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COMPETITIVE EQUILIBRIA IN ECONOMIES WITH MULTIPLE DIVISIBLE AND MULTIPLE DIVISIBLE COMMODITIES

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Competitive Equilibria in Economies with Multiple Divisible and Multiple Indivisible Commodities

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Abstract

In this paper we consider a general equilibrium model with a finite number of divisible and indivisible commodities. In models with indivisibilities it is typically assumed that there is only one perfectly divisible good, which serves as money. The presence of money in the model is used to transfer the value of certain amounts of indivisible goods. For such economies with one divisible commodity Danilov et al. showed the existence of a general equilibrium if the individual demands and supplies belong to a same class of discrete convexity. For economies with multiple divisible goods and money van der Laan et al. proved existence of a general equilibrium if the divisible goods are produced out of money using a linear production technology and no other producers are present in the model.

In the models to be presented in this paper we allow for multiple divisible commodities and a finite number of producers with non-increasing returns to scale technologies. Convexity is replaced by pseudoconvexity, while the indivisible parts of individual demands and supply should belong to some class of discrete convexity. In the first model money is present. Money is strictly desired by the consumers like in the other models, is indispensable for production and enough money should be present in the economy. To guarantee existence of a general equilibrium individual demands and supplies should be products of divisible and indivisible parts. In the second model there is no money, but at least one linear production technology is present in order to produce the divisible goods. Individual endowments being sufficiently large for production guarantee the existence of a competitive equilibrium.

Key words: indivisible commodities, divisible commodities, money, competitive equilibrium

JEL-code: D2, D4, D5, D6.
1 Introduction

Indivisible commodities have constituted a prominently important part of commercial commodities in most of the markets. Typical indivisible commodities are, to name a few, houses, cars, employees, airplanes, ships, trains, computers, machinery, and arts. Nowadays, even many divisible commodities are sold in indivisible quantities such as oil being sold in barrel as its smallest unit. Modelling economies with indivisibilities is therefore meaningful and realistic. However, studying such discrete economies stands in general a daunting challenge; see for example Koopmans and Beckman [12], Debreu [5], Henry [9], Kelso and Crawford [11], Gale [6], Quinzii [16], Shapley and Scarf [20], and Scarf [17, 18, 19], and more recently Kaneko and Yamamoto [10], Yamamoto [22], Shell and Wright [21], Garratt [7], Garratt and Qin [8], Ma [15], Bevia et al. [1], Bikhchandani and Mamer [2], van der Laan et al. [13], Yang [24]. In Danilov et al. [4] it was shown that discrete convex analysis is an appropriate tool to deal with indivisibles. Specifically, economies with indivisibles, money and no other perfectly divisible goods can be studied as continuous economies with divisible goods when individual demands and supplies for the indivisible goods belong to a same class of discrete convexity. Van der Laan et al. [14] consider economies with multiple indivisible goods, money and multiple indivisible goods. In their model the divisible goods are being produced from money by a linear production technology while other producers with non-increasing returns to scale technologies are not allowed in the model.

In this paper we extend the results of Danilov et al. [4] and van der Laan et al. [14] and obtain more general existence results in economies with both multiple indivisibles and multiple divisible goods and allowing for a finite number of producers with constant or decreasing returns to scale technologies. In the first model we require that one of the perfectly divisible goods is like money. As in all other models with money, there should be enough money owned by the consumers and consumers should have a strong desire for it. On the production side of the economy we assume that money is indispensable for production. Concerning discrete convexity it is assumed that the money equivalences of both the indifference levels and the cost functions are pseudoconvex and that the individual demands and supplies are the products of divisible and indivisible parts, the latter all belonging to a same class of discrete convexity. The former will guarantee that the convexified economy has a competitive equilibrium and the latter that this equilibrium induces a competitive equilibrium of the discrete economy.

In the second model, contrary to all the other models, money is not assumed to be present in the economy. Instead of money there is some producer with a production technology being linear for the divisible goods. Initial endowments are large enough for production and the divisible goods are all desirable. Preferences and production sets are pseudoconvex and the individual demands and supply for the invisibles should all belong
to a same class of discrete convexity. The former again guarantees that the convexified economy has a competitive equilibrium and the latter that this equilibrium induces a competitive equilibrium of the discrete economy.

The plan of the paper is as follows. In Section 2 the economic model with multiple divisible and multiple indivisible goods is introduced. In Section 3 the concept of discrete convexity is reviewed. Section 4 discusses the model with money and Section 5 the model without money. The existence proofs are given in Section 6.

2 Multiple divisible and indivisible goods

In this paper we deal with the problem of the existence of a competitive equilibrium in an exchange economy $E$ with consumption and production and with multiple divisible and multiple indivisible commodities. There is a finite set $K$ of discrete (indivisible) commodities and a finite set $M$ of perfectly divisible commodities. Bundles of commodities are denoted by elements of the set $\mathbb{Z}^K \times \mathbb{R}^L$. The set $J$ denotes the finite set of producers and $H$ denotes the finite set of consumers. A producer $j \in J$ is described by its input-output production set $C_j \subset \mathbb{Z}^K \times \mathbb{R}^L$. A vector $(Y, y) \in C_j$ means that producer $j$, $j \in J$, is able to produce the output vector $(Y, y)^+$, being the positive part of $(Y, y)$, from the input vector $-(Y, y)^-$, being minus the negative part of $(Y, y)$. Standard assumptions on $C_j$ are $C_j \cap \mathbb{Z}_+^K \times \mathbb{R}_+^L = \{0^K+L\}$, $C_j = C_j - (\mathbb{Z}_+^K \times \mathbb{R}_+^L)$ and $C_j$ is a closed set, for all $j \in J$.

The preferences of consumer $h$, $h \in H$, are described by a preference relation $\preceq_h$, being a monotone, continuous weak order on the consumption set $\mathbb{Z}_+^K \times \mathbb{R}_+^L$. Consumer $h \in H$ has a vector of initial endowments $\omega_h = (W_h, w_h) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L$ and is endowed with shares in the production: $\theta_{jh} \geq 0$, $j \in J$, is consumer $h$’s share in the production of producer $j$, where $\sum_{h \in H} \theta_{jh} = 1$ for all $j \in J$.

