_k^{G_T}[j] = T \). According to Lemma 2.1 there exist disjoint proper k-substars \( U_j \) at \( j, j \in K \), that cover \( T \). This implies that

\[ x(T) = \sum_{j \in K} x(U_j) \leq \sum_{j \in K} w_j = ce_k^w(T), \]

where the inequality follows from our assumption.

In the remainder of this section we focus on the balancedness of the dominating set games. The main theorem of this section provides a necessary and sufficient condition for non-emptiness of the core of the three games. It states that the games are balanced if and only if the fractional weighted k-domination number equals the weighted k-domination number.

**Theorem 4.2** Let \( G = (V,E) \) be a graph, \( k \in \mathbb{N} \) and \( w : V \to \mathbb{R}_+ \). Let \((N,cv_k^w), (N,ce_k^w)\) and \((N,c_k^w)\) be the corresponding dominating set games. The following statements are equivalent:

1. \( \gamma_k(G,w) = \gamma_k^w(G,w) \)
2. \((N,cv_k^w)\) is balanced
3. \((N,ce_k^w)\) is balanced
4. \((N,c_k^w)\) is balanced.

**Proof:** The equivalence of 1 and 2 follows from Theorem 2 in Deng et al. (1999). ”2 \Rightarrow 3” and ”3 \Rightarrow 4” follow from the observation that \( C(cv_k^w) \subset C(ce_k^w) \subset C(c_k^w) \). So we only need to show that ”4 \Rightarrow 2”. We will do this by showing that if \( C(c_k^w) \neq \emptyset \), then there exists an \( x \in C(c_k^w) \) such that \( x \geq 0 \). This implies, according to Theorem 4.1, that \( x \in C(cv_k^w) \).

Suppose that \( C(c_k^w) \neq \emptyset \). Consider the following algorithm. We will show that this algorithm produces a nonnegative core element.
**Algorithm 2**: Construction of a nonnegative core element of $C(c_k^\infty)$.

Step 1: Let $x \in C(c_k^\infty)$, $p = 1$ and $x^p = x$.

Step 2: If $x^p \geq 0$, then stop. Else go to step 3.

Step 3: Let $(i^p, j^p) \in \text{argmin}\{d_G(i, j) : x^p_i < 0, x^p_j > 0\}$. Let $e^p = \min\{x^p_{jp}, -x^p_{jp}\} > 0$. Let $x^p_{jp} = x^p_{jp} + e^p$, $x^p_{jp} = x^p_{jp} - e^p$ and $x^p_{jq} = x^p_{jq}$ for all $j \in N \setminus \{i^p, j^p\}$. Let $p = p + 1$, and return to step 2.

For showing that Algorithm 2 produces a nonnegative core element, we first show that $x^p \in C(c_k^\infty)$ for all $p$. Note that $x^1 \in C(c_k^\infty)$. Suppose that $x^p \in C(c_k^\infty)$, and that $x^p \not\geq 0$. We will show that $x^{p+1} \in C(c_k^\infty)$. Let $(i^p, j^p) \in \text{argmin}\{d_G(i, j) : x^p_i < 0, x^p_j > 0\}$. Let $(i^p, a_1, \ldots, a_m, j^p)$ be a shortest $i^p - j^p$ path. By definition of $i^p$ and $j^p$, it holds that $x_{a_1} = \ldots = x_{a_m} = 0$. Let $P = \{a_1, \ldots, a_m, j^p\}$ be the set of players on the shortest $i^p - j^p$ path excluding $i^p$.

According to Proposition 4.3, showing that $x^{p+1} \in C(c_k^\infty)$ boils down to showing that for all $l \in N$ and all $U \in \mathcal{P}_l$, it holds that $x^{p+1}_U \leq c_k^\infty(U)$. Because it holds that $x^p \in C(c_k^\infty)$, we only need to consider proper $k$-substars for which the allocated cost at $x^{p+1}$ is (strictly) larger than the allocated cost at $x^p$. In particular, we only need to consider coalitions containing $i^p$ and not containing $j^p$. So let $l \in N$ and let $U \in \mathcal{P}_l$ be such that $i^p \in U$ and $j^p \not\in U$. We distinguish between two cases.

**Case 1**: $U \cup P \in \mathcal{P}_l$.

From $x^p \in C(c_k^\infty)$ and the assumption that $U \cup P$ is a proper $k$-substar at $l$ we obtain that $\sum_{q \in U \cup P} x^p_q \leq w_l$. Because $x^p_q = 0$ for all $q \in P \setminus \{j^p\}$ it holds that

$$\sum_{q \in U} x^p_q \leq w_l - x^p_{jp}.$$  \hspace{1cm} (1)

Now it follows that

$$\sum_{q \in U} x^{p+1}_q = \sum_{q \in U \setminus \{i^p\}} x^{p+1}_q + x^p_{jp} = \sum_{q \in U \setminus \{i^p\}} x^p_q + x^{p+1}_{jp} = \sum_{q \in U} x^p_q - x^p_{jp} + x^{p+1}_{jp} \leq w_l - x^p_{jp} + x^{p+1}_{jp}$$

where the first inequality follows from (1) and the last inequality from $e^p \leq x^p_{jp}$.

