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Thijssen, J.J.J.; Huisman, K.J.M.; Kort, P.M.

Publication date:
2002

Citation for published version (APA):
Thijssen, J. J. J., Huisman, K. J. M., & Kort, P. M. (2002). *Symmetric Equilibrium Strategies in Game Theoretical Real Option Models*. (CentER Discussion Paper; Vol. 2002-81). Microeconomics.

No. 2002-81

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By Jacco J.J. Thijssen, Kuno J.M. Huisman, Peter M. Kort

August 2002

ISSN 0924-7815

Discussion paper

Symmetric Equilibrium Strategies in Game Theoretic Real Option Models *

Jacco J.J. Thijssen[†] Kuno J.M. Huisman[‡] Peter M. Kort[§]

August 21, 2002

Abstract

This paper considers the problem of investment timing under uncertainty in a duopoly framework. When both firms want to be the first investor a coordination problem arises. Here, a method is proposed to deal with this coordination problem, involving the use of symmetric mixed strategies.

The method is based on Fudenberg and Tirole (1985, Review of Economic Studies), where it was designed within a deterministic framework. The aim of our paper is to extend the applicability of this method to a stochastic environment. The need for this is exemplified by the fact that several recent contributions in multiple firm real option models make unsatisfactory assumptions to solve the coordination problem mentioned above. Moreover, our approach allows us to show that in many cases it is incorrect to claim that "the probability that both firms invest simultaneously, while it is only optimal for one firm to invest, is zero".

*The authors thank Dolf Talman, Eric van Damme, participants of the Seventh Viennese Workshop on Optimal Control, Dynamic Games and Non-linear Dynamics, and seminar participants at CentER for their constructive comments.

[†]Corresponding author. Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, PO Box 90153, 5000 LE Tilburg, The Netherlands. Tel: +31-13-4662824; fax: +31-13-4663280; e-mail: Thijssen@uvt.nl.

[‡]Centre for Quantitative Methods, PO Box 414, 5600 AK Eindhoven, The Netherlands.

[§]Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, The Netherlands, and Department of Economics, UFSIA, University of Antwerp, Antwerp, Belgium.

Keywords: Investment, Timing Game, Real Options, Preemption, Symmetric Strategies
JEL classification: C73, D81, L13

1 Introduction

The timing of an investment project is an important problem in capital budgeting. Many decision criteria have been proposed in the literature, the net present value (NPV) rule being the most famous one. In the past twenty years the real options literature emerged (cf. Dixit and Pindyck (1994)). In real option models uncertainty about the profitability of an investment project is explicitly taken into account. In the standard real option model the value of the investment project is assumed to follow a geometric Brownian motion. By solving the resulting optimal stopping problem one can show that it is optimal for the firm to wait longer with investing than when the firm uses the NPV approach.

A natural extension of the one firm real option model is to consider a situation where several firms have the option to invest in the same project. Important fields of application of the game theoretic real options approach are R&D competition, technology adoption and new market models. Restricting ourselves to a duopoly framework, the aim of this paper is to propose a method that solves the coordination problems which arise if there is a first mover advantage that creates a preemptive threat and erodes the option value. In the resulting preemption equilibrium, situations can occur where it is optimal for one firm to invest, but not for both. The coordination problem then is to determine which firm will invest. This problem is particularly of interest if both firms are *ex ante* identical. In the literature, several contributions (see Grenadier (2000) for an overview) solve this coordination problem by making explicit assumptions which are often unsatisfactory. In this paper, we propose a method, based on Fudenberg and Tirole (1985), to solve the coordination problem endogenously.

The basic idea of the method is that one splits the game into a timing game where the preemption moment is determined and a game, that is played either as soon as the preemption moment has been reached, or when the starting point is such that it is immediately optimal for one firm to invest but not for both. The outcome of the latter game determines which firm is the first investor. The first game is a game in continuous time where strategies are given by a cumulative distribution function. The second game is analogous to a repeated game in which firms play a fixed (mixed) strategy (invest or wait) until at least one firm invests.

As an illustration, a simplified version of the Smets (1991) model, that is presented in Dixit and Pindyck (1994, Section 9.3), is analyzed. In the preemption equilibrium situations occur where it is optimal for one firm to invest, but at the same time investment is not beneficial if both firms decide to do so. Nevertheless, contrary to e.g. Smets (1991) and Dixit and Pindyck (1994), we find that there are scenarios in which both firms invest at the same time, which leads to a low payoff for both of them. We obtain that such a coordination failure can occur with positive probability at points in time where the payoff of the first investor (or leader) is strictly larger than the follower's payoff. From our analysis it can thus be concluded that Smets' statement that "if both players move simultaneously, each of them becomes leader with probability one half and follower with probability one half" (Smets (1991, p. 12) and Dixit and Pindyck (1994, p. 313)) need not be true.

The point we make here extends to other contributions that include the real option framework in duopoly models. These papers, such as Grenadier (1996), Dutta et al. (1995), and Weeds (2002), make unsatisfactory assumptions with the aim to be able to ignore the possibility of simultaneous investment at points of time that this is not optimal. Grenadier (1996, pp. 1656-1657) assumes that "if each tries to build first, one will randomly (i.e., through the toss of a coin) win the race", while Dutta et al. (1995, p.568) assume that "If both [firms] i and j attempt to enter at any period t, then only one of them succeeds in doing so".

In a recent paper, Weeds (2002) uses Markov strategies resulting in two different investment patterns, where in one of them the firms invest sequentially. In these asymmetric preemption equilibria one of the firms is the first mover with probability one. This implies that the probability of a coordination failure is always zero. In her paper, this result holds because of the assumption that the value of the starting point of the geometric Brownian motion is lower than the value corresponding to the preemption point. For more general stochastic processes and arbitrary starting points this claim needs not be true. In our framework the two different outcomes that Weeds (2002) reports can be obtained by one pair of symmetric strategies, thereby solving the coordination problem endogenously. Furthermore, we show our result for more general stochastic processes.