Agents are assumed to be price takers. Given a price vector $p$, being a linear functional on $\mathbb{R}^K \times \mathbb{R}^L$, producer $j \in J$ solves the following maximization program:

$$\max_{(Y, y) \in C_j} p(Y, y). \tag{1}$$

The number $\pi_j(p) = \max_{(Y, y) \in C_j} p(Y, y)$ is the profit of producer $j$ and

$$S_j(p) = \text{Argmax}_{(Y, y) \in C_j} p(Y, y)$$

is producer $j$’s supply at price $p$. Consumer $h \in H$ seeks a best element with respect to his preference $\preceq_h$ in the budget set

$$B_h(p) = \{(X, x) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L \mid p(X, x) \leq \beta_h(p)\},$$
where at price vector $p$ consumer $h$’s income, $\beta_h(p)$, is defined by

$$\beta_h(p) = p(W_h, w_h) + \sum_{j \in J} \theta_{jh} \pi_j(p).$$

The demand of consumer $h$, $h \in H$, is the set $D_h(p)$ of best elements in the set $B_h(p)$ with respect to the preference $\preceq_h$.

An equilibrium is a tuple $(p, (X_h, x_h)_{h \in H}, (Y_j, y_j)_{j \in J})$ of a price vector $p$, individual demands $(X_h, x_h) \in D_h(p)$, $h \in H$, and individual supplies $(Y_j, y_j) \in S_j(p)$, $j \in J$, such that all markets clear:

$$\sum_{h \in H} (X_h, x_h) = \sum_{j \in J} (Y_j, y_j) + \sum_{h \in H} (W_h, w_h).$$

For the existence of an equilibrium we should require something like discrete convexity in the model, which replaces the usual convexity and continuity assumptions. The next section introduces the notion of discrete convexity, as was developed by Danilov and Koshevoy [3].

3 Discrete convexity

In this section a survey of the results by Danilov and Koshevoy [3] about discrete convexity is given. A first idea on convexity of discrete sets is to consider the convex hull $\text{co}(X)$ of a subset $X \subset \mathbb{Z}^K$, and require that $X = \text{co}(X) \cap \mathbb{Z}^K$. Such sets are called pseudoconvex. The reason, why such sets are called pseudoconvex and not convex, is that they may not satisfy the separation property, the cornerstone of Convex Analysis (and therefore, of Equilibrium Analysis). Consider the following example.

**Example 1.** Consider the two two-points pseudoconvex sets $A = \{(0,0), (1,1)\}$ and $B = \{(0,1), (1,0)\}$. These sets do not intersect, but their convex hulls intersect at the interior point $(1/2, 1/2)$. Thus the sets can not be separated by a linear functional on $\mathbb{R}^2$. 

The discrete convexity theory is constituted of classes of subsets of $\mathbb{Z}^K$ that are closed under Minkowski summation. The Minkowski sum of two subsets $A$ and $B$ in $\mathbb{R}^K$ is given by $A + B = \{a + b | a \in A, b \in B\}$.

**Definition 3.1** A class $\mathcal{D}$ of subsets of $\mathbb{Z}^K$ is a class of discrete convex sets if the following properties hold:

$DC1$. For any $A \in \mathcal{D}$ it holds that $A$ is pseudoconvex, $-A \in \mathcal{D}$, and $\text{co}(A)$ is a polyhedron;

$DC2$. For any $A$ and $B \in \mathcal{D}$ it holds that $A + B \in \mathcal{D}$.

One can easily check that sets of a class of discrete convexity $\mathcal{D}$ are well behaved with respect to the separation property. In fact, let $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$. Then $0^K \notin \mathcal{D}$.
A + (−B), A + (−B) ∈ \mathcal{D}, and so 0^K \not\in \text{co}(A + (−B)). Since the convex hull commutes with the Minkowski sum, we have 0^K \not\in \text{co}(A) + \text{co}(−B). Hence, \text{co}(A) and \text{co}(B) can be separated and so A and B.

In the previous example, with \(A = \{(0, 0), (1, 1)\}\) and \(B = \{(0, 1), (1, 0)\}\), we have 0^K \not\in A + (−B), but A + (−B) is not a pseudoconvex set, and so the convex hulls \text{co}(A) and \text{co}(B) cannot be separated. Therefore, there does not exist a class of discrete convexity which contains both sets.

Classes of discrete convexity are constructed as integer points of integral polyhedra. A polyhedron \(P \subset \mathbb{R}^K\) is said to be an integral polyhedron if \(P = \text{co}(P \cap \mathbb{Z}^K)\).

Let \(\mathcal{P}\) be a class of polyhedra with the following properties:

DCP1. Any polyhedron \(P \in \mathcal{P}\) is integral.

DCP2. For any polyhedra \(P, Q \in \mathcal{P}\), we have \(P \pm Q \in \mathcal{P}\) and

\[
(P \pm Q) \cap \mathbb{Z}^K = (P \cap \mathbb{Z}^K) \pm (Q \cap \mathbb{Z}^K).
\]

A class of polyhedra \(\mathcal{P}\) with properties DCP1 and DCP2 is said to be a class of discrete convexity. Because taking the convex hull commutes with adding up and subtracting sets and the sum of polyhedra is again a polyhedron, for any class \(\mathcal{P}\) of discrete convex polyhedra it holds that the class \(\mathcal{D}\) of subsets of \(\mathbb{Z}^K\) of the form \(P \cap \mathbb{Z}^K, P \in \mathcal{P}\), satisfies DC1 and DC2.

When \(|K| = 1\), the class of integral polyhedra, being segments with integral endpoints, is the only class of discrete convexity. This is, of course, not the case in higher dimensions.

**Example 2.** Hexagons. Consider a class \(\mathcal{H}\) of polyhedra in \(\mathbb{R}^2\), which consists of hexagons defined by inequalities \(a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, c \leq x_1 + x_2 \leq d\) with integral \(a_1, a_2, b_1, b_2, c\) and \(d\) (such a hexagon can be degenerated to a polyhedron with less than six vertices). It is easy to check that the vertices of such a hexagon are integral. Because the intersection of hexagons is again a hexagon, we conclude that \(\mathcal{H}\) is a class of discrete convexity.