**Case 2**: $U \cup P \notin \mathcal{P}_l$.

First suppose that $l = i^p$. Because of the assumption that $U \cup P$ is not a proper $k$-substar at $i^p$ it follows that $d_G(i^p, j^p) > k$. For all $j \in U$ it holds that $d_G(j^p, j) \leq k$. Hence, by definition of $i^p$ and $j^p$, it follows that $x^p_j \leq 0$ for all $j \in U \setminus \{i^p\}$. Since $x^{p+1}_j \leq 0$ it follows that $\sum_{j \in U} x^{p+1}_j \leq 0 \leq w_{ip}$.

So suppose that $l \neq i^p$. Let $W = \{j \in U : d_G(j, l) = d_G(j, i^p) + d_G(i^p, l)\}$ be the set of vertices in $U$ for which a shortest path to $l$ uses $i^p$. By Lemma A.1 of the Appendix it follows that $U \setminus W$ is a proper $k$-substar at $l$. This implies, using Lemma A.2 of the Appendix, that $d_G(q, i^p) < d_G(j^p, i^p)$ for all $q \in W$. Using the definition of $i^p$ and $j^p$ it follows that $x^{p+1}_q \leq 0$ for all $q \in W$. Therefore we have that
If a graph is sun-free chordal, then it is necessarily odd-sun-free chordal. Trees, line graphs of trees, interval graphs and block graphs are examples of sun-free chordal graphs (cf. Farber (1981)).

\[
\sum_{q \in U} x_q^{p+1} = \sum_{q \in W} x_q^{p+1} + \sum_{q \in W \not\subseteq \{p\}} x_q^{p+1} = \sum_{q \in U \setminus W} x_q^p + \sum_{q \in W, q \not\subseteq \{p\}} x_q^p + x_{ip}^{p+1} \\
\leq w_1 + \sum_{q \in W, q \not\subseteq \{p\}} x_q^p + x_{ip}^{p+1} \leq w_1,
\]

where the first inequality holds because \(U \setminus W\) is a proper \(k\)-substar at \(l\). The last inequality follows from \(x_q^p \leq 0\) for all \(q \in W\) and \(x_{ip}^{p+1} \leq 0\).

We conclude that \(x_{p+1} \in C(c_k^p)\). Hence, \(x_p \in C(c_k^p)\) for all \(p\). Now we show that the algorithm converges to a nonnegative core element. Let \(x_p\) and \(x_{p+1}\) be two core elements produced by the algorithm. Note that either \(x_{jp}^{p+1} = 0\) and \(x_{ip}^{p+1} \leq 0\), or \(x_{jp}^{p+1} \geq 0\) and \(x_{ip}^{p+1} = 0\). Hence, \(x_{p+1}\) contains at least one zero entry more than \(x_p\). Because \(x_p\) is a finite dimensional vector, the algorithm produces a nonnegative core element in a finite number of steps.

**Example 4.1** Let \(G, k, w_1\) and \(w_2\) be as in Example 2.2. Because it holds that \(\gamma_k(G, w_1) \neq \gamma_k^*(G, w_1)\) it follows, according to Theorem 4.2, that the corresponding dominating set games \((N, c_k^{w_1}), (N, c_k^{w_2})\) and \((N, c_k^{w_3})\) are not balanced. Because it holds that \(\gamma_k(G, w_2) = \gamma_k^*(G, w_2)\) we conclude that the cores of \((N, c_k^{w_2}), (N, c_k^{w_3})\) and \((N, c_k^{w_3})\) contain core elements.

The cores of the dominating set games are non-empty if and only if it holds that \(\gamma_k(G, w) = \gamma_k^*(G, w)\). Unfortunately, the problem of determining \(\gamma_k(G, w) = \gamma_k^*(G, w)\) is NP-complete on general graphs. Hence, it is difficult to determine whether \(\gamma_k(G, w) = \gamma_k^*(G, w)\). For some classes of graphs however, the \(k\)-domination problem is relatively easy to solve. For example, a special subclass of chordal graphs satisfies this property.

