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ESTIMATING MEAN AND VARIANCE THROUGH QUANTILES: AN EXPIRIMENTAL COMPARISON OF DIFFERENT METHODS

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Discussion paper

Estimating mean and variance through quantiles: an experimental comparison of different methods

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Abstract

If estimates of mean and variance are needed and only experts' opinions are available, the literature agrees that it is wise behaviour to ask only for their (subjective) estimates of quantiles: from these, estimates of the desired parameters are calculated. Quite a number of methods have been suggested up to now, using different sets of quantiles.

Four of these methods are compared here by means of an experiment in which 56 students acted as experts on four distributions. The accuracy of the estimation of both the quantiles and the parameters are discussed.

JEL-code: C13

Key-words: Estimation of mean, estimation of variance, quantiles, Pearson system, Johnson system

1 Introduction

Experts have a thorough knowledge of certain random processes, which however often is intuitive and not explicit. Their knowledge may be elicited by letting them make statements about probabilities concerning this random process. The present paper considers the case the expert's knowledge relates to the probability distribution of some random variable. His opinion on mean and variance/standard deviation of this distribution is needed.

Of course, the expert can be asked to give his subjective values for those two parameters directly. However, the literature seems to agree that this is a rather difficult task. In stead, subjective values for quantiles are asked for: apparently, their values are much more easily assessed. Of course, two interrelated questions immediately arise:

Which quantiles should be asked for?

How are mean and variance derived from them?

Even in recent years, many different answers have been given. Four of the proposed methods are included in this study.

To describe the four methods the following notation will be used. The variable of interest is X, having cumulative distribution function F, mean μ and variance σ^2. The quantile p_α of X satisfies

$$\alpha = P(X \leq p_\alpha) = F(p_\alpha)$$

Since only continuous X will be considered,

$$p_\alpha = F^{-1}(\alpha)$$

holds throughout.

Method 1. Extended Pearson-Tukey.

This method uses three quantiles, for α equal to 5, 50 and 95%; hence, the expert is asked the values of the quantiles

$$p_{0.05}, \quad p_{0.5}, \quad p_{0.95}$$

From these three numbers, estimates $\hat{\mu}$ and $\hat{\sigma}^2$ are obtained by means of following expressions:

$$\begin{cases} \hat{\mu} &= 0.630 p_{0.5} + 0.185(p_{0.05} + p_{0.95}) \\ \hat{\sigma}^2 &= 0.630(p_{0.5} - \hat{\mu})^2 + 0.185[(p_{0.05} - \hat{\mu})^2 + (p_{0.95} - \hat{\mu})^2] \end{cases} \tag{1.1}$$

Method 2. Extended Swanson-Megill.

Here, the three quantiles

$$p_{0.1}, \quad p_{0.5}, \quad p_{0.9}$$

are used to calculate estimate $\hat{\mu}$ and $\hat{\sigma}^2$ from

$$\begin{cases} \hat{\mu} &= 0.4 p_{0.5} + 0.3(p_{0.1} + p_{0.9}) \\ \hat{\sigma}^2 &= 0.4(p_{0.5} - \hat{\mu})^2 + 0.3[(p_{0.1} - \hat{\mu})^2 + (p_{0.9} - \hat{\mu})^2] \end{cases} \tag{1.2}$$

See KEEFER & VERDINI (1993) for a detailed discussion of both these methods.

Method 3. Selvidge-Lau-Ho.

The method uses the seven quantiles propagated by SELVIDGE (1980) and corresponding with the α-values

$$0.01, \quad 0.1, \quad 0.25, \quad 0.5, \quad 0.75, \quad 0.9 \quad 0.99$$

LAU, LAU & HO (1995) reported extensive simulation experiments in broad classes of probability distributions. As optimal estimation formulae they proposed

$$\begin{cases} \hat{\mu} &= 0.28871 p_{0.5} + 0.18584(p_{0.25} + p_{0.75}) \\ & \quad +0.13394(p_{0.1} + p_{0.9}) + 0.036128(p_{0.01} + p_{0.99}) \\ \hat{\sigma} &= 0.06611(p_{0.75} - p_{0.25}) + 0.16815(p_{0.9} - p_{0.1}) + 0.10794(p_{0.99} - p_{0.01}) \end{cases} \tag{1.3}$$

Note that σ is estimated rather than σ^2.

Method 4. Moors-Schuld-Mathijssen.