Observe, that the edges of the hexagons in Example 2 are parallel to the vectors \(e_1, e_2\) or \(e_1 - e_2\). These vectors have the following property: any pair of these vectors form a basis of the lattice \(\mathbb{Z}^2\). As we have seen in Example 1, if a class of integral polyhedra in \(\mathbb{R}^2\) contains polyhedra having edges being parallel to \(e_1 - e_2\) and to \(e_1 + e_2\), such a class fails to be a class of discrete convexity. The reason is that the pair of vectors \(e_1 - e_2\) and \(e_1 + e_2\) does not form a basis of \(\mathbb{Z}^2\). For example, points of the form \((2n + 1)e_1, n \in \mathbb{Z}\), can not be obtained as combinations of vectors \(e_1 - e_2\) and \(e_1 + e_2\) with integer coefficients. The property that every set of \(|K|\) linearly independent primitive vectors being parallel edges
of polytopes of some class of polyhedra forms a basis of the abelian group (lattice) $\mathbb{Z}^K$ is the decisive property for a class of polyhedra to be a class of discrete convexity. A collection $\mathcal{R}$ of vectors of $\mathbb{R}^K$ is said to be a \textit{unimodular system} if, for any subset $R \subset \mathcal{R}$, the abelian group $\mathbb{Z}(R) = \{ \sum_i a_i r_i \mid r_i \in R, a_i \in \mathbb{Z} \}$ coincides with the lattice $\mathbb{R}(R) \cap \mathbb{Z}^K$, where $\mathbb{R}(R) = \{ \sum_i a_i r_i \mid r_i \in R, a_i \in \mathbb{R} \}$. Now we have the following result (see Danilov and Koshevoy [3]).

**Theorem 3.2** Let $\mathcal{P}$ be a collection of pointed integral polyhedra of $\mathbb{R}^K$. Let $\mathcal{R}(\mathcal{P})$ denote the set of vectors in $\mathbb{Z}^K$ being parallel to edges of polyhedra of $\mathcal{P}^1$. Then $\mathcal{P}$ is a class of discrete convexity if and only if $\mathcal{R}(\mathcal{P})$ is a unimodular system.

The next example is a well-known unimodular system. 

**Example 3.** The set $\mathcal{A}_K := \{ \pm e_i, e_i - e_j, i, j \in K \}$ of vectors of $\mathbb{Z}^K$ is a unimodular system. Because $\mathcal{A}_K$ contains the standard basis, we need to show that any $|K|$ linear independent vectors of $\mathcal{A}_K$ form a basis of $\mathbb{Z}^K$. Let $B \subset \mathcal{A}_K$ be a basis of $\mathbb{R}^K$. Check that $B$ is a basis of $\mathbb{Z}^K$. One of $\pm e_i$, $i \in K$, belongs to $B$, otherwise $B$ is a subset of the hyperplane $\sum_{i \in K} x_i = 0$, and, hence, $B$ cannot be a basis of $\mathbb{R}^K$. Let $e_1 \in B$. If none of the vectors $\pm(e_i - e_1)$ belongs to $B$, then the set $B \setminus \{ e_1 \}$ is a subspace of the hyperplane $\{ x \in \mathbb{R}^K \mid x_1 = 0 \}$. By induction $B \setminus \{ e_1 \}$ forms a basis of $\mathbb{Z}^K \setminus \{ e_1 \}$. Hence $B$ is a basis of $\mathbb{Z}^K$. If $e_j - e_1$ belongs to $B$ for some $j \neq 1$, then, changing $e_j - e_1$ to $e_j = e_1 + (e_j - e_1)$, we obtain a new basis $B'$. Obviously, $B$ and $B'$ are either both bases or both not bases of $\mathbb{Z}^K$. Repeating the same argument, we may assume that none of the vectors $\pm (e_i - e_1)$ belongs to $B'$. Therefore, $B'$ is a basis of $\mathbb{Z}^K$, and, hence, so is $B$. \hfill \Box

The discrete convexity corresponding to the unimodular system of Example 3 is called \textit{polymatroidal} discrete convexity. It is interesting to note here, that nearly all known existence results with indivisibles fit into the polymatroidal discrete convexity (see Danilov et al. [4]).

## 4 The model with money

One way to deal with the problem of existence of a competitive equilibrium in an economy $\mathcal{E}$ with multiple indivisible and divisible commodities is to assume that among the set of perfectly divisible goods there is a commodity which can play the role of money. Specifically, we assume that there exists a commodity which is desirable and indispensable for production. Let $K$ denote again the set of indivisibles and let $L$ denote the set of perfectly indivisible nonmoney commodities. A bundle of commodities, with money as the last component, is denoted by an element in $\mathbb{Z}^K \times \mathbb{R}^L \times \mathbb{R}$. The production set of producer $j$, $j \in J$, \footnote{A vector $r$ belongs to $\mathcal{R}(\mathcal{P})$ if and only if there is a polyhedron $P \in \mathcal{P}$ which has an edge of the form $[x, x + ar]$ for some $a \in \mathbb{N}$ or $\{ y \mid y = x + br, b \in \mathbb{R} \}$ for some $x \in \mathbb{Z}^K$.}
is then a set \( C_j \subset \mathbb{Z}^K \times \mathbb{R}^L \times \mathbb{R}, \) where \((Y, y, m) \in C_j\) means that using the input vector \(-(Y, y, m)^-\) the producer \( j \) is able to produce the output vector \((Y, y, m)^+\). We assume \( C_j \cap \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ = \{0\}, \) \( C_j = C_j - (\mathbb{Z}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+)\) and \( C_j \) is a closed set, for all \( j \in J \).

Equivalently, we can represent the production set as \( \text{total endowment of all (nonmoney) goods is strictly positive: \( h \)} \text{Consumer function.} \)

\[ \text{Assumption M2} \]

For every consumer \( h, h \in H \), there exists a number \( \theta_{jh} \geq 0, j \in J \), with \( \sum_{h \in H} \theta_{jh} = 1 \) for any \( j \in J \). Endowments and preferences are such that every consumer prefers his initial bundle to any bundle without money. Moreover, the total initial endowments are strictly positive.