A graph is called **chordal** if it does not contain a circuit of length at least four as an induced subgraph. A **sun** is a chordal graph on \(2n\) vertices for some \(n \geq 3\), whose vertex set can be partitioned into two sets, \(W = \{w_1, \ldots, w_n\}\) and \(U = \{u_1, \ldots, u_n\}\) such that any two vertices of \(W\) are nonadjacent, and for each \(i, j \in \{1, \ldots, n\}\) such that any two vertices of \(W\) are nonadjacent, and for each \(i, j \in \{1, \ldots, n\}\), \(w_i\) is adjacent to \(u_j\) if and only if \(i = j\) or \(i = j + 1 (mod n)\). A graph is called an **odd (even) sun** if it is a sun on \(2n\) vertices, with \(n\) odd (even). An **odd (even)-sun-free chordal graph** is a chordal graph which does not contain a \((odd-)sun as an induced subgraph. Sun-free chordal graphs are called **strongly chordal** graphs in Farber (1981). The concept of an even sun is illustrated in the following example.

**Example 4.2** Let \(G\) be the graph depicted in Figure 3. Observe that \(G\) is chordal. Moreover, the sets \(W = \{w_1, \ldots, w_k\}\) and \(U = \{u_1, \ldots, u_6\}\) form a partition of the vertex set. Any two vertices of \(W\) are nonadjacent, and, \(w_i\) is connected to \(u_j\) if and only if \(i = j\) or \(i = j + 1 (mod 6)\). Hence, \(G\) is a sun. Because \(|U| = |W| = 6|\) we conclude that \(G\) is an even sun.

If a graph is sun-free chordal, then it is necessarily odd-sun-free chordal. Trees, line graphs of trees, interval graphs and block graphs are examples of sun-free chordal graphs (cf. Farber (1981)). Furthermore, Farber (1981) constructed a polynomial time test to determine if a graph is sun-free chordal. Theorem 4.3 states that odd-sun-free chordal graphs are characterized by the balancedness of their 1-neighbourhoodmatrices.

**Theorem 4.3** (Brouwer et al. (1984)) Let \(G = (V, E)\) be a graph. Then \(A_1(G)\) is balanced if and only if \(G\) is an odd-sun-free chordal graph.

Lubiw (1982) showed that powers of sun-free chordal graphs are sun-free chordal. Hence, for a sun-free chordal graph \(G\) it holds that \(A_k(G) = A_1(G^k)\) is balanced for all \(k \in \mathbb{N}\). Therefore it holds that \(\gamma_k(G, w) = \gamma_k^*(G, w)\) for all \(w : V \rightarrow \mathbb{R}_+\) and all \(k \in \mathbb{N}\). This leads to the following proposition.
Proposition 4.4 Dominating set games arising from sun-free chordal graphs are balanced for all \( w : V \rightarrow \mathbb{R}_+ \) and all \( k \in \mathbb{N} \).

It does not hold that powers of odd-sun-free chordal graphs are necessarily odd-sun-free chordal. For example, let \( G \) be the 6-sun depicted in Figure 3. Let \( H \) be the subgraph of \( G^2 \) induced by \{\( w_1, \ldots, w_6 \)\}. This subgraph is a circuit on 6 vertices, and hence not chordal. It follows that \( G^2 \) is not odd-sun-free chordal.

Trivially circuits with at least four vertices are not chordal graphs. Hence, these graphs are not sun-free chordal. However, Cornuéjols and Novick (1994) showed that \( A_1(C_6) \) and \( A_1(C_9) \) are ideal matrices. Moreover, they showed that \( A_k(C_n) \) are the only ideal matrices of the form \( A_k(C_n) \) with \( k, n \in \mathbb{N} \) such that \( k \leq \frac{n-2}{2} \). Note that if \( k > \frac{n-2}{2} \), then \( A_k(C_n) \) is the matrix with every entry a one.

Theorem 4.4 (Cornuéjols, Novick (1994)) It holds that \( A_1(C_6) \) and \( A_1(C_9) \) are ideal.

From Theorem 4.4 it follows that \( \gamma_1(C_6, w) = \gamma_1^*(C_6, w) \) and \( \gamma_1(C_9, w) = \gamma_1^*(C_9, w) \) for every \( w : V \rightarrow \mathbb{R}_+ \). Hence, we have the following proposition.

Proposition 4.5 Dominating set games arising from \( C_6 \) or \( C_9 \) are balanced for all \( w : V \rightarrow \mathbb{R}_+ \) and \( k = 1 \).

5 Concavity of dominating set games

In this section we consider concavity of dominating set games. For general cost functions on the vertices, dominating set games will not satisfy concavity. For the cost function \( w_i = 1 \) for all \( i \in V \) however, there exist interesting characterizations of concavity of dominating set games. Contrary to balancedness of dominating set games, the characterizations of concavity do not coincide for the three dominating set games.

First we derive a characterization of concavity of relaxed dominating set games. Let \( G = (V, E) \) be a graph, \( k \in \mathbb{N} \) and let \( w_i = 1 \) for all \( i \in V \). The corresponding relaxed dominating set game is concave if and only if there exists a vertex with distance at most \( k \) to the other vertices.