Again seven quantiles are used, with the α-values

$$i/8, \quad i = 1, 2, \ldots, 7$$

The corresponding quantiles divide the total area under the density of X in eight equal parts and were called octiles, accordingly. Both within the Pearson system of distributions and within the Johnson system, the seven octiles uniquely determine one probability distribution. A computer program was developed to find this unique distribution and its mean and variance. More details will follow in Section 3.

The theoretical aspects and relative (dis)advantages of the first three methods are discussed at length in the literature. However, for all four, practical experience is rather limited. Therefore, an experimental set-up was devised to study the merits of the different methods. In this experiment students were asked to give the quantiles (corresponding to one of the four methods) for four different random variables/frequency distributions. Since these four distributions were fully known to the authors, the exact quantiles and moments could be derived.

In view of the two above-mentioned questions that each method tries to answer, the performance of a method depends on two issues:

- the accuracy of assessing the quantiles,

- the accuracy of estimating mean and variance,

the latter being the most important, of course.

In Section 2 the four different cases are described for which the groups of students were asked to give their subjective values for either three or seven quantiles. Section 3 compares the quantile assessment accuracy of the four methods. The -most recent- Method 4 is considered in more detail in Section 4, while Section 5 compares the accuracy of the four methods in estimating mean and standard deviation. The final Section 6 discusses the results.

2 Experiment

To compare the different methods in practice, an experiment was devised among students in economics at Tilburg University. All students participating in the exam for the Statistics 2 course on January 7, 1997 were invited to take part in the experiment. The reward would be twofold:

- participants' exams would be corrected with priority: at the end of the experiment, participants would receive their mark;
- the 10% students performing best at the experiment would recieve a bonus of 50 Dutch guilders (about 25 US dollars).

One week later, on January 14, 56 participants entered the experiment. They were divided info four random groups, corresponding with the four methods of assessing quantiles described in the previous section. Each individual student was asked to assess the corresponding set of quantiles for two frequency distributions and two probability distributions. These four distributions were introduced in the following order; we give (a translation of) the original instruction. The full instruction (in Dutch) for Method 1 is added as Appendix 1. Note that all students are familiar with the concept of quantiles; nevertheless the instruction started with a refresher (not presented here) on this subject.

Case A (exam)

'The first question is about the Statistics 2 exam of January 7 - the result of which you will hear after finishing the experiment. We want to know how good you are in estimating the frequency distribution of the marks obtained by all participants. You only need to fill in on the form the three [c.q. seven] quantiles. It concerns the original, non-rounded, results.'

Note that in Holland exam results are expressed as a number between 0 and 10, with one decimal place - 10.0 being the best possible result. Rounding gives the final mark; to pass a 6 is needed.

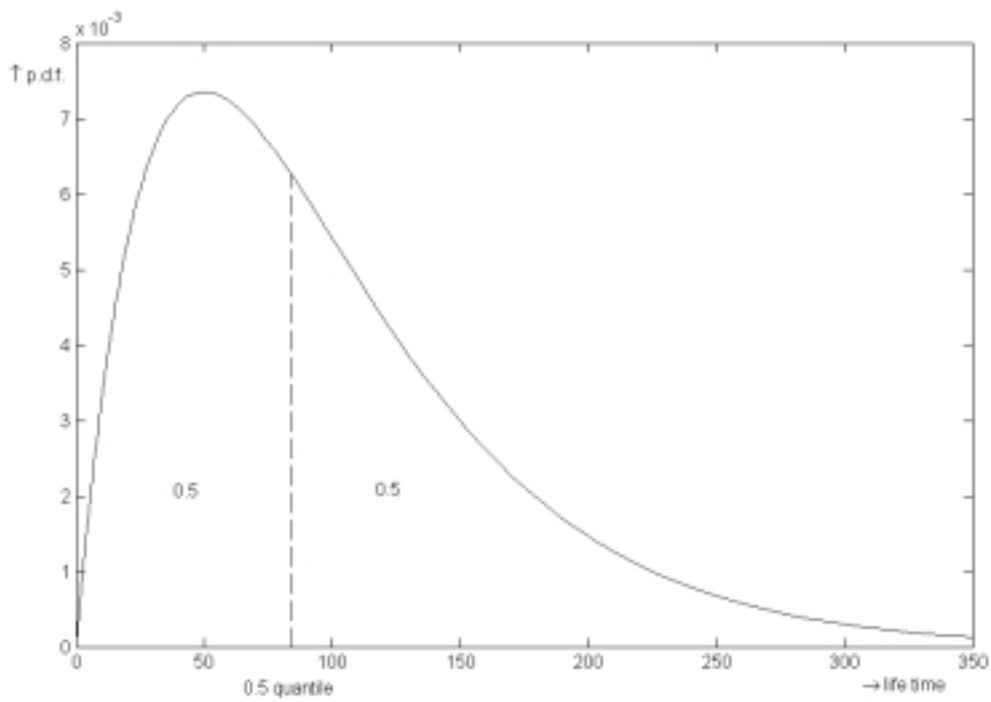
Case B (age)

'The second question is about the age distribution of Dutch woman on January 1, 1996. Round the quantiles to years (....). Fill in the three [c.q. seven] quantiles on the form.'