**ASSUMPTION M1.** For every agent \( h \in H \) there holds

\[ (W_h, w_h, m_h) \succ_h (Y, y, 0) \quad \forall (Y, y) \in \mathbb{Z}^K_+ \times \mathbb{R}^L. \]

The total endowment of all (nonmoney) goods is strictly positive: \( \sum_{h \in H} (W_h, w_h) > (0^K, 0^L) \).

Without loss of generality we assume that \( \sum_{h \in H} (W_h, w_h) \geq (1^K, 1^L) \). Next, we assume that every consumer has a strong desire for money.

**ASSUMPTION M2.** For any agent \( h \in H \) and any bundle \( (X, x, m) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \), there exists a certain amount of money \( m' \in \mathbb{R}_+ \) such that

\[ (0^L, 0^K, m') \succ_h (X, x, m). \]

Assumptions M1 and M2 together imply that any bundle of goods can be compared with money. Namely, let a preference relation \( \preceq \), being a continuous monotone weak order on \( \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \), satisfy Assumptions M1 and M2. Then, for any \( m \geq 0 \) and \( (X, x) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \), there exists a number \( q^m(X, x) \) such that \( (0^L, 0^K, m) \sim (X, x, q^m(X, x)) \) holds. The graph of \( q^m \) is exactly an indifference level of \( \preceq \) passing through the point \( (0^L, 0^K, m) \). Therefore, we can represent the preference \( \preceq \) by the family of functions \( q^m, m \geq 0 \), on \( \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \).

Denote by \( q^m_h, m \geq 0 \), the corresponding family of functions for consumer \( h, h \in H \), having preference \( \preceq_h \). We assume that all these functions are pseudoconvex and that all cost functions are pseudoconcave. A function \( f : \mathbb{Z}^K \times \mathbb{R}^L \rightarrow \mathbb{R} \) is called pseudoconvex if for any \( p : \mathbb{Z}^K \times \mathbb{R}^L \rightarrow \mathbb{R} \) and any \( (X, x) \in \text{co}(\text{Argmax}(p(Z, z) - f(Z, z))) \) there exists \( x' \in \mathbb{R}_+ \) such that \( (X, x') \in \text{Argmax}(p(Z, z) - f(Z, z)) \).

**ASSUMPTION M3.** The functions \( q^m_h, m \in \mathbb{R}_+, h \in H \), and the functions \( -c_l, l \in L \), are pseudoconvex functions.
This assumption allows us to convexify the economy, while no integer points are added to the integer part of individual demand and supply. We normalize prices such that the price of money is equal to 1. Thus, a price system is a linear functional $p: \mathbb{Z}^K \times \mathbb{R}^L \rightarrow \mathbb{R}$. Given a price $p$, producer $j \in J$ solves

$$
\max_{(Y,y,m) \in C_j} (p(Y,y) + m).
$$

(3)

For $j \in J$, denote $\pi_j(p) = \max_{(Y,y,m) \in C_j} p(Y,y) + m$, $S_j(p) = \text{Argmax}_{(Y,y,m) \in C_j} p(Y,y) + m$, and $S'_j(p) = \{(Y,y) \in \mathbb{Z}^K \times \mathbb{R}^L | \exists m \text{ such that } (Y,y,m) \in S_j(p)\}$, being the projection of $S_j(p)$ along the money coordinate.

Consumer $h \in H$ seeks a best element with respect to his preference $\preceq_h$ in the budget set

$$
B_h(p) = \{(X, x, m) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \mid p(X, x) + m \leq \beta_h(p)\},
$$

where the income, $\beta_h(p)$, is defined by

$$
\beta_h(p) = p(W_h, w_h) + m_h + \sum_{j \in J} \theta_{jh} \pi_j(p).
$$

Thus, given price $p$, the demand of consumer $h$ is the set $D_h(p)$ of best elements in the set $B_h(p)$ with respect to the preference $\preceq_h$. Denote $D'_h(p) := \{(Z, x) \mid \exists m \geq 0 \text{ such that } (Z, x, m) \in D_h(p)\}$, being the projection of $D_h(p)$ along the money coordinate.

An equilibrium is a tuple $(p, (X_h, x_h)_{h \in H}, (Y_j, y_j)_{j \in J})$ of a price $p$, individual demands $(X_h, x_h) \in D'_h(p)$, $h \in H$, and individual supplies $(Y_j, y_j) \in S'_j(p)$, $j \in J$, such that $\sum_{h \in H}(X_h, x_h) = \sum_{j \in J}(Y_j, y_j) + \sum_{h \in H}(W_h, w_h)$. Because preferences are strictly monotone in money and cost functions are monotone, it follows from Walras’ Law that there is also money balance.

If there is only one divisible good, money, then we are in the case of economies with only indivisibles and money as has been studied by Danilov et al. [4]. They have shown that discrete convexity of individual demands and supplies is a sufficient condition for the existence of a competitive equilibrium. In case other divisible goods than just money are present in the economy this condition becomes more complicated. Let us first convexify the economy.

The convexification $\text{co}(E)$ of the economy $E$ described above is an economy with commodity space $\mathbb{R}^K \times \mathbb{R}^L \times \mathbb{R}$, initial endowments $(W_h, w_h, m_h)$ and preferences $\text{co}(\preceq_h), h \in H$, and cost functions $\text{co}(c_l), l \in L$. The preference $\text{co}(\preceq)$ defined on $\mathbb{R}^K_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+$ is the convexification of a preference $\preceq$ defined on $\mathbb{Z}^K \times \mathbb{R}^L_+ \times \mathbb{R}_+$, so the indifference levels of $\text{co}(\preceq)$ come as boundaries of convex hulls of the weakly preferred sets of $\preceq$. The family of functions $\text{co}(q^m)$, $m \in \mathbb{R}_+$, describe the convexified preference. The convexified production set is given by $\text{co}C_l = \{(Y, y, m) \in \mathbb{R}^K \times \mathbb{R}^L \times \mathbb{R} | (Y, y) \in \text{dom}(\text{co}(c_l)) \text{ and } m \leq \text{co}(c_l)(Y, y)\}$, where $\text{dom}(\cdot)$ denotes the domain of a function. The latter sets may not have asymptotes.
Assumption M4. The production sets $\text{co}C_l$, $l \in L$, have no asymptotes (in all codimensions).