In this paper we extend the strategy spaces and equilibrium concepts as introduced in Fudenberg and Tirole (1985) to a stochastic framework. In a recent paper, Boyer et al. (2001) make a similar attempt. Their adaptation however is less suitable to model war of attrition situations as could arise in stochastic analogues of e.g. Hendricks et al. (1988), or models in which both preemption and war of attrition equilibria can arise (e.g. Thijssen et al. (2001)).

The contents of the paper is as follows. In Section 2 the equilibrium concept is presented and a symmetric equilibrium is derived, which is applied to the Dixit and Pindyck (1994, Section 9.3) model in Section 3. Section 4 concludes.

2 The General Model

The setting of the game is as follows. Two identical firms[1] $i = 1, 2$ both have access to an identical investment project. In the market there is some idiosyncratic risk or uncertainty about for example the profitability of the investment project. This creates an option value for the firms to postpone investment. On the other hand, strategic considerations push firms not to wait too long.

This section introduces the equilibrium notions and accompanying strategy spaces for timing games under uncertainty. We follow the approach that was introduced in Fudenberg and Tirole (1985) for the deterministic counterpart.

In our framework uncertainty is modelled in the following way. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq\infty}, P)$ be a filtered probability space satisfying the usual hypotheses, i.e. $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$

[1] Even with non-identical firms, the coordination problem that is analyzed in this paper can arise as is shown in Pawlina and Kort (2001).

is right-continuous[2]. Let $(J_t)_{0\leq t<\infty}$ be an adapted cadlag[3] process and let $(Z^1,\ldots,Z^d)$ be a vector of semimartingales with for all $j=1,\ldots,d$, $Z_0^j=0$ a.s. Furthermore, let for all $j=1,\ldots,d$, F_j be functional Lipschitz.[4] Then Protter (1995, Theorem 5.7) shows that there is a unique semimartingale $(Y_t)_{0\leq t<\infty}$, which, for all $\omega\in\Omega$, is the solution to the stochastic differential equation

$$Y_t(\omega)=J_t(\omega)+\sum_{j=1}^{d}\int_0^t F_j(Y(\omega))_{s-}dZ_s^j(\omega). \tag{1}$$

In the remainder, Y will be the stochastic process that governs the uncertainty about the profitability of the investment project. Examples of stochastic differential equations that can be used are for example the geometric Brownian motion or, more generally, Lévy processes.

Given the stochastic process $(Y_t)_{t\geq 0}$ we can define the payoff functions for the firms. The firm that moves first (i.e. that is the first to invest) is called the leader. When it invests at time t its discounted profit stream is given by $L(Y_t(\omega))$ for $\omega\in\Omega$ and $0\leq t<\infty$. The other firm is called the follower. When the leader invests at time t the optimal investment strategy of the follower leads to a discounted profit stream $F(Y_t(\omega))$. If both firms invest simultaneously at time t, the discounted profit stream for both firms is given by $M(Y_t(\omega))$. It is assumed that $L(\cdot)$ and $F(\cdot)$ are continuous functions.

---

[2] $\mathcal{F}_t=\bigcap_{u>t}\mathcal{F}_u$, all t, $0\leq t<\infty$.

[3] For all t, J_t is $\mathcal{F}_t$ measurable and the sample paths are right-continuous and have left limits.

[4] Let $\mathbb{D}^n$ denote the set of n-dimensional cadlag processes. An operator F mapping $\mathbb{D}^n$ to $\mathbb{D}^1=\mathbb{D}$ is functional Lipschitz if for any X, Y in $\mathbb{D}^n$ the following two conditions are satisfied:

1. for any stopping time T, $X^{T-}=Y^{T-}$ implies $F(X)^{T-}=F(Y)^{T-}$;
2. there exists an increasing (finite) process $K=(K_t)_{t\geq 0}$ such that

$$|F(X)_t-F(Y)_t|\leq K_t\|X_t-Y_t\|_t^* \quad \text{a.s.}, \forall t\geq 0,$$

where $\|X\|_t^*=\sup\limits_{0\leq u\leq t}\|X_u\|$.

2.1 The Equilibrium Concept

As in Fudenberg and Tirole (1985), the aim is to find an equilibrium that is the continuous time analogue of a subgame perfect Nash equilibrium. To define the strategy spaces and equilibrium concept we extend and slightly adapt the concepts introduced in Fudenberg and Tirole (1985) path-wise.[5] First we define a simple strategy for the subgame starting at t_0.

Definition 1 *A* simple strategy *for player $i = 1, 2$ in the subgame starting at $t_0 \in [0, \infty)$ is given by a tuple of real-valued functions $(G_i^{t_0}, \alpha_i^{t_0}) : [t_0, \infty) \times \Omega \to [0, 1] \times [0, 1]$, such that for all $\omega \in \Omega$*

1. *$G_i^{t_0}(\cdot; \omega)$ is non-decreasing and right-continuous with left limits;*
2. *$\alpha_i^{t_0}(\cdot; \omega)$ is right differentiable and right-continuous with left limits;*
3. *if $\alpha_i^{t_0}(t; \omega) = 0$ and $t = \inf\{u \geq t_0 | \alpha_i^{t_0}(u; \omega) > 0\}$, then the right derivative of $\alpha_i^{t_0}(t; \omega)$ is positive.*

Denote the strategy set of simple strategies of player i in the subgame starting at t_0 by $S_i^s(t_0; \omega)$. Furthermore, define the strategy space by $S^s(t_0; \omega) = \prod_{i=1,2} S_i^s(t_0; \omega)$ and denote the strategy at $t \in [t_0, \infty)$ by $s^{t_0}(t; \omega) = \big(G_i^{t_0}(t; \omega), \alpha_i^{t_0}(t; \omega)\big)_{i=1,2}$.