Proposition 5.1 Let \( G = (V, E) \) be a graph, \( k \in \mathbb{N} \) and \( w_i = 1 \) for all \( i \in V \). Let \( (N, cv_k^1) \) be the corresponding relaxed dominating set game. It holds that \( (N, cv_k^1) \) is concave if and only if \( k \geq r(G) \).
Proof: It holds that \( k \geq r(G) \) if and only if there is an \( x \in V \) with \( N_k[x] = V \). We will show that \((N, cv_k^1)\) is concave if and only if there is an \( x \in V \) with \( N_k[x] = V \).

First we show the ”if” part. Suppose there is an \( x \in V \) such that \( N_k[x] = V \). Then it holds that \( cv_k^1(S) = 1 \) for all \( S \subset N, S \neq \emptyset \). Hence, \((N, cv_k^1)\) is concave.

Now we prove the ”only if” part. Suppose that for all \( x \in V \) it holds that \( N_k[x] \neq V \). We will show that \((N, cv_k^1)\) is not concave.

Let \( v \in V \) be such that \( N_k[v] \) is a maximal \( k \)-neighbourhood in the sense that it is not a proper subset of any other \( k \)-neighbourhood. Note that each graph contains such a vertex, so in particular does \( G \) contain such a vertex. Because of our assumption it holds that \( N_k[v] \neq V \). Now let \( y \in V \) be such that \( d_G(v, y) = k + 1 \). Hence, \( y \notin N_k[v] \). It follows that \( N_k[v] \cap N_k[y] \neq \emptyset \).

This implies that \( cv_k^1(N_k[v]) = 1 \), \( cv_k^1(N_k[y]) = 1 \) and \( cv_k^1(N_k[v] \cap N_k[y]) = 1 \). Because it holds that \( N_k[v] \) is a maximal \( k \)-neighbourhood and \( y \notin N_k[v] \) it follows that \( cv_k^1(N_k[v] \cup N_k[y]) = 2 \). Hence, \( cv_k^1(N_k[v] \cup N_k[y]) + cv_k^1(N_k[v] \cup N_k[y]) = 3 > 2 = cv_k^1(N_k[v]) + cv_k^1(N_k[y]) \). We conclude that \((N, cv_k^1)\) is not concave.

\( \Box \)

Example 5.1 Let \( G \) be the graph depicted in Figure 4. It holds that \( r(G) = 2 \). From Proposition 5.1 it follows that the corresponding relaxed dominating set game \((N, cv_k^1)\) is concave if and only if \( k \geq 2 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure4.png}
\caption{A graph for which \( r(G) = 2 \) and \( \Delta(G) = 3 \).}
\end{figure}

Now we consider concavity of rigid dominating set games. We will characterize concavity for these games in case \( w_i = 1 \) for all \( i \in V \). For this we need to introduce the concept of block graphs.

A vertex is called a cutvertex if the subgraph \((V \setminus \{v\}, E_{V \setminus \{v\}})\) consists of more components than \( G \). A bridge is an edge \( e \in E \) with the same property, i.e. if \((V, E \setminus \{e\})\) has more components than \( G \). A graph is called 2-connected if it has at least three vertices and contains no cutvertex. A subgraph \( B \) is called a block if it is a bridge or a maximal 2-connected subgraph. A graph is a block graph if every block is complete. Note for example that a tree is a block graph. The concept of block graphs is illustrated in the following example.

Example 5.2 Let \( G \) be the graph depicted in Figure 5. The vertices 5, 6 and 9 are cutvertices, and the edge \((5, 6)\) is a bridge. The blocks are \( \{1, 2, 3, 4, 5\}, \{5, 6\}, \{6, 7, 8, 9\} \) and \( \{9, 10, 11\} \). Because each of these blocks forms a complete subgraph, it follows that \( G \) is a block graph.

The following lemma, which is needed for our characterization of concave rigid dominating set games, provides a relation between the radius and the diameter of a blockgraph.

Lemma 5.1 Let \( k \in \mathbb{N} \) and let \( G = (V, E) \) be a block graph satisfying \( \Delta(G) \leq 2k \). Then it holds that \( r(G) \leq k \).

Proof: Block graphs are 3-sun-free chordal graphs. For 3-sun free chordal graphs it holds that \( r(G) = \lceil \frac{\Delta(G)}{3} \rceil \) (cf. Theorem 3.6 in Chang and Nemhauser(1984)). Hence, if \( G \) is a blockgraph satisfying \( \Delta(G) \leq 2k \), then it holds that \( r(G) \leq k \). \( \Box \)
Proof: We conclude that \((N, c_k^1)\) is concave if and only if \(G\) is a block graph satisfying \(\Delta(G) \leq 2k\).

**Proposition 5.2** Let \(G = (V, E)\) be a graph, \(k \in \mathbb{N}\) and \(w_i = 1\) for all \(i \in V\). Let \((N, c_k^1)\) be the corresponding rigid dominating set game. It holds that \((N, c_k^1)\) is concave if and only if \(G\) is a block graph satisfying \(\Delta(G) \leq 2k\).