Assumption M4 holds for example if $\text{co}C_l$ are polyhedra for all $l \in L$. The final assumption says money is indispensable for production.

Assumption M5. There exists $m_0 \in \mathbb{R}_+$ such that if $m < m_0$ then for any $(Y, y) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L$ it holds that $(Y, y, -m) \notin C_j$, $j \in J$. For any $m \geq m_0$ there exists $b(m) > 0$ such that for any $(Y, y) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L$ with $||| (Y, y) ||_2 > b(m)$, where $|| \cdot ||_2$ denotes the Euclidean norm in $\mathbb{R}_+^K \times \mathbb{R}_+^L$, it holds that $(Y, y, -m) \notin C_j$, $j \in J$.

Without loss of generality we assume that $m_0 = 0$. Under the Assumptions M1–M5 we are able to prove that the convexified economy $\text{co}(\mathcal{E})$ has a competitive equilibrium.

Proposition 4.1 Let $\mathcal{E}$ be a discrete economy and let the Assumptions M1–M5 hold, then there exists a competitive equilibrium in the convexified economy $\text{co}(\mathcal{E})$.

This proposition will be proven in Section 6. The idea of the proof is similar as in [4]. A complication is due to the presence of multiple divisible goods and production. To guarantee that the discrete economy $\mathcal{E}$ itself has a competitive equilibrium, we have to assume that all the individual demands and supplies are the products of divisible and indivisible parts and that the latter parts belong to some class of discrete convexity.

Assumption M6. For each $h \in H$ and $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$, the set $D'_h(p)$ is the product of an indivisible and a divisible part, that is $D'_h(p) = D^\text{ind}_h(p) \times D^\text{div}_h(p)$ for some $D^\text{ind}_h(p) \subset \mathbb{Z}_+^K$ and $D^\text{div}_h(p) \subset \mathbb{R}_+^L$. All sets $D^\text{ind}_h(p)$, $h \in H$ and $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$, belong to some class of discrete convexity $\mathcal{D}$. For each $j \in J$ and $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$, the set $S_j^p(p) = S_j^\text{ind}(p) \times S_j^\text{div}(p)$ is the product of an indivisible and a divisible part, and all sets $S_j^\text{ind}(p)$, $j \in J$ and $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$, belong to the same class of discrete convexity $\mathcal{D}$.

In Section 6 we will prove the following existence result.

Theorem 4.2 Let Assumptions M1–M6 be satisfied. Then there exists an equilibrium in the economy $\mathcal{E}$.

As we will see, we can obtain equilibria in the model using equilibria in the convexified economy. Specifically, let $p^*$ be an equilibrium price in the convexified economy $\text{co}(\mathcal{E})$. Then under the assumptions made above there exists an equilibrium in the original economy $\mathcal{E}$ with the same price vector $p^*$.

5 Model without money

In this section we will not distinguish some good of the divisible commodities as money. In the economy $\mathcal{E}$ consumers have preference relations $\preceq_h$ on the consumption set $\mathbb{Z}_+^K \times \mathbb{R}_+^L$, etc.
$h \in H$, and producers have production sets $C_j \subset \mathbb{Z}^K \times \mathbb{R}^L$, $j \in J$, satisfying the conditions of Section 3. Instead of the presence of money in the economy we assume that there is at least one producer having a production technology being linear in the divisible goods.

**Assumption T1.** There is one production technology being linear in the divisible part. That means that there exists a producer, say $j = 1$, such that for any $p$, $S_1(p) = S_1^{ind}(p) \times T$, where $T \subset \mathbb{R}^L$ is a linear subspace of codimension 1. □

Because of Assumption T1 the equilibrium prices of the divisible goods are completely determined by the rule $p^{div}(x) = 0$ for any $x \in T$. Because of our assumptions it holds that $p^{div} \in \mathbb{R}^L_+$. Therefore, only the appropriate prices of indivisible goods can equilibrate demands and supplies. Let us normalize the prices of the divisible goods such that $p^{div}(1^L) = 1$. The preferences of the consumers are such that the divisible goods are more desirable than the indivisible goods.

**Assumption T2.** For each $(X, x) \in \mathbb{Z}^K \times \mathbb{R}^L$ and $h \in H$ there exists $x_h \in \mathbb{R}^L$ such that $(X, x) \preceq_h (0^K, x_h)$. □

Furthermore, we assume that all production sets and preferences are pseudoconvex and that production sets have no asymptotes.

**Assumption T3.** For every $h \in H$ and any tuple of bundles $(X, x) \sim_h (X_1, x_1) \sim_h \ldots \sim_h (X_r, x_r)$ in $\mathbb{Z}_+^K \times \mathbb{R}_+^L$ such that $X = \sum_i \alpha_i X_i \in \mathbb{Z}_+^K$, $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, r$, it holds that $(X, x) \succeq_h (X, \sum_i \alpha_i x_i)$. For every $j \in J$ and any tuple of bundles $(Y_1, y_1), \ldots, (Y_r, y_r)$ in $C_j$ and $Y \in \mathbb{Z}^K$ such that $Y = \sum_i \alpha_i Y_i$, $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, r$, there exists $y \in \mathbb{R}^L$ such that $(Y, y) \in C_j$ and $p^{div}(y) \geq \sum_i \alpha_i p^{div}(y_i)$. Moreover, the production sets $coC_l$, $l \in L$, have no asymptotes (in all codimensions). □

The next assumption requires that total endowment is strictly positive and that each consumer has enough initial endowment.

**Assumption T4.** The total endowment is strictly positive: $\sum_{h \in H} (W_h, w_h) > (1^K, 1^L)$. For every $h \in H$, it is possible to produce from the initial endowment $(W_h, w_h)$ a vector of goods which is strictly preferred by consumer $h$ to any vector without indivisible goods. □

In the model of van der Laan et al. [14] it is assumed that there is also money in the model and that there is only one producer who produces the divisible non-money goods using money as an input. For a price system $p \in \mathbb{R}^K \times \mathbb{R}^L$, let $S_j(p) = \text{Argmax}_{(Y,y)\in C_j} p(Y,y)$ be the supply of producer $j \in J$ and let $S_j^{ind}(p) = \{Y \in \mathbb{Z}^K \mid \exists y \in \mathbb{R}^L : (Y, y) \in S_j(p)\}$ be the projection of $S_j(p)$ along the divisible goods coordinates. Similarly, let $D_h(p)$ be the demand of consumer $h \in H$ at price vector $p$, and let $D_h^{ind}(p) = \{X \in \mathbb{Z}_+^K \mid \exists x \in \mathbb{R}_+^L : (X, x) \in D_h(p)\}$ be the projection of $D_h(p)$ along the divisible goods coordinates.