The function $G_i^{t_0}$ can be seen as a cumulative distribution function in the sense that $G_i^{t_0}(t; \omega)$ is the probability that a firm has invested before or at time t. The function $\alpha_i^{t_0}(\cdot; \omega)$ describes a sequence of atoms. Simple strategies allow for several investment strategies. First, the cumulative distribution function G allows for continuous investment strategies (used in war of attrition models) and single jumps. The atom function α allows for coordination between firms in cases where investment by at least one firm is optimal. Suppose for example that it is optimal for only one firm to invest. Taking $G^1 = G^2 = 1$ leads to certain investment by both firms which is suboptimal. If one only uses the cumulative distribution function (as in Dutta and Rustichini (1995) for example) symmetric strategies will lead to a suboptimal outcome a.s.

The atom function replicates discrete time results that are lost by modelling in continuous time (see Fudenberg and Tirole (1985, p. 390)). To see

[5] The equilibrium concept of Fudenberg and Tirole (1985) is also linked to the concept used by Simon (1987a) and Simon (1987b).

this, consider the following game in discrete time, depicted in Figure 1. In

		Firm 2	
		α_2	$1-\alpha_2$
Firm 1	α_1	$(M(Y_t), M(Y_t))$	$(L(Y_t), F(Y_t))$
	$1-\alpha_1$	$(F(Y_t), L(Y_t))$	repeat game

Figure 1: Payoffs (first for firm 1 and second for firm 2) and strategies of matrix game.

each period firm i invests with probability α_i, given that the realisation of the stochastic process remains constant until at least one of the firms has invested. Let Δt be the size of a period and let T_Δ be such that for some constant T, $T_\Delta/\Delta t = T$. Then if we take $\Delta t = 1$ we get for instance that the probability that both firms invest simultaneously before time T, denoted by $\mathbb{P}(1,2|T)$, is given by

$$\begin{aligned}&\mathbb{P}(1,2|T)\\ &\quad = \alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2)\alpha_1\alpha_2 + \cdots + (1-\alpha_1)^{T-1}(1-\alpha_2)^{T-1}\alpha_1\alpha_2.\end{aligned}$$

Letting $\Delta t \downarrow 0$ we get a result that is independent of T and that represents the probability that firms invest simultaneously at any time τ in our continuous time model,

$$\begin{aligned}\mathbb{P}(1,2|\tau) &= \lim_{\Delta t \downarrow 0} \sum_{t=0}^{T_\Delta/\Delta t - 1} (1-\alpha_1)^t(1-\alpha_2)^t\alpha_1\alpha_2\\ &= \alpha_1\alpha_2 \sum_{t=0}^{\infty} \big((1-\alpha_1)(1-\alpha_2)\big)^t\\ &= \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2-\alpha_1\alpha_2} < 1.\end{aligned} \tag{2}$$

In the continuous time setting it is assumed that if $\alpha_i^{t_0}(t;\omega) > 0$, the game depicted in Figure 1 is played, where firm i invests with probability $\alpha_i^{t_0}(t;\omega) > 0$. Furthermore, it is assumed that playing the game consumes no time[6] and if firm 1 chooses row 2 and firm 2 chooses column 2 the game is repeated instantaneously. As soon as at least one firm has invested this game is over and time continues. Define the time of the first "sequence of atoms" in player i's strategy in the subgame starting at t_0 by

$$\tau_i(t_0;\omega) = \begin{cases} \infty & \text{if } \alpha_i(t;\omega) = 0, \ \forall t \geq t_0, \\ \inf\{t \geq t_0 | \alpha_i(t;\omega) > 0\} & \text{otherwise,} \end{cases}$$

and $\tau(t_0;\omega) = \min\{\tau_1(t_0;\omega), \tau_2(t_0;\omega)\}$. So, at time τ at least one firm invests. It can be the case that only one firm invests, but it is also possible that both firms invest simultaneously. Note that from eq. (2) it follows that with symmetric strategies $\alpha_1(\tau) = \alpha_2(\tau) = \alpha$ we get

$$\mathbb{P}(1,2|\tau) = \frac{\alpha^2}{2\alpha - \alpha^2} = \frac{\alpha}{2-\alpha} \geq 0. \tag{3}$$

Hence, if $\alpha > 0$ the probability that both firms invest simultaneously is strictly positive. This is not in accordance with many contributions in the literature where (unsatisfactory) assumptions are made to sustain the claim that only one firm invests (e.g. Weeds (2002) and Grenadier (1996)).

Our definition of simple strategies differs from Fudenberg and Tirole (1985). Firstly, the cumulative distribution function in Fudenberg and Tirole (1985) is conditional on the fact that no firm has invested yet. This implies that in their approach simple strategies do not describe the behaviour of each firm for the entire range of time. As soon as firm i has invested, the simple strategy cannot describe firm j's strategy anymore. Furthermore, it implies that the strategy of a firm depends on the strategy of the other firm which is conceptually undesirable because in principle each firm must be able to choose its strategy without taking into account the strategy of its competitor. Secondly, Fudenberg and Tirole (1985) take the following restriction on the atom function in their definition of simple strategies: $\alpha_i^{t_0}(t;\omega) > 0 \Rightarrow G_i^{t_0}(t;\omega) = 1$, which results from G being the cumulative distribution function of a conditional distribution. However, it is not guaranteed that firm i invests with probability one if its atom function is strictly

[6] This implies that $Y_t(\omega)$ remains constant and that there is no discounting.

positive. This requirement also implicates that the investment strategy of firm i cannot be described by simple strategies if firm j has invested. The analysis in Fudenberg and Tirole (1985) uses the (G, α) pair to describe a preemption situation in a deterministic analogue to our stochastic model. They conclude that at the preemption moment (to be made precise later) one firm invests and the other waits. However, since G is based on the condition that no firm has invested yet, the (G, α) pair cannot describe the strategy of the firm that has not invested at the preemption moment. In our model, the cumulative distribution G is unconditional and hence can describe a firm's strategy for the entire time span.