**Proof:** First we show the ”only if” part. Suppose that \(G\) does not satisfy \(\Delta(G) \leq 2k\). Let \(v, w \in V\) be such that \(d_G(v, w) = 2k + 1\). Let \((v, a_1, \ldots, a_{2k}, w)\) be a shortest \(v - w\) path in \(G\). It holds that 
\[
c_k^1(\{a_1, \ldots, a_{2k}\}) = c_k^1(\{v, a_1, \ldots, a_{2k}\}) = c_k^1(\{a_1, \ldots, a_{2k}, w\}) = 1\]
and 
\[
c_k^1(\{v, a_1, \ldots, a_{2k}, w\}) = 2.
\]
Hence, 
\[
c_k^1(\{a_1, \ldots, a_{2k}\}) + c_k^1(\{v, a_1, \ldots, a_{2k}, w\}) = 3 > 2 = c_k^1(\{v, a_1, \ldots, a_{2k}\}) + c_k^1(\{a_1, \ldots, a_{2k}, w\}).
\]
We conclude that \((N, c_k^1)\) is not concave.

Now suppose that \(G\) is not a block graph. Then there is a block which is not complete. Hence, \(G\) contains \(C_m\) with \(m \geq 4\) or the graph depicted in Figure 6 as an induced subgraph. We distinguish between three cases.

**Case 1:** \(G\) contains \(C_m\) as an induced subgraph with \(4 \leq m \leq 2k + 2\).

Let \(x\) and \(y\) be vertices of \(C_m\) with \((x, y) \notin E\). In \(C_m\) there are two disjoint paths \((x, a_1, \ldots, a_i, y)\) and \((x, b_1, \ldots, b_j, y)\). Because \((x, y) \notin E\), it holds that \(i \geq 1\) and \(j \geq 1\). Therefore, 
\[
c_k^1(\{x, y\}) = 2, \quad c_k^1(\{x, a_1, \ldots, a_i, y\}) = 1, \quad c_k^1(\{x, b_1, \ldots, b_j, y\}) \geq 1
\]
Thus, 
\[
c_k^1(\{x, y\}) + c_k^1(\{x, a_1, \ldots, a_i, y, b_1, \ldots, b_j\}) \geq 3 > 2 = c_k^1(\{x, a_1, \ldots, a_i, y\}) + c_k^1(\{x, b_1, \ldots, b_j, y\}).
\]
Therefore, \((N, c_k^1)\) is not concave.

**Case 2:** \(G\) contains \(C_m\) as an induced subgraph with \(m > 2k + 2\).

Let \(x, y, w\) be vertices of \(C_m\) such that \((x, w) \in E\), and \((w, y) \in E\). Let \(H = C_m \setminus \{w\}\) be the induced subgraph of \(C_m\) obtained by deleting \(w\). Then \(G\) contains an induced subgraph \(H\) and two

![Figure 5: A block graph.](image)

![Figure 6: A graph for which \((N, c_k^1)\) is not concave.](image)
vertices $x, y \in H$ such that $d_H(x, y) \geq 2k + 1$. It follows that $(N, c_k^1)$ is not concave.

**Case 3:** $G$ contains the graph depicted in Figure 6 as an induced subgraph.

Then it holds that $c_k^1(\{2, 4\}) = 2$, $c_k^1(\{1, 2, 4\}) = 1$, $c_k^1(\{2, 3, 4\}) = 1$. Hence, $c_k^1(\{2, 4\}) + c_k^1(N) = 3 > 2 = c_k^1(\{1, 2, 4\}) + c_k^1(\{2, 3, 4\})$. Therefore, $(N, c_k^1)$ is not concave.

Now we show the "if" part. Let $G = (V, E)$ be a block graph and let $k \in \mathbb{N}$. Suppose that $G$ satisfies $\Delta(G) \leq 2k$. Let $(N, c_k^1)$ be the corresponding rigid dominating set game. If $T \subset N$ is such that $G_T$ is a connected subgraph of $G$, then it holds that $G_T$ is again a block graph satisfying $\Delta(G_T) \leq 2k$. Hence it follows from Lemma 5.1, that there is an $x \in T$ with $d_{G_T}(x, y) \leq k$ for every $y \in T$. As a result, $c_k^1(T) = 1$ for every connected $T \subset N$.