Case C (life time)

'This question, as well as the last, is on probability distributions. The life time of a certain kind of light bulb depends on chance. Figure 1 shows the probability density of the life time of such a light bulb; for clarification, the median is indicated.

Figure 1. Density life time light bulb.



The question now is about the total life time of *two* of these light bulbs. Each of the two has the density from the figure; the two life times are independent. Fill in on the form the values that the three [c.q. seven] quantiles of this total life time have, according to you.'

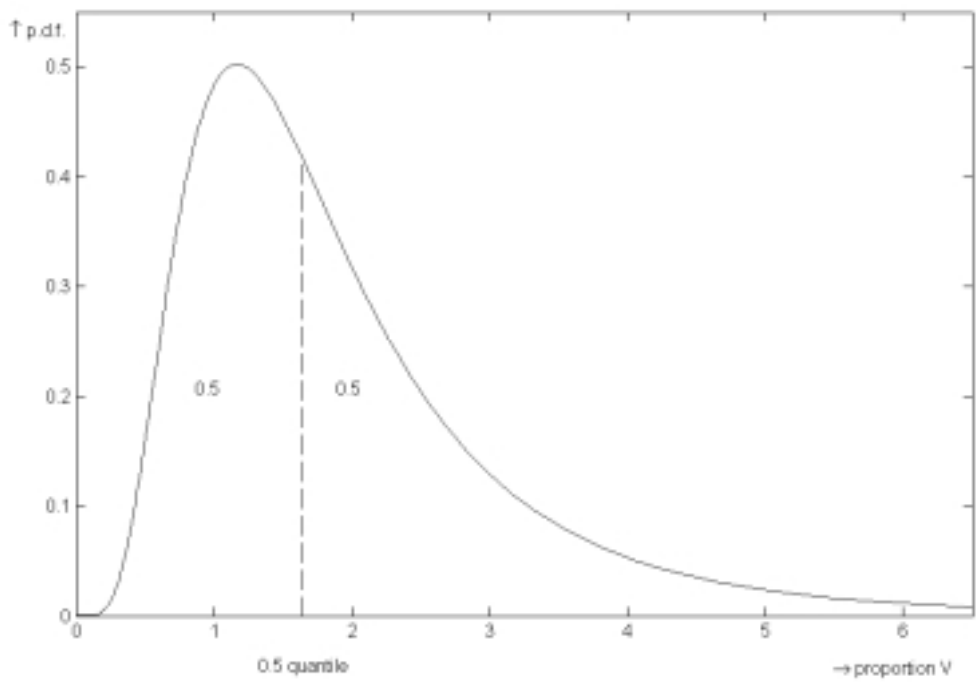
Case D (amount A)

'Two persons, A and B, are allowed to split up between them 100 Dutch guilders. The proportion between the amount A receives and B's amount is determined by chance; this proportion

$$V = \frac{\text{amount } A}{\text{amount } B}$$

therefore is a random variable, which probability density is shown in Figure 2. As indicated, the probability that V exceeds 1.64 equals 0.5.

Figure 2. Density proportion V.



The amount that A receives is a random variable as well. E.g., if $v = 4$ is the outcome of the random drawing, A receives four times B's amount; hence A receives 80 guilders. In case $v = 1/3$, B receives thrice as much as A, so that A gets only 25 guilders. The general rule is

$$\text{amount } A = 100\frac{v}{v+1}$$

(in guilders). The question is about the probability distribution of the amount that A will obtain. Fill in on the form the - according to you - correct values of the three [c.q. seven] quantiles.'

For the group corresponding to method 1, the form that had to be filled in looked like this:

FORM

Name:

Administration number:

Case	A (exam)	B (age)	C (life time)	D (amount A)
0.05-quantile				
0.5-quantile				
0.95-quantile				

Similar forms were handed to the other three groups.

