Enumeration of Cospectral Graphs

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Enumeration of cospectral graphs

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Didicated to the memory of Jaap Seidel

Abstract

We have enumerated all graphs on at most eleven vertices and determined their spectra with respect to various matrices, such as the adjacency matrix and the Laplacian matrix. We have also counted the numbers for which there is at least one other graph with the same spectrum (a cospectral mate). In addition we consider a construction for pairs of cospectral graphs due to Godsil and McKay, which we call GM switching. It turns out that for the enumerated cases a large part of all cospectral graphs comes from GM switching, and that the fraction of graphs on n vertices with a cospectral mate starts to decrease at some value of $n < 11$ (depending on the matrix). Since the fraction of cospectral graphs on n vertices constructible by GM switching tends to 0 if $n \to \infty$, the present data give some indication that possibly almost no graph has a cospectral mate. We also derive asymptotic lower bounds for the number of graphs with a cospectral mate from GM switching.

Key words: Graph, eigenvalue, enumeration. *AMS subject classifications*:05C50.

1 Introduction

In a sense the present paper is a sequel to Godsil and McKay's article [5] on cospectral graphs. In there two graphs are called cospectral whenever their adjacency matrices have the same spectrum. Godsil and McKay present several methods for constructing pairs of non-isomorphic cospectral graphs. One of these methods uses an operation on graphs that leaves the spectrum of the adjacency matrix invariant. We shall call this operation GM switching. With some extra requirements GM switching also applies to other matrices like the Laplacian matrix and the sign-less Laplacian matrix (see Section 2). This leads to lower bounds for the number of cospectral graphs with respect to the various matrices.

In addition, Godsil and McKay enumerated by computer all graphs on at most 9 vertices, computed their adjacency spectrum and determined the number of graphs for which there exists at least one cospectral mate. Here we extend the computer enumeration to the other types of matrices mentioned, and to 10 and 11 vertices.

We should mention that Lepović [7] has enumerated all connected graphs on 10 vertices and determined many data, including the number of graphs with exactly i cospectral mates for all relevant values of i. His results are consistent with ours.

2 The matrix

Throughout, A will be the adjacency matrix of a graph G on n vertices, and D is the diagonal matrix containing the degrees $d_1, \ldots, d_n$ of G (A and D have the same vertex ordering). The matrix $L = D - A$ is known as the Laplacian matrix of G. We shall also consider $|L| = D + A$ and call it the sign-less Laplacian matrix. The matrix $\overline{A} = J - A - I$ (as usual, J is the all-ones matrix, and I is the identity matrix) is the adjacency matrix of the complement of G, and the Seidel matrix S is defined by $S = \overline{A} - A = J - 2A - I$. For the Seidel matrix the following operation, called Seidel switching ([8]), gives the Seidel matrix $\widetilde{S}$ of another graph, cospectral with S. Let Δ be a diagonal matrix with diagonal entries ± 1. Then $\widetilde{S} = \Delta S \Delta$. Since $\Delta = \Delta^{-1}$, $\widetilde{S}$ is similar to S, and hence cospectral with S. For a given graph on $n > 1$ vertices almost all (and at least one) of the possible switchings changes the number of edges and therefore lead to non-isomorphic cospectral mates with respect to the Seidel matrix. However, for the other mentioned matrices having a cospectral mate seems exceptional (see [3] for a survey).

Remark that for $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$, two square matrices M and $\widetilde{M}$ are cospectral if and only if $\alpha M + \beta I$ and $\alpha\widetilde{M} + \beta I$ are. Moreover if the all-ones vector $\mathbf{1}$ is an eigenvector of M and $\widetilde{M}$ then M and $\widetilde{M}$ are cospectral if and only if $\alpha M + \beta I + \gamma J$ and $\alpha\widetilde{M} + \beta I + \gamma J$ are cospectral. In particular the Laplacian matrix L and the Laplacian matrix $\overline{L} = nI - J - L$ of the complement behave the same as far as cospectrality is concerned. Non-regular graphs, however, may be cospectral with respect to A, but not with respect to $\overline{A}$ (see Figure 4 for an example). An interesting result is the following theorem of Johnson and Newman [6] (see also [3]).