The convexified economy $co(E)$ of $E$ is obtained by replacing demands and supplies of $E$ by their convex hulls. In Section 6 it will be shown that under the Assumptions T1–T4 a competitive equilibrium in the convexified economy exists.
Proposition 5.1 Let $\mathcal{E}$ be a discrete economy and let the Assumptions T1–T4 hold, then there exists a competitive equilibrium in the convexified economy $\text{co}(\mathcal{E})$.

To guarantee that the discrete economy $\mathcal{E}$ itself has a competitive equilibrium we have to assume that the individual demands and supplies for the indivisibles belong to a same class of discrete convexity.

Assumption T5. The sets $D_h^{\text{ind}}(p)$, $h \in H$, and $S_j^{\text{ind}}(p)$, $j \in J$, belong for every $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$ to the same class of discrete convexity $\mathcal{D}$.

Theorem 5.2 Let Assumptions T1–T5 be satisfied. Then there exists a competitive equilibrium in the economy $\mathcal{E}$.

In the next section the propositions and theorems of this section are proved.

6 Proofs of Existence of Equilibrium

In this section we prove Propositions 4.1 and 5.1 and Theorems 4.2 and 5.2.

6.1 Proof of Proposition 4.1

Let us outline the idea of the proof. We take a price cube

$$Q = \{ p \in \mathbb{R}^K \times \mathbb{R}^L \mid 0 \leq p_i \leq M, \forall i \in K \cup L \},$$

for some $M > 0$ and define a price correspondence $P$ from $Q$ to $Q$. This $P$ maps a price $p$ to the set of equilibrium prices in an economy $\mathcal{E}(p)$ with transferable utilities. The economy $\mathcal{E}(p)$ is constructed as follows. For every agent $h \in H$, we pick an indifference level which “touches” the budget set $B_h(p)$. Then we set a new preference to agent $h$ such that the indifference levels of this preference are parallel translations (with respect to the money coordinate) of the “touching” indifference level. We will show then that the correspondence $P$ has a fixed point and that this fixed point determines an equilibrium in the convexified economy.

To choose $M$ denote $(W, w) = \sum_{h \in H}(W_h, w_h)$. Let $Z^* \in \mathbb{R}_+^K$ and $z^* \in \mathbb{R}_+^L$ be such that for any $(Z, z)$ in the sets $(\mathbb{R}_+^K \times \mathbb{R}_+^L) \setminus ((\{Z^*\} - \mathbb{R}_+^K) \times \mathbb{R}_+^L)$ or $(\mathbb{R}_+^K \times \mathbb{R}_+^L) \setminus (\mathbb{R}_+^K \times (\{z^*\} - \mathbb{R}_+^L))$ the aggregate cost of producing $(Z, z)$ is more than the total initial amount of money in the economy, i.e., $C(Z, z) > \sum_{h \in H} m_h$ for any $(Z, z)$ in one of the above sets, where $C$ denotes the aggregate cost function defined by

$$C(Z, z) = \min\{ \sum_{j \in J} c_j(Z_j, z_j) \mid \sum_{j \in J}(Z_j, z_j) = (Z, z), (Z_j, z_j) \in \mathbb{R}_+^K \times \mathbb{R}_+^L \}.$$
Because of Assumption M5 such \( Z^* \) and \( z^* \) exist. From Assumption M2 it follows that there exists \( T_h > 0 \) satisfying \( (0^K, 0^L, T_h) \sim_h (W + Z^*, w + z^*, \sum_{h \in H} m_h) \), i.e., \( q^{T_h}_h \) is the function which defines the indifference level of the preference \( \preceq_h \) which passes through the point \( (W + Z^*, w + z^*, \sum_{h \in H} m_h) \), or, equivalently, through the point \( (0^K, 0^L, T_h) \). Then \( M \) is taken to be equal to \( \sum_{h \in H} T_h \).

Because of the chosen \( T_h \), we may assume\(^2\) that, for \( m > T_h \), the indifference level of \( \preceq_h \), passing through the point \( (0^K, 0^L, m) \) is a parallel translation of \( q^{T_h}_h \), i.e.,

\[
q^m_h(X, x) = q^{T_h}_h(X, x) + m - T_h, \quad (X, x) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L.
\]

To construct the economy \( E(p) \), for every \( h \in H \), we pick an indifference level which “touches” the budget set \( B_h(p) \). Formally, for every \( h \in H \), let us choose \( m_h(p) \) to be equal to

\[
\inf \{ m \mid q^m_h(X, x) \geq \beta_h(p) - p(X, x), (X, x) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L, (X, x) \leq (W, w) + (Z^*, z^*) \}.
\]

Now, for every \( h \in H \), we take a new preference on \( \mathbb{Z}_+^K \times \mathbb{R}_+^L \) defined by the utility function

\[
u^p_h(X, x) = m_h(p) - q^{m_h(p)}_h(X, x), (X, x) \in \mathbb{Z}_+^K \times \mathbb{R}_+^L.
\]

According to the assumptions, \( \nu^p_h \) is pseudoconcave for every \( p \) and \( h \in H \).

So, the \( h \)th consumer of the economy \( E(p) \) is specified by the transferable utility function \( \nu^p_h: \mathbb{Z}_+^K \times \mathbb{R}_+^L \to \mathbb{R}_+ \), \( \nu^p_h(0^K, 0^L) = 0 \), \( h \in H \). The producers of \( E(p) \) are the same as in the economy \( E \).