Of course, the exact frequency distribution for the case A and B was known, so that quantiles and moments could be found exactly. Furthur, Figure 1 pictures the function

$$f(x) = \frac{1}{2500} x e^{-x/50}, \ x > 0$$

i.e. the density of the chi-squared distribution with 4 degrees of freedom and non-centrality parameter 25. In other words, the distribution of the life time X of an individual light bulb is given by

$$X \sim \mathcal{X}_4^2(25)$$

Consequently, X has median 83.9, mean 100 and variance 5000. Now, the total life time T of two independent light bulbs has the distribution

$$T \sim \mathcal{X}_8^2(25)$$

Hence, moments and quantiles for case C can be found by standard methods.

Beta distributions of type 2 have the density

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{x^{p-1}}{(1+x)^{p+q}}, \ x > 0$$

where $p > 0$ and $q > 0$ are parameters; notation:

$$X \sim B2(p, q)$$

Figure 2 shows the density

$$f(v) = 3960 v^7/(1+v)^{13}, \ v > 0$$

of $V \sim B2(8, 5)$, having 1.64 as median. Now, standard distributional theory gives

$$Y = \frac{V}{V+1} \sim Be(8, 5)$$

where $Be(p, q)$ denotes the wellknown beta distribution itself; the corresponding density is

$$f(y) = 3960y^7(1-y)^4,\ 0 < y < 1$$

Finally, the amount S that A receives equals $100\,Y$. So, moments and quantiles for case D can be found easily.

Table 1 presents all 13 quantiles occurring in this paper for each of the four cases; means and standard deviations are added.

Table 1. Quantiles and standard deviations mean for cases A-D.

α (in %)	Case A (exam)	Case B (age)	Case C (life time)	Case D (amount A)
1	0.5	0.84	41.2	30.2
5	1.4	4.1	68.3	39.1
10	2.0	8.2	87.2	44.1
12.5	2.4	10.3	94.9	45.9
25	3.3	21.2	126.8	52.7
37.5	4.0	29.4	155.0	57.7
50	4.4	37.1	183.6	62.1
62.5	5.2	45.6	215.6	66.4
75	5.8	55.2	255.5	71.0
87.5	6.5	68.7	315.9	76.7
90	6.7	71.7	334.0	78.1
95	7.4	78.8	387.7	81.9
99	8.8	88.1	502.3	87.9
mean	4.50	38.8	200.0	61.5
stand. dev.	2.70	22.8	100.0	13.0

Since the densities of X in Case C and the density of V in Case D are displayed, students have some knowledge about the distributions of T and S: in a way, they can taken to be 'experts' in those two fields. Of course, they also have some intuitive knowledge about the distributions in Cases A and B.

3 Quantile accuracy

The 56 participating students were given a three-digit number; the first digit (1-4) corresponds with the method that had to be applied, the latter two (1-14) identify a specific student. E.g., 307 is the seventh student who had to give quantile values according to Method 3.

Appendix 2 shows all subjective quantile values $\hat{p}_\alpha$. However, Case D was apparently not fully understood by everyone: nine students gave quantile values exceeding 100, e.g. These observations have been omitted in the remainder of the paper and will be indicated by an asterisk *. Further, the observations of student 406 for Case B and of student 409 for Case C have been omitted, because the number of quantiles given was not equal to 7.

The question to be considered in this section is whether some sets of quantile values are easier to assess than other. To answer this question, two measures of accuracy were calculated for each student and every case, the Mean Absolute Deviation and the Mean Squared Error:

$$\text{MAD} = \frac{1}{K_g}\sum_{k=1}^{K_g}|\hat{p}_{\alpha k} - p_{\alpha k}| \tag{3.1}$$

$$\text{MSE} = \frac{1}{K_g}\sum_{k=1}^{K_g}(\hat{p}_{\alpha k} - p_{\alpha k})^2 \tag{3.2}$$

Here K_g denotes the number of quantiles $p_{\alpha k}$ needed for method g; hence, K_g equals either 3 or 7. These values are presented in Tables 2A-2D; maximum (bold) and minimum (italics) values per column are indicated. Table 3 gives the values, averaged over the students.

Table 2A. Quantile accuracy for Case A (exam).