The expected discounted value for firm $i = 1, 2$ of the subgame starting at t_0, given strategy $s^{t_0}(\omega) \in S^s(t_0; \omega)$, is denoted by $V_i(t_0, s^{t_0}(\omega))$ and is given by,[7]

$$
\begin{aligned}
V_i(t_0,&s^{t_0}(\omega)) = \\
&\Big[\int_{t_0}^{\tau(t_0;\omega)-} L(Y_t(\omega))(1 - G_j^{t_0}(t;\omega))dG_i^{t_0}(t;\omega) + \int_{t_0}^{\tau(t_0;\omega)-} F(Y_t(\omega))\times \\
&\times (1 - G_i^{t_0}(t;\omega))dG_j^{t_0}(t;\omega) + \sum_{t<\tau(t_0;\omega)} \lambda_i(t;\omega)\lambda_j(t;\omega)M(Y_t(\omega))\Big] \\
&+ \big(1 - G_i^{t_0}(\tau(t_0;\omega)-)\big)\big(1 - G_j^{t_0}(\tau(t_0;\omega)-)\big)W_i(\tau(t_0;\omega), s^{t_0}(\omega)),
\end{aligned}
$$

where $\lambda_i(t;\omega) = G_i^{t_0}(t;\omega) - G_i^{t_0}(t-;\omega)$ and

$$
\begin{aligned}
W_i(\tau,&s^{t_0}(\omega)) = \\
&\left(\frac{\lambda_j(\tau;\omega)}{1 - G_j^{t_0}(\tau-;\omega)}\right)\big((1 - \alpha_i^{t_0}(\tau;\omega))F(Y_\tau(\omega)) + \alpha_i^{t_0}(\tau;\omega)M(Y_\tau(\omega))\big) \\
&+ \left(\frac{1 - G_j^{t_0}(\tau;\omega)}{1 - G_j^{t_0}(\tau-;\omega)}\right)L(Y_\tau(\omega)),
\end{aligned}
$$

if $\tau_j(t_0;\omega) > \tau_i(t_0;\omega)$;

$$
\begin{aligned}
W_i(\tau,&s^{t_0}(\omega)) = \\
&\left(\frac{\lambda_i(\tau;\omega)}{1 - G_i^{t_0}(\tau-;\omega)}\right)\big((1 - \alpha_j^{t_0}(\tau;\omega))L(Y_\tau(\omega)) + \alpha_j^{t_0}(\tau;\omega)M(Y_\tau(\omega))\big) \\
&+ \left(\frac{1 - G_i^{t_0}(\tau;\omega)}{1 - G_i^{t_0}(\tau-;\omega)}\right)F(Y_\tau(\omega)),
\end{aligned}
$$

[7] In the remainder let for $t \in [0, \infty)$, $t-$ be defined by $t- = \lim_{\Delta t \uparrow 0} t + \Delta t$.

if $\tau_i(t_0;\omega) > \tau_j(t_0;\omega)$; and

$W_i(\tau, s^{t_0}(\omega)) =$

$$\begin{cases} M(Y_\tau(\omega)) \qquad \text{if } \alpha_i^{t_0}(\tau;\omega) = \alpha_j^{t_0}(\tau;\omega) = 1, \\ \Big[\alpha_i^{t_0}(\tau;\omega)(1-\alpha_j^{t_0}(\tau;\omega))L(Y_\tau(\omega)) + \alpha_j^{t_0}(\tau;\omega)(1-\alpha_i^{t_0}(\tau;\omega))F(Y_\tau(\omega)) \\ +\alpha_i^{t_0}(\tau;\omega)\alpha_j^{t_0}(\tau;\omega)M(Y_\tau(\omega))\Big]/ \\ \Big(\alpha_i^{t_0}(\tau;\omega) + \alpha_j^{t_0}(\tau;\omega) - \alpha_i^{t_0}(\tau;\omega)\alpha_j^{t_0}(\tau;\omega)\Big) \qquad \text{if } 2 > \alpha_i^{t_0}(\tau;\omega) + \alpha_j^{t_0}(\tau;\omega) > 0, \\ \frac{(\alpha_i^{t_0})'(\tau;\omega)L(Y_\tau(\omega)) + (\alpha_j^{t_0})'(\tau;\omega)F(Y_\tau(\omega))}{(\alpha_i^{t_0})'(\tau;\omega) + (\alpha_j^{t_0})'(\tau;\omega)} \qquad \text{if } \alpha_i^{t_0}(\tau;\omega) = \alpha_j^{t_0}(\tau;\omega) = 0. \end{cases}$$

if $\tau_i(t_0;\omega) = \tau_j(t_0;\omega)$.

In the last case, thus where $\tau_i = \tau_j$, the function $W_i(\cdot)$ has been obtained by using limiting arguments similar to the one used in eq. (2). For $\alpha_i(\cdot) = \alpha_j(\cdot) = 0$, $W_i(\cdot)$ has been derived by applying L'Hopital's rule. Here one uses the assumptions on existence and positivity of the right-derivative of the atom function $\alpha(\cdot)$.