Theorem 1 *For the adjacency matrix A of a graph, define $\mathcal{A} = \{A + \alpha J | \alpha \in \mathbb{R}\}$. If G and $\widetilde{G}$ are cospectral with respect to two matrices in $\mathcal{A}$, then G and $\widetilde{G}$ are cospectral with respect to all matrices in $\mathcal{A}$.*

In particular, if two graphs G and $\widetilde{G}$ are cospectral, and so are their complements (so they are cospectral with respect to A and $A - J$), then G and $\widetilde{G}$ are cospectral with respect to any matrix of the form $\alpha A + \beta I + \gamma J$.

3 GM switching

We will formulate GM switching as an operation on (certain) matrices, which enables us to apply it to A, L and $|L|$.

Theorem 2 *Let N be a $(0,1)$-matrix of size $b \times c$ (say) whose column sums are 0, b or $b/2$. Define $\widetilde{N}$ to be the matrix obtained from N by replacing each column $\mathbf{v}$ with $b/2$ ones by its complement $\mathbf{1} - \mathbf{v}$. Let B be a symmetric $b \times b$ matrix with constant row (and column) sums, and let C be a symmetric $c \times c$ matrix. Put*

$$M = \begin{bmatrix} B & N \\ N^\top & C \end{bmatrix} \text{ and } \widetilde{M} = \begin{bmatrix} B & \widetilde{N} \\ \widetilde{N}^\top & C \end{bmatrix}.$$

Then M and $\widetilde{M}$ are cospectral.

Proof. Define $Q = \begin{bmatrix} \frac{2}{b}J - I_b & O \\ O & I_c \end{bmatrix}$. Then $Q^{-1} = Q$ and $QMQ^{-1} = \widetilde{M}$. □

The matrix partition used in [5] is more general than the one presented here. But this simplified version suffices for our purposes. Notice that in case all columns of N have $b/2$ ones, GM switching is the same as Seidel switching.

If $M = A$ is the adjacency matrix of a graph G, we see that the subgraph G_B, induced by the vertices of B, must be regular and every vertex in G_C (the subgraph corresponding to C) must be adjacent to all, to none, or to exactly half of the vertices of G_B. If this is the case we will say that the subgraph has the GM property. The switched graph $\widetilde{G}$ clearly has the same number of edges, but not necessarily the same vertex degrees. It is obvious that the corresponding subgraph of the complement of G also has the GM property, and switching leads to the complement of $\widetilde{G}$. Thus GM switching produces pairs of cospectral graphs for which also the complement is cospectral. Hence, by Theorem 1, G and $\widetilde{G}$ are cospectral with respect to any matrix of the form $\alpha A + \beta I + \gamma J$.

If, in addition to the GM property, we assume that the vertices of G_B have the same degree in G, we say that G_B has the GM* property. In this case the hypothesis of Theorem 2 are fulfilled for all matrices of the form $A + \delta D$, and switching doesn't change the degrees, so after switching the diagonal entries of D remain the row sums of A. Hence, if the

GM* property is fulfilled, switching gives cospectral graphs for all matrices of the form $\alpha A + \beta I + \delta D$. This includes the Laplacian and the sign-less Laplacian matrix.

However, if $M = -L = A - D$, the row sums of M are all equal to 0, so it suffices to require that N has constant row sums. Then it follows that B has constant row sums, even if G_B is not regular. So for the Laplacian matrix the GM* condition may be weakened. It is sufficient that the vertices of G_B all have the same number of neighbours in G_C, and (of course) every vertex of G_C has 0, b or $b/2$ neighbours in G_B.