Method i	1		2		3		4	
Student	MAD	MSE	MAD	MSE	MAD	MSE	MAD	MSE
i01	0.60	0.60	0.63	0.68	0.89	1.04	0.74	0.72
i02	0.77	0.64	0.57	0.52	*0.38*	0.26	*0.46*	*0.31*
i03	0.93	1.09	**1.67**	**2.94**	**1.31**	**2.00**	0.94	0.94
i04	0.77	0.64	1.33	2.05	0.66	0.61	0.54	0.32
i05	1.07	1.41	0.67	*0.47*	0.49	0.31	1.21	1.88
i06	1.13	1.38	*0.47*	0.57	0.66	0.63	**1.27**	**1.97**
i07	1.00	1.14	0.87	0.78	0.50	0.41	0.93	1.11
i08	0.67	0.49	1.37	1.93	0.57	0.47	0.47	0.34
i09	0.87	0.90	0.50	0.66	0.84	0.77	0.70	0.51
i10	*0.53*	0.46	1.30	2.18	0.93	1.04	0.97	1.12
i11	1.27	1.65	1.47	2.27	0.39	*0.24*	0.99	1.18
i12	**1.43**	**2.78**	1.07	1.21	0.93	1.08	1.13	1.31
i13	0.93	1.55	0.97	1.05	0.99	1.39	0.63	0.49
i14	0.60	*0.36*	0.80	0.77	1.07	1.35	1.19	1.67

Table 2B. Quantile accuracy for Case B (age).

Method i	1		2		3		4	
Student	MAD	MSE	MAD	MSE	MAD	MSE	MAD	MSE
i01	**9.27**	**128.5**	3.47	22.0	3.78	20.5	8.07	91.6
i02	5.60	50.5	2.80	9.4	*1.78*	*5.0*	7.64	68.6
i03	*0.67*	*0.6*	6.47	62.0	4.95	40.3	*1.30*	*3.7*
i04	8.27	83.8	3.40	29.0	2.67	16.9	4.56	25.9
i05	1.67	4.0	2.47	7.8	5.18	32.7	**8.93**	**110.9**
i06	3.07	17.6	3.33	12.7	6.75	72.3	*	*
i07	1.40	3.7	2.20	5.6	5.47	42.7	7.50	67.1
i08	5.73	37.2	3.87	16.6	7.16	85.9	4.61	28.3
i09	9.00	89.4	7.13	67.5	4.95	29.0	4.50	25.6
i10	7.53	77.9	**9.67**	119.6	4.18	28.1	4.90	38.3
i11	1.17	2.0	4.33	40.4	4.55	28.1	5.50	62.9
i12	2.60	7.0	*0.47*	*0.3*	**7.95**	**100.7**	5.36	33.5
i13	4.70	31.5	6.73	63.8	2.38	13.6	5.90	49.3
i14	1.40	3.3	9.47	**121.4**	4.12	20.1	5.13	37.9

Table 2C. Quantile accuracy for Case C (life time).

Method i	1		2		3		4	
Student	MAD	MSE	MAD	MSE	MAD	MSE	MAD	MSE
i01	65.4	6563	85.6	10874	*30.1*	*1198*	82.1	**14003**
i02	66.5	6994	95.6	9342	81.3	14921	34.6	1502
i03	88.1	8561	42.3	2074	49.1	3755	89.0	12828
i04	36.4	1428	44.8	3800	58.7	4867	34.6	1441
i05	*28.7*	*999*	42.3	2098	52.8	4154	*25.9*	*1019*
i06	71.4	9333	55.6	4915	71.4	7570	43.3	2538
i07	**118.5**	**16698**	*34.1*	*1690*	**106.6**	**22143**	44.2	5375
i08	48.7	4281	63.3	5264	69.6	8444	96.0	9440
i09	59.9	6718	52.9	3234	61.0	5878	*	*
i10	51.4	4503	**120.9**	**28653**	78.4	7077	97.2	9898
i11	34.7	1390	68.9	5280	38.5	2431	29.0	1270
i12	102.2	11319	102.3	23863	38.2	2539	52.0	4455
i13	84.1	12095	42.3	2098	58.9	4712	**101.0**	10688
i14	92.7	16000	65.0	4705	59.6	5735	43.3	2234

Table 2D. Quantile accuracy for Case D (amount A).