The definition of simple strategies does not *a priori* exclude the possibility that both firms choose an atom function α that turns out to be inconsistent with the cumulative distribution function G. In equilibrium it should naturally be the case that inconsistencies of this kind do not occur. Therefore, we introduce the notion of α-*consistency.*

Definition 2 *A tuple of simple strategies* $\big((G_i^{t_0}, \alpha_i^{t_0})\big)_{i=1,2}$ *for the subgame starting at* $t_0 \geq 0$ *is* α-consistent *if for* $i = 1, 2$ *it holds that*

$$\alpha_i^{t_0}(t;\omega) - \alpha_i^{t_0}(t-;\omega) \neq 0 \Rightarrow G_i^{t_0}(t;\omega) - G_i^{t_0}(t-;\omega) = \\ = \Big(1 - G_i^{t_0}(t-;\omega)\Big)\frac{\alpha_i^{t_0}(t;\omega)}{\alpha_i^{t_0}(t;\omega) + \alpha_j^{t_0}(t;\omega) - \alpha_i^{t_0}(t;\omega)\alpha_j^{t_0}(t;\omega)}.$$

Definition 2 requires that if for either firm there is a jump in the atom function, then the jump in the cumulative distribution function for both firms should be equal to the probability that the firm invests by playing the game as depicted in Figure 1. Note that if $\alpha_i^{t_0}(t;\omega) - \alpha_i^{t_0}(t-;\omega) \neq 0$ and $\alpha_i^{t_0}(t;\omega) = 1$, then α-consistency implies that $G_i^{t_0}(t;\omega) = 1$.

A Nash equilibrium for the subgame starting at t_0 is then defined as follows.

Definition 3 *Given $\omega \in \Omega$, a tuple of simple strategies $s^* \in S^s(t_0;\omega)$ is a* Nash equilibrium *for the subgame starting at t_0 if s^* is α-consistent and*

$$\forall_{i\in\{1,2\}}\forall_{s_i\in S_i^s(t_0;\omega)} : V_i(t_0,s^*) \geq V_i(t_0,s_i,s^*_{-i}).$$

To define subgame perfect equilibrium, the notion of closed loop strategy is needed.

Definition 4 *A* closed loop strategy *for player $i \in \{1,2\}$ is a collection of simple strategies $\{\big(G_i^t(\cdot;\omega),\alpha_i^t(\cdot;\omega)\big)_{0\leq t<\infty}|(G_i^t(\cdot;\omega),\alpha_i^t(\cdot;\omega)) \in S_i^s(t;\omega)\}$, that satisfies the following intertemporal consistency conditions:*

1. $\forall_{0\leq t\leq u\leq v<\infty} : v = \inf\{\tau > t|Y_\tau = Y_v\} \Rightarrow G_i^t(v;\omega) = G_i^u(v;\omega)$;
2. $\forall_{0\leq t\leq u\leq v<\infty} : v = \inf\{\tau > t|Y_\tau = Y_v\} \Rightarrow \alpha_i^t(v;\omega) = \alpha_i^u(v;\omega)$.

The set of closed loop strategies for player $i \in \{1,2\}$ is denoted by $S_i^{cl}(\omega)$. As before, we define the strategy space to be $S^{cl}(\omega) = \prod_{i\in\{1,2\}} S_i^{cl}(\omega)$.

The intertemporal consistency conditions differ from their deterministic counterparts in Fudenberg and Tirole (1985). Taking their conditions without adaptation implies that one requires that for each firm the probability of having invested before time v starting at time t equals the probability of not having invested before time u starting from t times the probability of having invested before time v starting at time u. This is not a sensible requirement in the stochastic case. Consider for example a situation with one firm and suppose that investment is more optimal for higher values of Y. So, the probability of investment is monotonic in Y. Let $0 \leq t \leq \tau \leq u \leq v$ and suppose that for certain $\omega \in \Omega$, Y is strictly increasing on $[t,\tau]$, strictly decreasing on $(\tau,u]$ and strictly increasing on $(u,v]$. Furthermore suppose that $Y_t = Y_v$, $G^t(t) = 1/4$ and $G^t(\tau) = G^t(u) = G^t(v) = 1/2$. Intertemporal consistency in the Fudenberg and Tirole (1985) sense would then imply that $G^u(v) = 0$, whereas one would expect $G^u(v) = G^t(t) = 1/4$. That is, due to randomness, first passage times are important, not time as such. A subgame perfect equilibrium is now defined in the standard way.

Definition 5 *A tuple of closed loop strategies $s^* \in S^{cl}(\omega)$ is a* subgame perfect equilibrium *if for every $t \in [0,\infty)$, the corresponding tuple of simple strategies $\big(G_i^t(\cdot;\omega),\alpha_i^t(\cdot;\omega)\big)_{i=1,2}$ is a Nash equilibrium.*

Let in the remainder $\omega \in \Omega$ be fixed. For notational convenience we drop ω as an argument.

2.2 Preemption Games

The coordination problem that we want to consider only arises in cases that there exists an incentive to be the first mover. These games are called preemption games, because there is an incentive to preempt the competitor. Apart from the assumptions already made we introduce five additional assumptions:

1. $M(\cdot)$ is continuous;
2. there exists a unique Y_F such that $L(Y) = F(Y) = M(Y)$ for all $Y \geq Y_F$ and $F(Y) > M(Y)$ for all $Y < Y_F$;
3. $L(Y_0) < F(Y_0)$;
4. $F(\cdot)$ is strictly increasing for $Y < Y_F$;
5. $L(\cdot) - F(\cdot)$ is strictly quasi-concave on $[0, Y_F)$.

Note that in Fudenberg and Tirole (1985) $L(\cdot)$, $F(\cdot)$ and $M(\cdot)$ are functions of time, whereas in this model they are functions of the stochastic process Y.