In Figures 1, 2 and 3 we have three examples of pairs of cospectral graphs produced by GM switching. In all cases $b = 4$ and the upper vertices correspond to G_B and the lower vertices to G_C. In the example of Figures 1 and 3, G_B satisfies the GM condition and therefore the graphs are cospectral with respect to the adjacency matrix A, but also with respect to the adjacency matrix of the complement $\overline{A}$ and any other matrix of the form $\alpha A + \beta I + \gamma J$. In the example of Figures 2 and 3 all vertices of G_B have the same number of neighbours in G_C, and the graphs are cospectral with respect to the Laplacian matrix L. In Figure 3, G_B satisfies the GM* condition. This implies that GM switching gives cospectral graphs with respect to any matrix of the form $\alpha A + \beta I + \delta D$, including the sign-less Laplacian matrix $|L|$. The matrices B and N in the last example have more structure than in the first two. This is the reason why we have no example on eight vertices. In fact ten is the smallest number of vertices for which GM switching produces non-isomorphic cospectral graphs with respect to $|L|$ (see Table 1).

Figure 1: Non-isomorphic cospectral graphs with respect to A and $\overline{A}$

Figure 2: Non-isomorphic cospectral graphs with respect to L

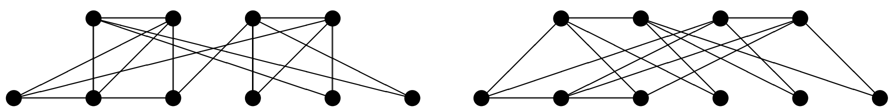

Figure 3: Non-isomorphic cospectral graphs with respect to $|L|$, L, A and $\overline{A}$

4 Lower bounds

GM switching gives lower bounds for cospectral graphs with respect to several types of matrices. We use the notation of Theorem 2. It is intuitively clear that the larger b is, the less likely it is that a given graph has one of the properties required for GM switching. Any pair of vertices in any graph satisfy the GM condition, but GM switching just interchanges the two corresponding vertices, and the switched graph is isomorphic to the original one. Four (ordered) vertices in a graph G satisfy the GM condition with probability 2^{1-n} (Indeed, for the first three vertices mutual adjacencies as well as the adjacencies with G_C can be chosen arbitrarily, then all adjacencies with the fourth vertex are fixed), but switching almost always produces non-isomorphic graphs. To see this we make some requirements on G and the chosen subgraph G_B on four vertices. Firstly, we require that G_B is the only 4-vertex subgraph with the GM condition. Secondly we require that G_C has no non-trivial automorphism. Thirdly we need that for every partition of the vertex set of G_B into two pairs, there is at least one vertex in G_C adjacent to the vertices in one part of the partition, but not to the vertices in the other part. Now suppose that $\widetilde{G}$, the graph obtained after switching is isomorphic to G. Then the isomorphism must fix the partition (by the first assumption), it must fix G_C point-wise (by the second assumption), hence there must be a permutation of the rows of N, with matrix P say, such that $PN = \widetilde{N}$. This is impossible because of the third assumption. Almost all pairs (G, G_B) with the GM property satisfy the above three conditions and therefore the number of these pairs equals $\binom{n}{4} g_n 2^{1-n}(1 - o(1)) = n^3 g_{n-1}(\frac{1}{24} - o(1))$, where g_k denotes the number of nonisomorphic graphs on k vertices. So we have the following lower bound.

Theorem 3 *The number of graphs G on n vertices for which there exists a graph $\widetilde{G}$ which is cospectral, but non-isomorphic, with G with respect to the adjacency matrix and the adjacency matrix of the complement (and hence with respect to any matrix of the form $\alpha A + \beta I + \gamma J$) is at least*

$$n^3 g_{n-1}(\tfrac{1}{24} - o(1)).$$

According to the abstract in [5], Godsil and McKay were aware of this bound; they just didn't work out the details. There is a more direct, but less accurate, way to obtain the above formula. Start with a graph G' on $n-1$ vertices. Fix a set X of three vertices. There is a unique way to extend G' by one vertex x to a graph G, such that $X \cup \{x\}$ induces a regular subgraph in G, and every vertex not in $X \cup \{x\}$ has an even number of neighbours in $X \cup \{x\}$. Thus $X \cup \{x\}$ satisfies the GM property. This implies that from a graph G on $n-1$ vertices one can make $\binom{n-1}{3}$ graphs on n vertices with a 4-vertex subgraph that satisfies the GM property, and each of these can be constructed in four ways. Ignoring possible isomorphisms leads to the required formula.