Now let $i, j \in N$, with $i \neq j$, and $S \subset N \setminus \{i, j\}$. Let $S_1, \ldots, S_p$ be the components of $S$ in $G$. Obviously, $c_k^1(S) = p$. Let $I \subset \{1, \ldots, p\}$ be the index set of the components that are connected via vertex $i$. That is, for all $l \in I$ there is a $m \in S_l$ with $(m, i) \in E$. Similarly, let $J \subset \{1, \ldots, p\}$ be the index set of the components that are connected via vertex $j$. First we will show that $|I \cap J| \leq 1$. Suppose that $|I \cap J| \geq 2$. Then at least two components, say $S_1$ and $S_2$, are connected both via vertices $i$ and $j$. Let $m_1, m_2 \in S_1$ such that $(m_1, i) \in E$ and $(m_2, j) \in E$, and let $m_3, m_4 \in S_2$ such that $(m_3, i) \in E$ and $(m_4, j) \in E$. Let $P_1 \subset S_1$ be the set of vertices corresponding to a shortest $m_1 - m_2$ path in $G_{S_1}$, and let $P_2 \subset S_2$ be the set of vertices corresponding to a shortest $m_3 - m_4$ path in $G_{S_2}$. It holds that the subgraph induced by $P_1 \cup P_2 \cup \{i, j\}$ forms a 2-connected subgraph of $G$. Because $G$ is a block graph, it follows that this subgraph is complete. This implies that $(m_3, m_4) \in E$, contradicting that $S_1$ and $S_2$ are disconnected. We conclude that $|I \cap J| \leq 1$.

Note that it holds that $c_k^1(S \cup \{i\}) = p - |I| + 1$ and $c_k^1(S \cup \{j\}) = p - |J| + 1$. If it holds that $|I \cap J| = 0$ and $(i, j) \not\in E$ or that $|I \cap J| = 1$ and $(i, j) \in E$, then it holds that $c_k^1(S \cup \{i, j\}) = p - |I| - |J| + 2$. If it holds that $|I \cap J| = 0$ and $(i, j) \in E$ or that $|I \cap J| = 1$ and $(i, j) \not\in E$, then it follows that $c_k^1(S \cup \{i, j\}) = p - |I| - |J| + 1$. We conclude that $c_k^1(S \cup \{i, j\}) \leq p - |I| - |J| + 2$. Therefore,

$$c_k^1(S \cup \{i\}) + c_k^1(S \cup \{j\}) = 2p - |I| - |J| + 2 \geq c_k^1(S) + c_k^1(S \cup \{i, j\}).$$

The previous proposition is illustrated in Example 5.3.

**Example 5.3** Let $G$ be the graph depicted in Figure 7. Clearly, $G$ is a blockgraph and $\Delta(G) = 3$. From Proposition 5.2 it follows that the corresponding rigid dominating set game $(N, c_k^1)$ is concave if and only if $k \geq 2$.

![Figure 7: A graph for which $(N, c_k^1)$ is concave for $k \geq 2$.](image)

The final part of this section is dedicated to concavity of intermediate dominating set games. We provide a characterization of concavity if $w_i = 1$ for all $i \in V$. Let $G = (V, E)$ be a graph, and let $w_i = 1$ for all $i \in V$. First note that for $k = 1$ the corresponding intermediate dominating set
game coincides with the rigid dominating set game. From Theorem 4.1 it follows that \((N, ce^1_k)\) is concave if and only if \(G\) is a block graph satisfying \(\Delta(G) \leq 2\). However, for \(k \geq 2\), the intermediate dominating set game does not necessarily coincide with the rigid dominating set game. Therefore the characterizations of concavity of these two games are not the same. In fact, intermediate dominating set games are concave if and only if the diameter of \(G\) is at most \(k\).

**Proposition 5.3** Let \(G = (V, E)\) be a graph, \(k \geq 2\), and \(w_i = 1\) for all \(i \in V\). Let \((N, ce^1_k)\) be the corresponding intermediate dominating set game. It holds that \((N, ce^1_k)\) is concave if and only if \(\Delta(G) \leq k\).

**Proof:** We note that \(\Delta(G) \leq k\) if and only if \(N_k[v] = V\) for all \(v \in V\). First we show the "if" part. Suppose that \(G\) is such that \(N_k[v] = V\) for all \(v \in V\). Then it holds that \(ce^1_k(S) = 1\) for all \(S \subset N\). Trivially, \((N, ce^1_k)\) is concave.

Now we show the "only if" part. Suppose that there is an \(x \in V\) such that \(N_k[x] \neq V\). Then there exists a \(y \in V\) such that \(d_G(x, y) = k + 1\). Let \((x, a_1, \ldots, a_k, y)\) be a shortest \(x - y\) path. Note that, because \(k \geq 2\), it holds that \(a_1 \neq a_k\). Therefore \(ce^1_k([x, y]) = 2\), \(ce^1_k([x, a_1, y]) = 1\), \(ce^1_k([x, a_k, y]) = 1\) and \(ce^1_k([x, a_1, a_k, y]) = 1\). It follows that \(ce^1_k([x, a_1, a_k, y]) + ce^1_k([x, y]) = 3 > 2 = ce^1_k([x, a_1, y]) + ce^1_k([x, a_k, y])\).