Because of these assumptions there exists a unique $Y_P < Y_F$ such that $L(Y) < F(Y)$ for all $Y < Y_P$, $L(Y_P) = F(Y_P)$ and $L(Y_P) > F(Y_P)$ for all $Y_P < Y < Y_F$. Furthermore, define Y_L to be the maximum location of the discounted leader value.[8] It is assumed that $Y_L > Y_P$, i.e. no war of attrition arises. For the remainder define the following stopping times $T_P^t = \inf(u \geq t | Y_u \geq Y_P)$ and $T_F^t = \inf(u \geq t | Y_u \geq Y_F)$.[9]

In case firms are identical, coordination on a non-symmetric equilibrium is hard to establish in a noncooperative setting. Therefore, we concentrate

[8] In case of exogenous firm roles, i.e. where the firms know beforehand which one of them is the first investor, it is optimal for the leader to invest at the moment that $Y_t = Y_L$, when $Y_0 \leq Y_L$.

[9] The existence of Y_L and Y_F need not be assumed but can be obtained from solving optimal stopping problems. For the general semimartingale case existence is not readily guaranteed. If the solution to eq. (1) satisfies the Markov property and some other technical conditions, existence and uniqueness of Y_L and Y_F can be obtained (see Shiryaev (1978)).

on equilibria that are supported by symmetric strategies.[10] In the appendix the following theorem is proved.

Theorem 1 *A symmetric subgame perfect equilibrium is given by the tuple of closed-loop strategies* $(\bar{s}^t)_{0\le t<\infty} = \big((G_1^t,\alpha_1^t),(G_2^t,\alpha_2^t)\big)_{0\le t<\infty}$, *where for* $i=1,2$ *and* $t\ge 0$

$$G_i^t(u) = \begin{cases} 0 & \textit{if } u < T_P^t, \\ \frac{L(Y_{T_P^t})-M(Y_{T_P^t})}{L(Y_{T_P^t})-2M(Y_{T_P^t})+F(Y_{T_P^t})} & \textit{if } T_P^t \le u < T_F^t, \\ 1 & \textit{if } u \ge T_F^t, \end{cases} \tag{4}$$

$$\alpha_i^t(u) = \begin{cases} 0 & \textit{if } u < T_P^t, \\ \frac{L(Y_{T_P^t})-F(Y_{T_P^t})}{L(Y_{T_P^t})-M(Y_{T_P^t})} & \textit{if } T_P^t \le u < T_F^t, \\ 1 & \textit{if } u \ge T_F^t. \end{cases} \tag{5}$$

3 Dixit and Pindyck (1994, Section 9.3) Model

In this section we apply the equilibrium concept introduced in the previous section to the model analyzed in Dixit and Pindyck (1994, Section 9.3).[11] This model considers an investment project with sunk costs $I>0$. After the investment is made, the firm can produce one unit of product at any point in time. Since the number of firms is two, market supply is $Q\in\{0,1,2\}$. The price is given by market demand $D(Q)$ multiplied by a shock Y which satisfies a geometric Brownian motion process. So, $P=YD(Q)$. In Figure 2 the three value functions are plotted. If the leader invests at $Y<Y_F$ the follower's value is maximized when the follower invests at Y_F.

The equilibrium outcome depends on the value Y_0, which is the initial value of $(Y_t)_{0\le t<\infty}$. To determine the outcomes, three regions have to be distinguished. The first region is defined by $Y_0\le Y_P$. If we restrict ourselves to symmetric equilibrium strategies, it follows from Theorem 1 that there

[10] The focus on symmetric strategies is not a priori clear. There is a growing literature on equilibrium selection started by Harsanyi and Selten (1988) that shows that in games with symmetric players, asymmetric equilibria can survive (cf. Kandori et al. (1993) and Young (1998)).

[11] This model is a simplified version of the Smets (1991) model and is also extensively analyzed in Nielsen (2002), Huisman and Kort (1999), and Huisman (2001).

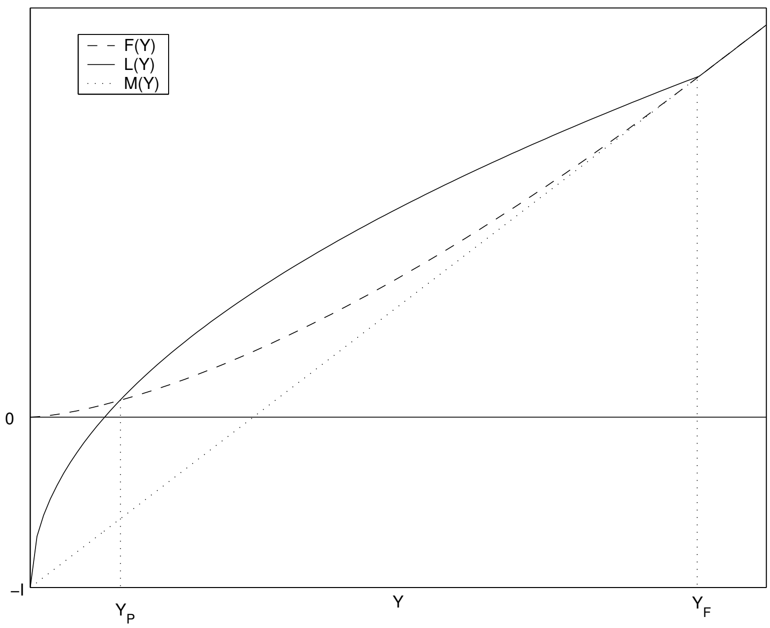

Figure 2: Value functions in the Dixit and Pindyck (1994, Section 9.3) model.