The set \( P(p) \) is the set of equilibrium prices in \( E(p) \). Specifically, let \( U^p \) be the aggregate utility function on \( \mathbb{R}_+^K \times \mathbb{R}_+^L \) given by the co-convolution

\[
U^p(d) = \max_{\sum_{h \in H} d_h = d} \{ \sum_{h \in H} ca(u^p_h)(d_h) \}, \quad d \in \mathbb{R}_+^K \times \mathbb{R}_+^L.
\]

The aggregate cost function \( C \) on \( \mathbb{R}_+^K \times \mathbb{R}_+^L \) is given by the convolution

\[
C(s) = \min_{\sum_{j \in J} s_j = s} \{ \sum_{j \in J} co(c_j)(s_j) \}, \quad s \in \mathbb{R}_+^K \times \mathbb{R}_+^L.
\]

Here \( ca(u) \) denotes the concavification of a function \( u \), \( ca(u) = -co(-u) \), and \( co(c) \) denotes the convexification of a function \( c \).

Let \( t^p \) be a solution to

\[
\max \{ U^p((W, w) + t) - C(t) \mid t \in \mathbb{R}_+^K \times \mathbb{R}_+^L, t \geq -(W, w) \}.
\]

Such a solution exists (for details see [4]), because \( C(s) \to +\infty \) with \( ||s|| \to \infty \) and because of Assumption M5. Since each \( \nu^p_h(\cdot) \) is bounded by \( T_h \), we have that \( U^p(\cdot) \) is

\(^2\)Or we take a modified preference \( \preceq_h \) by setting its indifference levels satisfying this property, see [4].
bounded by $M = \sum_h T_h$. Denote $N = U^p((W, w) + t^p) - C(t^p)$. Then, because $U^p$ is a concave function and $C$ is a convex function and since for any $(Z, z) \in \mathbb{R}^K \times \mathbb{R}^L$ it holds that $C(Z, z) + N \geq U^p(W + Z, w + z)$, there exists a separating function $p'$, i.e., we have $C(Z, z) + N \geq p'(Z, z) + N \geq U^p((W + Z, w + z)$, for any $(Z, z) \in \mathbb{R}^K \times \mathbb{R}^L$.

Define $P(p)$ to be the set of all such separation linear functionals $p'$, i.e., these functionals form the set of equilibrium prices in $\mathcal{E}(p)$. Let us check that $P$ maps $Q$ to itself. Monotonicity of $C$ and $U^p$ implies that $p' \in \mathbb{R}_+^K \times \mathbb{R}_+^L$. On the other hand, due to the monotonicity of $U^p$ and because of individual rationality, for any $p' \in P(p)$ it holds that $M \geq U^p((W, w) + t^p) \geq p'((W, w) + t^p) \geq p'(W, w)$. Since we assumed that $(W, w) \geq (1^K, 1^L)$, we obtain that $p'_j < M$ for all $j \in K \cup L$. Therefore, $P(p) \subset Q$.

Because the set $P(p)$ is the set of separation linear functionals, the correspondence $P$ has convex compact images. Moreover, $U^p$ is continuous with respect to $p$, since all $u^p_h$ are continuous with respect to $p$, and, hence, $P$ is a closed mapping. Therefore, by Kakutani fixed point theorem, $P$ has a fixed point, i.e., there exists $p^* \in Q$ such that $p^* \in P(p^*)$.

Obviously, $t^p < (Z^*, z^*)$. Hence, the following equality is satisfied

$$\text{Argmax}_{x \geq 0, t^* + \kappa} u^n_s p^*(x) \geq p^*(x) = \text{co}(D_h(p^*))$$

Moreover, for any $p' \in Q$, we have $S(p') = \sum_{j \in J} \text{co}(S_j(p'))$, see [4]. Because $U^{p^*}$ and $C$ are aggregate utility and cost functions (the co-convolution and the convolution, respectively), we have

$$(W, w) \in \sum_{h \in H} \text{co}(D_h(p^*)) - \sum_{j \in J} \text{co}(S_j(p^*)) \quad (5)$$

This yields an equilibrium in the convexified economy $\text{co}(\mathcal{E})$. Q.E.D.

### 6.2 Proof of Proposition 5.1

Here we also construct an auxiliary economy. Because of Assumption T4, the production set $\sum_j C_j$ of the aggregate producer is a closed convex set.\(^3\) Now, we explain how to aggregate consumers. Pick some price $p \in \mathbb{R}_+^K$. For each $h \in H$, we consider an indifference level “touching” the budget set $B_h(p, p^{\text{div}})$. Denote by $I_h(p)$ this indifference level. First we set the preference $\tilde{x}_h$ of the $h$th consumer such that the indifference levels are parallel translations of the “touching” level by the vector $h(0^K, 1^L)$, $\lambda \in [\lambda_h, +\infty)$, where $\lambda_h$ is such that the translation of the indifference level by the vector $\lambda_h(0^K, 1^L)$ passes through the endowment vector $(W_h, w_h)$. Note that $\lambda_h \leq 0$. Now set indifference levels of a preference

\(^3\)In general, the sum of convex closed sets might not be closed, but because of our assumptions the sum $\sum_j C_j$ is a closed set.
\( \preceq (p) \) of the aggregate consumer, endowed with the aggregate vector \( (W, w) = \sum_h (W_h, w_h) \), by the rule
\[
\sum_h (I_h(p) - \lambda_h t(0^K, 1^L)), \quad \text{if } t \in [-1, 0],
\]
and
\[
\sum_h (I_h(p) + t(0^K, 1^L)), \quad \text{if } t \geq 0.
\]
Because there exists an indifference level of \( \preceq (p) \) which is passing through \( (W, w) \), this list of indifference levels suffices to set up the preference due to individual rationality. Note also that any indifference level is well defined since all \( I_h(p) \) belong to the cone \( \mathbb{R}_+^K \times \mathbb{R}_+^L \).

We define \( P(p) \) as the set of equilibrium prices in the economy \( \mathcal{E}(p) \) with one producer with production set \( C = \sum_j C_j \) and one consumer with preference relation \( \preceq (p) \). The equilibrium prices come of the form of the separating functionals between the set \( C \) and a translation on the vector \(- (W, w)\) of the set being the sum of the indifference level of \( \preceq (p) \) passing through the point \( (W, w) + y(p) \) and the positive orthant \( \mathbb{R}_+^K \times \mathbb{R}_+^L \), where \( y(p) \in \text{Argmax}_{y \in C} (p, p \text{div})(y) \), i.e., we translate the set with respect to vectors of the form \( a(0^K, 1^L) \), \( a \geq -1 \), such that the production set and the translated set touch each other.