To get a lower bound for cospectral graphs with respect to L and $|L|$ we need the following lemma.

Lemma 1 *The number of* $(0,1)$*-matrices of size* $4\times c$*, for which each column sum is even and all row sums are equal, is at least* $\kappa 2^{3c}/c\sqrt{c}$ *for some constant* $\kappa>0$.

Proof. There are eight possible columns, six of which have two zeros and two ones. Each of these six columns should occur the same number of times as its complement. So the required number equals the number of sequences of length c with symbols $1,2,\ldots,8$, where $n_1=n_2$, $n_3=n_4$ and $n_5=n_6$ (n_i denotes the number of i's is the sequence). Put $k=n_1$, $\ell=n_3$ and $m=n_5$, then we obtain the following formula for the number of these sequences.

$$\sum_{k+\ell+m\le c/2}\frac{c!}{(2k)!(2\ell)!(2m)!(c-2k-2\ell-2m)!}\binom{2k}{k}\binom{2\ell}{\ell}\binom{2m}{m}2^{c-2k-2\ell-2m}.$$

With Stirling's formula we have $\binom{2k}{k}>2^{2k}/2\sqrt{k}$, so the above number is greater than

$$\sum_{k+\ell+m\le c/2}N_{k,\ell,m}\frac{1}{2\sqrt{k}}\frac{1}{2\sqrt{\ell}}\frac{1}{2\sqrt{m}}\ge\frac{1}{(2\sqrt{c/2})^3}\sum_{k+\ell+m\le c/2}N_{k,\ell,m},$$

where $N_{k,\ell,m}=\frac{c!}{(2k)!(2\ell)!(2m)!(c-2k-2\ell-2m)!}2^c$, which is just the number of these sequences for which $n_1+n_2=2k$, $n_3+n_4=2\ell$ and $n_5+n_6=2m$. Therefore $\sum_{k+\ell+m\le c/2}N_{k,\ell,m}\ge 8^{c-3}$ and the claim follows. □

From the formula above it follows that the probability that a 4-vertex subgraph satisfies the GM* condition equals $\kappa/(2^n n\sqrt{n})$ for some constant $\kappa>0$. Now we apply the same reasoning as above with the GM condition replaced by the GM* condition, and find the following result.

Theorem 4 *The number of graphs* G *on* n *vertices for which there exist is a non-isomorphic graph* $\widetilde{G}$ *which is cospectral with* G *with respect to all matrices of the form* $\alpha A+\beta I+\delta D$ *is at least*

$$\kappa n\sqrt{n}g_{n-1},$$

for some constant $\kappa>0$.

The above Lemma applies to the adjacency matrix (for which we have a better bound already), the Laplacian and the sign-less Laplacian matrix. We saw that for the Laplacian matrix, a weaker version of the GM* condition suffices. But this only leads to a bigger constant κ.

5 Enumeration

To determine the cospectrality of graphs we first of all had to generate the graphs by computer and then determine their characteristic polynomials. These would have to be stored on disc and then compared. To reduce the amount of storage space required we used the fact that graphs which are cospectral (with respect to the considered matrices) must have the same number of edges. However, in the case of graphs on 11 vertices a further sub-division had to be made. For example, there are $106,321,628$ graphs on 11 vertices with 27 (and 28) edges, and this number proved to be too great to deal with on account of the disc space that was available, not so much for the graphs themselves, as it was not necessary to store them (at least in the case of the spectrum A), but rather for the characteristic polynomials that were required to determine cospectrality. Not only must graphs cospectral with respect to A have the same number of edges, they must also have the same number of triangles. This was useful in reducing the maximum number of graphs (on 11 vertices) to be considered at any one time to around $15,000,000$. Thus, for the spectrum of A a procedure was written that generated the graphs according to the number of edges and triangles.