are three possible equilibrium outcomes. In the first outcome firm 1 is the leader and invests at Y_P and firm 2 is the follower and invests at Y_F. Note that in this case the symmetric equilibrium strategies lead to an asymmetric equilibrium outcome. In this particular outcome firm 1 is the first one to invest in the repeated game depicted in Figure 1. The second outcome is the symmetric counterpart: firm 2 is the leader and invests at Y_P and firm 1 is the follower and invests at Y_F. The third possibility is that both firms invest simultaneously at Y_P, i.e. both firms invest in the same round of the repeated game depicted in figure 1. However, this equilibrium arises with probability zero. To see this, note that since a geometric Brownian motion has continuous sample-paths and at Y_P it holds that $L(Y_P) = F(Y_P)$, it can be concluded from eq. (5) that $\alpha(\tau) = \alpha(T_P) = 0$. Now from eq. (3) it directly follows that the probability of joint investment is zero. Using the same kind of limiting argument as in eq. (2), we obtain that the probability that player i is the only one who invests at $t = \tau$, $\mathbb{P}(i|\tau)$, is given by

$$\mathbb{P}(i|\tau) = \frac{\alpha_i(\tau)(1-\alpha_j(\tau))}{\alpha_i(\tau)+\alpha_j(\tau)-\alpha_i(\tau)\alpha_j(\tau)}. \tag{6}$$

By applying L'Hopital's rule, while imposing symmetry, one then obtains that in equilibrium

$$\mathbb{P}(1|\tau) = \mathbb{P}(2|\tau) = \frac{\alpha_i'(T_P)}{\alpha_i'(T_P) + \alpha_j'(T_P)} = \frac{1}{2}. \tag{7}$$

So, with equal probability either firm becomes the leader. Due to the definition of Y_P, i.e. $L(Y_P) = F(Y_P)$, it follows that the expected value of each player equals

$$V_i(\tau, \bar{s}^\tau) = W_i(\tau, \bar{s}^\tau) = \tfrac{1}{2}(L(Y_P) + F(Y_P)) = F(Y_P).$$

In the second region it holds that $Y_P < Y_0 < Y_F$. There are three possible outcomes. Since $L(Y)$ exceeds $F(Y)$ in case $Y \in (Y_P, Y_F)$, it can be obtained from eq. (5) that $\alpha_i(0) > 0$. Due to $\tau = 0$ and eq. (6) we know that with probability $\frac{\alpha_i(0)(1-\alpha_j(0))}{\alpha_i(0)+\alpha_j(0)-\alpha_i(0)\alpha_j(0)}$ firm i invests at $t = 0$ and firm j invests at T_F. Equation (2) implies that the firms invest simultaneously at $t = 0$ with probability $\frac{\alpha_1(0)\alpha_2(0)}{\alpha_1(0)+\alpha_2(0)-\alpha_1(0)\alpha_2(0)} > 0$, leaving them with a low value of $M(Y_0) < F(Y_0)$. The expected payoff of each firm then equals

$$\begin{aligned} &W_i(\tau, \bar{s}^\tau) = \\ &\quad \frac{\alpha_i(0)(1-\alpha_j(0))L(Y_0) + \alpha_j(0)(1-\alpha_i(0))F(Y_0) + \alpha_i(0)\alpha_j(0)M(Y_0)}{\alpha_i(0) + \alpha_j(0) - \alpha_i(0)\alpha_j(0)} \\ &\quad = F(Y_0), \end{aligned}$$

where the latter equation follows from eq. (5). Since there are first mover advantages in this region, each firm is willing to invest with positive probability. However, this implies via eq. (2) that the probability of simultaneous investment, leading to a low payoff $M(Y_0)$, is also positive. Since the firms are both assumed to be risk neutral, they will fix the probability of investment such that their expected value equals $F(Y_0)$, which is also their payoff if they let the other firm invest first.

When Y_0 is in the third region $[Y_F, \infty)$, according to eqs. (2) and (5) the outcome exhibits joint investment at Y_0. The value of each firm is again $F(Y_0)$.

4 Conclusion

At present, only a few contributions deal with the effects of strategic interactions on the option value of waiting associated with investments under uncertainty.[12] However, due to the importance of studying the topic of investment under uncertainty in an oligopolistic setting, it can be expected that more papers will appear in the immediate future. This paper proposes a method to solve a coordination problem frequently occurring in such oligopoly models. This is especially important, since in those papers that already exist, this coordination problem is not treated in a satisfactory way. For instance, Weeds (2002) explicitly makes the assumption that the stochastic process always starts at a value lower than the preemption value, i.e. $Y_0 < Y_P$. Furthermore, her result does not hold for stochastic processes with non-continuous sample-paths. In Nielsen (2002), Grenadier (1996) and Dutta et al. (1995) it is assumed that at the preemption point only one firm can succeed in investing. There are two reasons for this assumption to be unsatisfactory. Firstly, in all these contributions the firms are assumed to be identical so there is no *a priori* ground for this assumption. Secondly, the firms can invest simultaneously if it is optimal for both, so it seems unsatisfactory to exclude this possibility simply because of the fact that it is not optimal for both firms to invest.

The reason for our outcomes to be more realistic is the following. When there is an incentive to be the first to invest ($L > F > M$) both firms are willing to take a risk. Since they are both assumed to be risk neutral they will risk so much that their expected value equals F, which equals their payoff if they allow the other firm to invest first. Employing the results of Section 2 learns that in this case there is a positive probability that both firms invest exactly at the same time, leaving them with the low payoff M. In our framework, this risk is explicitly taken into account by the firms, as opposed to most contributions in this field. In order to obtain realistic conclusions in game theoretic real option models it is inevitable that all aspects concerning the option effect and the strategic aspect should be taken into account.

[12]For a survey see Grenadier (2000).