**Example 5.4** Let \(G\) be the graph depicted in Figure 4. It holds that \(\Delta(G) = 3\). From Proposition 5.3 it follows that the corresponding intermediate dominating set game \((N, ce^1_k)\) is concave if and only if \(k \geq 3\).

### 6 Concluding remarks

In this section we make some final remarks on the structure of the core of dominating set games. We show that a special subclass of relaxed dominating set games satisfies the CoMa-property and we make some suggestions for further research.

A balanced game is said to satisfy the CoMa-property if it holds that the extreme points of the core are marginal vectors. Obviously, all concave games satisfy the CoMa-property. The contrary does not necessarily hold (cf. Kuipers (1993) and Hamers et al. (2002)).

**Proposition 6.1** Let \(G = (V, E)\) be a graph, \(k \in \mathbb{N}\) and \(w_i = 1\) for all \(i \in V\). Let \((N, cv^1_k)\) be the corresponding relaxed dominating set game. If \(G^k\) is odd-sun-free chordal, then it holds that \((N, cv^1_k)\) satisfies the CoMa-property.

**Proof:** We need to show that each extreme point of the core of \((N, cv^1_k)\) is a marginal vector. In order to do so, we first show that each extreme point of \(C(cv^1_k)\) is integer, i.e. consists of ones and zeros.

Because \(G^k\) is odd-sun-free chordal, it follows from Theorem 4.3 that \(A_1(G^k) = A_k(G)\) is balanced. This implies that each extreme point of the polyhedron \(Q = \{x \in \mathbb{R}^n : A_k(G)x \leq 1, x \geq 0\}\) is integer (cf. Berge (1972)). Because of the balancedness of \(A_k(G)\) we have that \(cv^1_k(N) = \min\{\sum_{i \in N} y_i : yA_k(G) \geq 1, y \in \{0, 1\}^n\} = \min\{\sum_{i \in N} y_i : yA_k(G) \geq 1, y \geq 0\}\). From the Duality theorem of linear programming it follows that \(\min\{\sum_{i \in N} y_i : yA_k(G) \geq 1, y \geq 0\} = \max\{\sum_{i \in N} x_i : A_k(G)x \leq 1, x \geq 0\}\). This implies that for all \(x \in Q\) it holds that \(\sum_{i \in N} x_i \leq cv^1_k(N)\). From Theorem 4.1 it follows that

\[
C(cv^1_k) = Q \cap \{x \in \mathbb{R}^n : x(N) = cv^1_k(N)\}.
\]
It holds that \( C(cv_k^1) \) is a facet of \( Q \), and hence that \( C(cv_k^1) \) has only integral extreme points.

Now let \( x \) be an extreme point of \( C(cv_k^1) \). Then each entry of \( x \) is equal to 0 or 1. Let \( S \subset N \) be such that \( i \in S \) if and only if \( x_i = 1 \). Furthermore, let \( \sigma : N \to \{1, \ldots, n\} \) be a bijection such that \( \sigma(i) \leq |S| \) for all \( i \in S \). That is, \( \sigma \) begins with the members of \( S \) and ends with the members of \( N \setminus S \). Note that \( cv_k^1(N) = \sum_{i \in S} x_i \). We will show that \( m^\sigma(v) = x \).

Let \( j \in S \). Because \( x \) is a core element it holds that \( cv_k^1([j, \sigma]) \geq \sum_{i \in [j, \sigma]} x_i = |[j, \sigma]| \). Trivially it holds that \( |[j, \sigma]| \geq cv_k^1([j, \sigma]) \). Hence, \( cv_k^1([j, \sigma]) = |[j, \sigma]| \). Similarly it follows that \( cv_k^1((j, \sigma)) = |(j, \sigma)| \). We conclude that \( m_2^\sigma(cv_k^1) = cv_k^1([j, \sigma]) - cv_k^1((j, \sigma)) = |[j, \sigma]| - |(j, \sigma)| = 1 = x_j \).

Now let \( j \in N \setminus S \), and let \( i \in S \) be such that \( \sigma(i) = |S| \). It holds that \( cv_k^1(S) = |[i, \sigma]| = cv_k^1(N) \).

From the monotonicity of \( (N, cv_k^1) \) it follows that \( cv_k^1([j, \sigma]) = cv_k^1((j, \sigma)) = cv_k^1([i, \sigma]) = |S| \). We conclude that \( m_2^\sigma(cv_k^1) = cv_k^1([j, \sigma]) - cv_k^1((j, \sigma)) = |[i, \sigma]| - |[i, \sigma]| = 0 = x_j \).

Hence, \( m^\sigma(cv_k^1) = x \). Therefore \( (N, cv_k^1) \) satisfies the CoMa-property.

Unfortunately, Proposition 6.1 does not extend to arbitrary cost functions. This is illustrated in the following example.