In the case of L and $|L|$ another method had to be adopted since it is possible for cospectrality to occur between graphs that have different numbers of triangles (see e.g. the graphs in Figure 4). Here we used the fact that cospectral graphs must have the same $\sum d_i$ and $\sum d_i^2$. The method used was to generate the graphs according to the number of edges m as above ($\frac{1}{2}\sum d_i$), but without any restriction on the number of triangles. As each graph was generated, $\sigma = \sum d_i^2$ was calculated and the graph was then stored on disc (in a compressed form), using a different file for each value of σ. Since graphs in different files could not be cospectral, it was only necessary to determine the characteristic polynomials of graphs in the same file and to compare them. This meant that even for graphs on 11 vertices and 27 edges we only had to consider at most around $10,000,000$ at any one time. Because of the compression of the graphs the amount of disc space used to store all the graphs with a fixed number of edges was at most 1.17 GBytes, approximately. The disadvantage of this method is that it involved a lot of disc activity.

The graphs generated were then fed to GAP [4] in such a way that strings were produced comprising the coefficients of the characteristic polynomials, separated by commas, and these were stored in a file, one to each line. This file was then sorted using the Unix procedure **sort**, after which it was an easy matter to count the number of non-unique lines (the number of cospectral graphs).

To avoid duplication of effort the two cases A and $A\,\&\,\overline{A}$ were dealt with simultaneously. As each graph G was generated, GAP was programmed to produce the coefficients of the characteristic polynomials of G and $\overline{G}$ and these were stored in separate files, again one to each line. Then, as above, using **sort** on one of the files, the numbers cospectral with respect to A were readily determined. For the case $A\,\&\,\overline{A}$, the two files were pasted together before being sorted.

The results are in Table 1. The last three columns give the numbers of graphs with a cospectral mate, which can be constructed by GM switching. Column GM gives the number of graphs G with the GM property for which $\widetilde{G}$ is non-isomorphic to G. So it gives a lower bound for column $A\,\&\,\overline{A}$ (and, of course, for column A). Column GM* is

n	# graphs	A	$A\,\&\,\overline{A}$	L	$\|L\|$	GM	GM-L	GM*
2	2	0	0	0	0	0	0	0
3	4	0	0	0	0	0	0	0
4	11	0	0	0	2	0	0	0
5	34	2	0	0	4	0	0	0
6	156	10	0	4	16	0	0	0
7	1044	110	40	130	102	40	72	0
8	12346	1722	1166	1767	1201	1054	1082	0
9	274668	51038	43811	42595	19001	38258	30266	0
10	12005168	2560516	2418152	1412438	636607	2047008	958680	9480
11	1018997864	215331677	212264372	91274836	38966935	176895408	60944708	1297220

Table 1: Numbers of graphs with cospectral mates

defined analogously with the GM* property, and gives lower bound for $|L|$ (and for L, A and $A\,\&\,\overline{A}$). Column GM-L is a lower bound from GM switching for column L. These numbers were obtained by computer enumeration of the graphs with the required partition. The graphs were only stored on disc (in compressed form, one to each line as in the case of the characteristic polynomials) if their standard form and that of the switched graph were different. The list was finally sorted to remove multiple entries. The reason for this is that organising isomorph rejection in the initial search was more expensive in time.

n	# graphs	A	$A\,\&\,\overline{A}$	L	$\|L\|$	GM	GM-L	GM*
2	2	0	0	0	0	0	0	0
3	4	0	0	0	0	0	0	0
4	11	0	0	0	0.182	0	0	0
5	34	0.059	0	0	0.118	0	0	0
6	156	0.064	0	0.026	0.103	0	0	0
7	1044	0.105	0.038	0.125	0.098	0.038	0.069	0
8	12346	0.139	0.094	0.143	0.097	0.085	0.088	0
9	274668	0.186	0.160	0.155	0.069	0.139	0.110	0
10	12005168	0.213	0.201	0.118	0.053	0.171	0.080	0.001
11	1018997864	0.211	0.208	0.090	0.038	0.174	0.060	0.001