Appendix

Proof of Theorem 1

First notice that since Y is a semimartingale and hence right-continuous, continuity of $L(\cdot)$, $F(\cdot)$ and $M(\cdot)$ implies that G and α are right-continuous. Also, for each $\omega \in \Omega$ the strategy $(\bar{s}^t)_{0 \leq t < \infty}$ satisfies the intertemporal consistency conditions of Definition 4. Hence, the closed loop strategies are well-defined. Furthermore, for all t, $\bar{s}^t$ satisfies α-consistency.

Let $t \geq 0$. The expected value of $\bar{s}^t$ for player i can be obtained by considering two cases.

1. $Y_\tau \geq Y_F$
   This implies $\alpha_i^t(\tau) = \alpha_j^t(\tau) = 1$. The expected value is then given by $V_i(t, \bar{s}^t) = M(Y_\tau) = F(Y_{T_P^t})$.

2. $Y_P < Y_\tau < Y_F$
   This implies $\alpha_i^t(\tau) = \alpha_j^t(\tau) = \alpha \in (0, 1]$. The expected value is then given by

$$\begin{aligned}
V_i(t, \bar{s}^t) &= W_i(\tau, \bar{s}^t) \\
&= \frac{1}{2-\alpha}\Big[(1-\alpha)L(Y_\tau) + (1-\alpha)F(Y_\tau) + \alpha M(Y_\tau)\Big] \\
&= F(Y_{T_P^t}).
\end{aligned}$$

3. $Y_\tau = Y_P$
   This implies $\alpha_i^t(\tau) = \alpha_j^t(\tau) = \alpha(\tau^t) = 0$. The expected value is given by

$$\begin{aligned}
V_i(t, \bar{s}^t) &= W_i(\tau, \bar{s}^t) \\
&= \frac{\alpha'}{2\alpha'}[L(Y_\tau) + F(Y_\tau)] \\
&= F(Y_{T_P^t}),
\end{aligned}$$

   since Y_P is defined such that $L(Y_P) = F(Y_P)$.

So, $V_i(t, \bar{s}^t) = F(Y_{T_P^t})$.

Take any strategy for player i, (G_i^t, α_i^t) and denote $\tilde{s}^t = \big((G_i^t, \alpha_i^t), \bar{s}_j^t\big)$. Then it holds that

$$V_i(t, \tilde{s}^t) = \int_t^{\tau-} L(Y_s)dG_i^t(s) + \big(1 - G_i^t(\tau-)\big)W_i(\tau, \tilde{s}^t).$$

Consider the following cases

1. $\tau_i < \tau_j$
   In this case $W_i(\tau, \tilde{s}^t) = L(Y_\tau) = L(Y_{\tau_i})$. Hence, $V^i(t, \tilde{s}^t) \leq F(Y_{T_P^t})$, since for all $u < T_P^t (= \tau_j)$ it holds that $F(Y_u) > L(Y_u)$.

2. $\tau_i > \tau_j$
   For this case it holds that
   $$\begin{aligned}
   \big(1 - G_i^t(\tau-)\big)W_i(\tau, \tilde{s}^t) &= \big(1 - G_i^t(\tau-)\big)W_i(\tau_j, \tilde{s}^t) \\
   &= \big(1 - G_i^t(\tau-)\big)\frac{G_i^t(\tau) - G_i^t(\tau-)}{1 - G_i^t(\tau-)}\Big[(1 - \alpha_j^t(\tau))L(Y_\tau) + \alpha_j^t(\tau)M(\tau)\Big] \\
   &\quad + \big(1 - G_i^t(\tau-)\big)\frac{1 - G_i^t(\tau)}{1 - G_i^t(\tau-)}F(Y_\tau) \\
   &= \big(G_i^t(\tau) - G_i^t(\tau-)\big)F(Y_\tau) + \big(1 - G_i^t(\tau)\big)F(Y_\tau) \\
   &= \big(1 - G_i^t(\tau-)\big)F(Y_\tau).
   \end{aligned}$$
   So,
   $$\begin{aligned}
   V_i(t, \tilde{s}^t) &= \int_t^{T_P^t -} L(Y_s)dG_i^t(s) + \big(1 - G_i^t(T_P^t -)\big)F(Y_{T_P^t}) \\
   &\leq F(Y_{T_P^t}).
   \end{aligned}$$

3. $\tau_i = \tau_j$, $a_i \equiv \alpha_i^t(\tau) \neq \alpha_j^t(\tau) \equiv a_j$
   In this case
   $$\begin{aligned}
   W_i(t, \tilde{s}^t) &= \frac{a_i(1 - a_j)L(Y_\tau) + a_j(1 - a_i)F(Y_\tau) + a_i a_j M(Y_\tau)}{a_i + a_j - a_i a_j} \\
   &= \frac{1}{a_i(L(Y_\tau) - M(Y_\tau)) + (1 - a_i)(L(Y_\tau) - F(Y_\tau))} \times \\
   &\quad \times \Big[a_i(F(Y_\tau) - M(Y_\tau))L(Y_\tau) + (1 - a_i)(L(Y_\tau) - F(Y_\tau))F(Y_\tau) \\
   &\quad + a_i(L(Y_\tau) - F(Y_\tau))M(Y_\tau)\Big] \\
   &= F(Y_\tau).
   \end{aligned}$$
   So,
   $$\begin{aligned}
   V_i(t, \tilde{s}^t) &= \int_t^{\tau -} L(Y_s)dG_i^t(s) + \big(1 - G_i^t(\tau-)\big)F(Y_\tau) \\
   &\leq F(Y_{T_P^t}).
   \end{aligned}$$

Therefore, we conclude $V_i(t, \tilde{s}^t) \leq V_i(t, \bar{s}^t)$. Hence, $\bar{s}^t$ is a Nash equilibrium for the subgame starting at time t. □

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