**Example 6.1** Let \( G \) be the graph depicted in Figure 8 and let \( k = 1 \). Clearly, \( G \) is an odd-sun-free chordal graph. Let the cost function be given by \( w = (6, 4, 4, 4) \). For the relaxed dominating set game \( (N, cv_1^w) \) it holds that \( cv_1^w(S) = 4 \) if \( S \in \{1, 2, 3, 4, 12, 13, 14\} \) and \( cv_1^w(S) = 6 \) if \( S \in \{23, 24, 34, 123, 124, 134, 234, 1234\} \). It is straightforward to check that \( (3, 1, 1, 1) \in ext(C(cv_1^w)) \).

Moreover, \( x \) is not a marginal vector.

![Figure 8](image)

Figure 8: A graph for which \( (N, cv_1^w) \) does not satisfy the CoMa-property for all \( w : V \to \mathbb{R}_+ \).

In the proof of Proposition 6.1 we only use that \( A(G) \) is balanced and that the cost function equals \( w_i = 1 \) for all \( i \in V \). Hence, the proof can be used to show that combinatorial optimization games with cost function consisting of ones and a balanced matrix satisfy the CoMa-property. This gives rise to the question whether there exist other relations between the structure of a \( \{0, 1\} \)-matrix and the structure of the core of a combinatorial optimization games.

Another interesting question is whether dominating set games and intermediate dominating set games satisfy CoMa for odd-sun-free chordal graphs and cost function \( w_i = 1 \) for all \( i \in V \).

## 7 References


Farber M. 1981. Applications of LP duality to problems involving independence and domination. Technical Report 81-13, Department of Computer Science, Simon Fraser University, Burnaby, Canada.


Appendix

Lemma A.1 Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Let $i^p, l \in V$ be distinct and let $U$ be a proper $k$-substar at $l$ with $i^p \in U$. Let $W = \{ j \in U : d_{G_U}(j, l) = d_{G_U}(j, i^p) + d_{G_U}(i^p, l) \}$. Then it holds that $U \setminus W$ is a proper $k$-substar at $l$.

Proof: Suppose that $U \setminus W$ is not a proper $k$-substar at $l$. Then there is a $q \in U \setminus W$ with $d_{G_{U \setminus W}}(q, l) > k > d_{G_U}(q, l)$, where the second inequality holds because $U$ is a proper $k$-substar at $l$. Because the length of every shortest $q - l$ path in $G_{U \setminus W}$ is strictly larger than the length of every shortest $m - l$ path in $G_U$, it must hold that every shortest $m - l$ path in $G_U$ uses an element $j \in W$. Hence, $d_{G_U}(m, l) = d_{G_U}(m, j) + d_{G_U}(j, l)$. Because $j \in W$ it follows by definition of $W$ that $d_{G_U}(j, l) = d_{G_U}(j, i^p) + d_{G_U}(i^p, l)$. We conclude that $d_{G_U}(q, l) = d_{G_U}(q, j) + d_{G_U}(j, i^p) + d_{G_U}(i^p, l)$, which implies that there is a shortest $q - l$ path in $G_U$ which uses $i^p$. Therefore $q \in W$, contradicting $q \in U \setminus W$. As a result we have that $U \setminus W$ is a proper $k$-substar at $l$. □

Lemma A.2 Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Let $i^p, j^p, l \in V$ be distinct and let $U$ be a proper $k$-substar at $l$ with $i^p \in U$ and $j^p \notin U$. Let $(i^p, a_1, \ldots, a_m, j^p)$ be a shortest $i^p - j^p$ path in $G$, and let $P = \{ a_1, \ldots, a_m, j^p \}$. Finally, let $W = \{ j \in U : d_{G_U}(j, l) = d_{G_U}(j, i^p) + d_{G_U}(i^p, l) \}$. Suppose that $U \cup P$ is not a proper $k$-substar at $l$. Then it holds that $d_G(q, i^p) < d_G(i^p, j^p)$ for all $q \in W$.

Proof: Let $q \in W$. Because $U$ is a proper $k$-substar it holds that $d_{G_U}(q, l) \leq k$. It holds that $U \cup P$ is not a proper $k$-substar at $l$, hence, $d_{G_{U \cup P}}(j^p, i^p) + d_{G_{U \cup P}}(i^p, l) > k$. This implies that

\[ d_{G_{U \cup P}}(j^p, i^p) + d_{G_{U \cup P}}(i^p, l) > k \geq d_{G_U}(q, l) \]

\[ = d_{G_U}(q, i^p) + d_{G_U}(i^p, l) \geq d_{G_U}(q, i^p) + d_{G_{U \cup P}}(i^p, l), \] (2)

where the equality holds because $q \in W$. From (2) we obtain that $d_{G_{U \cup P}}(j^p, i^p) > d_{G_U}(q, i^p)$. This implies that

\[ d_G(q, i^p) \leq d_{G_U}(q, i^p) < d_{G_{U \cup P}}(j^p, i^p) = d_G(j^p, i^p). \] □