In order to get a fixed point of \( P \), we take a cube \( Q = \{ p \in \mathbb{R}^K | 0 \leq p_k \leq M \} \) for some \( M > 0 \) such that \( P \) maps every \( p \in Q \) to a subset of \( Q \). The number \( M \) is determined as follows. Given the initial endowments, there exist bounds for the maximal production of each good due to Assumptions T4 (we may exclude the linear producer, having fixed \( p \text{div} \)). Let \( (B, b) \in \mathbb{R}_+^K \times \mathbb{R}_+^L \) be a vector which is in every coordinate larger than the maximal production of the good corresponding to this coordinate, and for \( h \in H \) let \( T_h \) be the cost \( p \text{div}(x_h) \) of producing at price \( p \text{div} \) the vector \( (0^K, x_h) \in \mathbb{R}_+^K \times \mathbb{R}_+^L \) satisfying \( (0^K, x_h) \sim_h (W_h + B, w_h + b) \). Then we take \( M \) equal to \( \sum_h T_h \).

Because any \( p' \in P(p) \) is a separating functional, we have that \( M \geq p'(W) \), and since \( W \geq 1^K \), we obtain \( p'_k \leq M \) for every \( k \in K \). Clearly, \( P \) has compact convex images and is a closed mapping. Therefore, by Kakutani fixed point theorem, \( P \) has a fixed point. Since due to Walras' law at a fixed point \( p^* \) of \( P \) the vector \( p^* \) supports the indifference level \( \sum_h I_h(p^*) \), a fixed point of \( P \) yields an equilibrium of the convexified economy. Q.E.D.

### 6.3 Proof of Theorem 4.2

In Proposition 4.1 the existence of an equilibrium of the convexified economy \( \text{co}(\mathcal{E}) \) was shown. Let \( p^* \) be any equilibrium price for \( \text{co}(\mathcal{E}) \) and let \( (\tilde{y}_j, y_j) \in \text{co}(S'_j(p^*)) \), \( j \in J \), and \( (\tilde{x}_h, x_h) \in \text{co}(D_h(p^*)) \), \( h \in H \), be the corresponding supplies and demands. Hence,
\[
\sum_{j \in J} \tilde{y}_j + \sum_{h \in H} \tilde{x}_h = \sum_{h \in H} W_h.
\]
Since $\sum_{h \in H} W_h \in \mathbb{Z}^K$ and due to Assumption M6, we can find integer vectors $Y_j \in S_j^{\text{ind}}(p^*)$, $h \in H$, and $X_h \in D_h^{\text{ind}}(p^*)$, $h \in H$, satisfying
\[
\sum_{j \in J} Y_j + \sum_{h \in H} X_h = \sum_{h \in H} W_h, \tag{6}
\]
and
\[
(Y_j, y_j) \in S_j^{\text{ind}}(p^*), \quad j \in J, \quad (X_h, x_h) \in D_h^{\text{ind}}(p^*), \quad h \in H. \tag{7}
\]
In fact, (6) holds because the sets $S_j^{\text{ind}}(p^*)$, $j \in J$, and $D_h^{\text{ind}}(p^*)$, $h \in H$, belong to a same class of discrete convexity. The inclusions (7) hold due to the product structure of the sets $S_j^{\text{ind}}(p^*)$, $j \in J$, and $D_h^{\text{ind}}(p^*)$, $h \in H$.

Now, one can check that the prices $p^*$, the supplies $(Y_j, y_j, \pi_j(p^*) - p^*(Y_j, y_j)) \in S_j(p^*)$, $j \in J$, the demands $(X_h, x_h, \beta_h(p^*) - p^*(X_h, x_h)) \in D_h(p^*)$, $h \in H$, form an equilibrium in the economy $E$. Indeed, the latter vectors satisfy the market clearing condition, and that they are supplies and demands follows without any further conditions for the supplies and with help of Assumption M1 for the demands. Q.E.D.

### 6.4 Proof of Theorem 5.2

In Proposition 5.1 we proved the existence of an equilibrium in the convexified economy. Now let us assume we have an equilibrium in $\text{co}(E)$, that is a tuple of prices $p^* : \mathbb{R}_{+}^T \to \mathbb{R}$, supplies $(z^*_j, y^*_j) \in \text{co}(S_j(p^*))$, $j \in J$, and demands $(t^*_h, x^*_h) \in \text{co}(D_h(p^*))$, $h \in H$, satisfying $\sum_h t^*_h + \sum_j z^*_j = \sum_h W_h$ and $\sum_h x^*_h + \sum_j y^*_j = \sum_h w_h$. Therefore, we have
\[
\sum_{h \in H} W_h \in \sum_{h \in H} \text{co}(D_h^{\text{ind}}(p^*)) + \sum_{j \in J} \text{co}(S_j^{\text{ind}}(p^*)).
\]

By Assumption T5, there exist $T^*_h \in D_h^{\text{ind}}(p^*)$, $h \in H$, and $Z^*_j \in S_j^{\text{ind}}(p^*)$, $j \in J$, satisfying $\sum_h T^*_h + \sum_j Z^*_j = \sum_h W_h$. Let $x_h$, $h \in H$, and $y_j$, $j \in J$, be such that $(T^*_h, x_h) \in D_h(p^*)$, $h \in H$, and $(Z^*_j, y_j) \in S_j(p^*)$, $j \in J$.

By Walras' law we have
\[
\sum_{h \in H} (p^*(T^*_h) + p^\text{div}(x_h)) = \sum_{h \in H} p(W_h, w_h) + \sum_{j \in J} (p^*(Z^*_j) + p^\text{div}(y_j)).
\]

Because of the balance of the indivisible goods, $\sum_h T^*_h + \sum_j Z^*_j = \sum_h W_h$, we have $p^\text{div}(\sum_h x_h - \sum_j y_j) = 0$. Define the new production plan of the producer 1 as $(Z^*_1, y'_1)$, where $y'_1 := y_1 + (\sum_h x_h - \sum_j y_j)$. By Assumption T3, $(Z^*_1, y'_1)$ belongs to $S_1(p^*)$, and with this modification for the first producer, we obtain a competitive equilibrium of the economy $E$. Q.E.D.
References