Table 2: Fractions of graphs with cospectral mates

Table 2 exhibits the same results as Table 1, but expressed as fractions of the total number of graphs on n vertices. An interesting observation from this table is that the fractions of graphs with a cospectral mate is nondecreasing for small n, but starts to decrease at $n = 10$ for A, at $n = 9$ for L, and already at $n = 4$ for $|L|$. In addition, the last three columns show that the majority of graphs with cospectral mates with respect to $A\,\&\,\overline{A}$ and L comes from GM switching (at least for $n \geq 7$). If this tendency continues, the lower bounds given in Theorems 3 and 4 will be asymptotically tight and almost all graphs will be determined by their spectrum for all cases in the table. Indeed, the fraction of graphs that admit a non-trivial GM switching tends to zero as n tends to infinity (see also [5]). The conclusion may be that the present data give some indication that, with respect to all matrices considered in the enumeration, the fraction of non-isomorphic cospectral pairs tends to zero as n tends to infinity.

As mentioned, the enumeration has been carried out for each possible number of edges. The data, differentiated according to the number m of edges is presented in Table 3 and 4. Note that for the columns $A \,\&\, \overline{A}$ and L, a graph and its complement give the same number of cospectral graphs, so these columns are palindromic.

We end with some explicit examples of cospectral pairs. For each of the considered matrices, we give the smallest (with respect to (n, m), in alphabetic order) pair of cospectral graphs in Figure 4. The first pair is the standard example of a pair of cospectral graphs, first presented by Cvetković [1]. We like to call it the Saltire pair (because the two pictures superposed give the Scottish flag: Saltire). The pair cospectral with respect to A and $\overline{A}$ can be obtained by GM switching: take for G_B the coclique of size 4. The pair with cospectral Laplacian matrices was first given by Van Dam [2]; note that one is bipartite, and the other one not. The last picture gives graphs which have the same line graph. This implies that they are cospectral with respect to $|L|$, see for example [3].

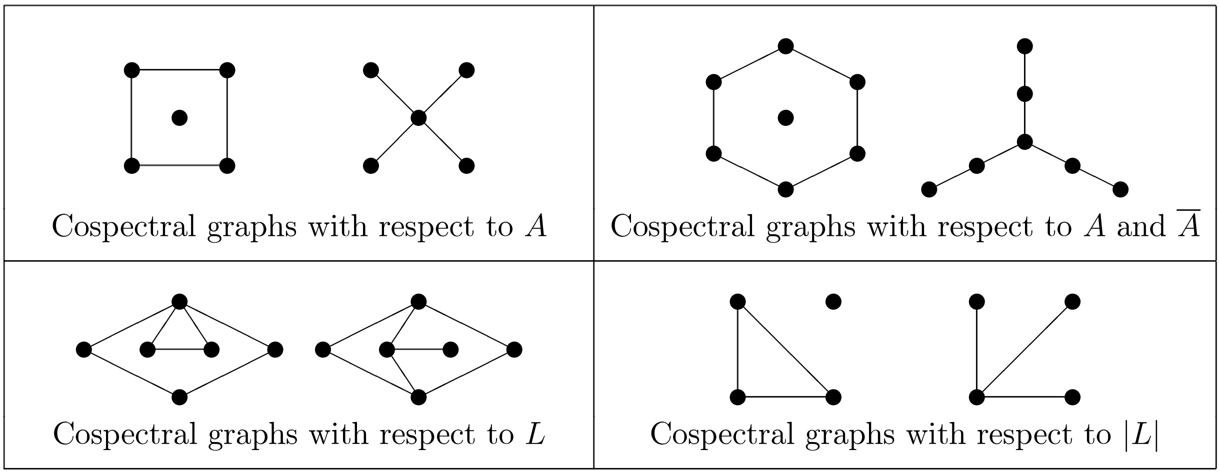

Figure 4:

Memorial

This paper is dedicated to the memory of Professor Jaap Seidel. We have both been greatly influenced by him, not only mathematically, but also in our private lives, and the present paper reflects these two aspects. Jaap taught us about switching, which is a basic concept in this paper, and more importantly, it was Jaap who introduced us to each other, which led to a friendship that goes further than mathematics.

Graphs on 4 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	3	0	0	0	2
4	2	0	0	0	0

Graphs on 5 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	4	0	0	0	2
4	6	2	0	0	0
5	6	0	0	0	0
6	6	0	0	0	0
7	4	0	0	0	2
8	2	0	0	0	0

Graphs on 6 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	9	4	0	0	2
5	15	2	0	0	0
6	21	2	0	0	0
7	24	2	0	2	4
8	24	0	0	2	4
9	21	0	0	0	0
10	15	0	0	0	0
11	9	0	0	0	2
12	5	0	0	0	2
13	2	0	0	0	0

Graphs on 7 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	10	4	0	0	2
5	21	3	0	0	2
6	41	14	2	5	2
7	65	8	2	10	6
8	97	18	4	18	12
9	131	12	6	20	14
10	148	16	6	12	11
11	148	13	6	12	14
12	131	12	6	20	12
13	97	4	4	18	6
14	65	2	2	10	2
15	41	2	2	5	2
16	21	2	0	0	2
17	10	0	0	0	2
18	5	0	0	0	0
19	2	0	0	0	0

Table 3:
Numbers of graphs with a non-isomorphic cospectral mate with respect to matrices A, $A\,\&\,\overline{A}$, L and $|L|$ for all non-trivial numbers m of edges up to 9 vertices.

Graphs on 8 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	11	4	0	0	2
5	24	5	0	0	4
6	56	21	2	7	7
7	115	26	4	14	16
8	221	63	10	39	36
9	402	68	30	69	58
10	663	164	55	91	79
11	980	148	87	115	93
12	1312	219	133	170	100
13	1557	223	173	233	142
14	1646	219	178	291	167
15	1557	210	173	233	141
16	1312	151	133	170	98
17	980	91	87	115	86
18	663	64	55	91	66
19	402	30	30	69	51
20	221	10	10	39	26
21	115	4	4	14	14
22	56	2	2	7	5
23	24	0	0	0	4
24	11	0	0	0	2
25	5	0	0	0	2
26	2	0	0	0	0

Graphs on 9 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	11	4	0	0	2
5	25	5	0	0	4
6	63	24	2	7	9
7	148	51	6	19	24
8	345	136	18	60	50
9	771	204	61	156	93
10	1637	512	158	306	179
11	3252	740	383	535	293
12	5995	1419	764	954	433
13	10120	2065	1469	1656	688
14	15615	3282	2342	2642	1109
15	21933	4331	3557	3597	1635
16	27987	5513	4624	4373	1958
17	32403	6338	5619	4674	2086
18	34040	6404	5805	4637	2071
19	32403	5990	5619	4674	2075
20	27987	4930	4624	4373	1940
21	21933	3695	3557	3597	1619
22	15615	2458	2342	2642	1081
23	10120	1500	1469	1656	656
24	5995	789	764	954	408
25	3252	395	383	535	278
26	1637	162	158	306	149
27	771	63	61	156	84
28	345	18	18	60	38
29	148	8	6	19	22
30	63	2	2	7	7
31	25	0	0	0	4
32	11	0	0	0	2
33	5	0	0	0	2
34	2	0	0	0	0

Graphs on 10 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	11	4	0	0	2
5	26	5	0	0	4
6	66	26	2	7	11
7	165	62	6	21	31
8	428	191	22	75	80
9	1103	412	86	237	155
10	2769	1068	278	568	338
11	6759	1994	831	1279	681
12	15772	4843	2178	2722	1307
13	34663	8874	5380	5455	2344
14	71318	18747	11811	10428	4362
15	136433	31852	24094	18826	8069
16	241577	56827	44229	31373	13909
17	395166	87986	75358	47972	21814
18	596191	133350	116870	68692	31495
19	828728	181236	166403	92350	42534
20	1061159	233250	217639	119163	54427
21	1251389	273336	260561	145233	65430
22	1358852	294399	283328	161818	72165
23	1358852	291391	283328	161818	72181
24	1251389	266294	260561	145233	65338
25	1061159	221659	217639	119163	54290
26	828728	168717	166403	92350	42342
27	596191	118267	116870	68692	31312
28	395166	76093	75358	47972	21660
29	241577	44628	44229	31373	13716
30	136433	24288	24094	18826	7919
31	71318	11928	11811	10428	4202
32	34663	5432	5380	5455	2224
33	15772	2188	2178	2722	1182
34	6759	840	831	1279	590
35	2769	290	278	568	259
36	1103	94	86	237	128
37	428	26	22	75	60
38	165	6	6	21	27
39	66	2	2	7	9
40	26	0	0	0	4
41	11	0	0	0	2
42	5	0	0	0	2
43	2	0	0	0	0

Table 4:
Numbers of graphs with a non-isomorphic cospectral mate with respect to matrices A, $A\,\&\,\overline{A}$, L and $|L|$ for all non-trivial numbers m of edges on 10 and 11 vertices.

Graphs on 11 vertices

m	# graphs	A	$A\,\&\,\overline{A}$	L	$\lvert L\rvert$
2	2	0	0	0	0
3	5	0	0	0	2
4	11	4	0	0	2
5	26	5	0	0	4
6	67	26	2	7	11
7	172	65	6	21	33
8	467	220	24	79	93
9	1305	558	100	276	205
10	3664	1617	367	793	487
11	10250	3601	1266	2128	1092
12	28259	10088	4032	5511	2536
13	75415	21915	12057	13095	5436
14	192788	56851	33149	29242	11713
15	467807	118099	85356	62858	25201
16	1069890	269166	202525	130173	53338
17	2295898	529579	446061	257453	107427
18	4609179	1054698	912308	483437	203086
19	8640134	1892069	1731073	858901	360278
20	15108047	3292566	3059756	1450889	602866
21	24630887	5279190	5034203	2325548	956313
22	37433760	8001477	7703505	3508525	1436639
23	53037356	11264629	10974450	4926016	2033387
24	70065437	14903754	14598628	6390926	2686072
25	86318670	18373280	18101952	7659772	3290167
26	99187806	21108349	20865502	8546114	3739927
27	106321628	22561018	22365864	8985654	3973874
28	106321628	22519077	22365864	8985654	3973697
29	99187806	20975573	20865502	8546114	3739550
30	86318670	18177143	18101952	7659772	3289555
31	70065437	14648996	14598628	6390926	2685316
32	53037356	11003951	10974450	4926016	2032407
33	37433760	7721507	7703505	3508525	1435463
34	24630887	5043664	5034203	2325548	954838
35	15108047	3065242	3059756	1450889	601329
36	8640134	1733808	1731073	858901	358747
37	4609179	913741	912308	483437	201801
38	2295898	446687	446061	257453	106447
39	1069890	202832	202525	130173	52593
40	467807	85509	85356	62858	24695
41	192788	33221	33149	29242	11291
42	75415	12079	12057	13095	5152
43	28259	4048	4032	5511	2261
44	10250	1272	1266	2128	936
45	3664	369	367	793	378
46	1305	100	100	276	173
47	467	24	24	79	71
48	172	8	6	21	29
49	67	2	2	7	9
50	26	0	0	0	4
51	11	0	0	0	2
52	5	0	0	0	2
53	2	0	0	0	0

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