Method	1		2		3		4	
Student	MAD	MSE	MAD	MSE	MAD	MSE	MAD	MSE
i01	*	*	4.37	22.9	14.89	350.5	**17.46**	**395.8**
i02	*	*	*	*	6.88	84.1	12.04	188.8
i03	9.11	196.6	7.94	125.5	*3.19*	*20.3*	6.62	91.6
i04	5.28	34.1	4.96	28.9	7.17	83.7	9.95	126.5
i05	*1.28*	2.1	*1.13*	*3.2*	**24.35**	**644.5**	1.50	2.8
i06	*	*	**23.13**	**655.3**	10.17	174.7	3.77	25.0
i07	16.01	**444.8**	9.79	198.4	*	*	*	*
i08	**18.45**	422.9	21.27	540.8	8.41	110.3	14.66	308.0
i09	7.32	76.8	19.97	554.1	*	*	*	*
i10	1.89	4.2	12.79	244.4	15.74	332.5	1.63	6.4
i11	*1.28*	*2.0*	5.96	48.7	5.46	52.8	*1.23*	*2.2*
i12	6.28	70.1	5.04	46.1	*	*	2.16	8.5
i13	7.95	140.7	5.04	43.8	10.27	124.8	11.80	162.1
i14	7.45	124.6	12.71	257.5	9.23	178.8	15.20	404.2

Table 3. Mean quantile accuray.

	MAD					MSE			
Case	A	B	C	D	Case	A	B	C	D
Method					Method				
1	0.90	*4.43*	**67.8**	*7.48*	1	1.08	38.4	7635	*138*
2	**0.98**	4.70	65.4	10.32	2	**1.29**	41.3	**7706**	**213**
3	*0.76*	4.71	61.0	**10.52**	3	*0.83*	*38.3*	6816	196
4	0.87	**5.68**	*59.4*	8.17	4	0.99	**49.5**	*5899*	143

The maximum value in each column is printed boldly, the minimum in italics. The accuracy in assessing the four sets of quantiles does not show much spread; perhaps the accuracy of method 2 is slightly lower.

4 More about Method 4

Based on a new interpretation of kurtosis, MOORS (1986, 1988) proposed a quantile measure for this distributional feature. Writing $E_i = p_{i/8}$, $i = 1, 2, \ldots, 7$ for simplicity, this measure T was defined as

$$T = (E_7 - E_5 + E_3 - E_1)/(E_6 - E_2)$$

Note that it extends in a natural way the familiar series of quantile measures for location, spread and skewness:

$$\begin{cases} Q : \text{the median } E_4, \\ R : \text{the half interquantile range } (E_6 - E_2)/2, \\ S : \text{Bowley's skewness measure } (E_6 - 2E_4 + E_2)/(E_6 - E_2). \end{cases}$$

It was shown in MOORS *et al.* (1996) that there is a one-to-one relation between the foursome (Q, R, S, T) and all distributions within either the Pearson or the Johnson system of distributions; note that the same holds for the moment-based foursome $(\mu, \sigma, \beta_1, \beta_2)$: compare STUART & ORD (1987). This opened a new possibility to estimate mean and standard deviation from subjective octiles, consisting of three steps:

1. choose either the Pearson or the Johnson system,

2. determine within the chosen system the unique distribution corresponding to the foursome (Q, R, S, T),

3. calculate μ and σ for this distribution.

A computer programme was developed for performing steps 2 and 3. Details can be found in MOORS *et al.* (1995).

The Pearson and Johnson systems of distributions will be considered now in some more detail. Both systems mainly consist of types of four-parameter distributions; these parameters will be denoted by the foursome (a, b, c, d), of which (a, b) is the location-scale parameter. These types are separated by so-called transition types having only (the first) three parameters. Tables 4 and 5 show these types for the two systems; in the densities, the location-scale parameters have been suppressed for simplicity, as well as the normalizing constants for the Pearson system.

Table 4. Outline of the Pearson system.

Name	Type	Density*	Range	Parameters
Beta 1	I	$x^{c-1}(1-x)^{d-1}$	[0,1]	$c, d > 0$
Student	VII	$(1+x^2/c)^{-(c+1)/2}$	$\mathbb{R}$	$c > 0$
Arctan	IV	$(1+x^2)^{-c}\exp[d \arctan x]$	$\mathbb{R}$	$c > 1/2, d \in \mathbb{R}$
Inverse gamma	V	$x^{-(c+1)}e^{-1/x}$	$\mathbb{R}^+$	$c > 0$
Beta 2	VI	$x^{c-1}/(x+1)^{c+d}$	$\mathbb{R}^+$	$c, d > 0$
Gamma	III	$x^{c-1}e^{-x}$	$\mathbb{R}^+$	$c > 0$

* up to a normalizing constant.

Roman figures correspond with the original codes by Pearson. The (missing) type II is the class of symmetric Beta 1 distributions with $c = d$.

Table 5. Outline of the Johnson system

Name	Type	Density*	Range	Parameters
Lognormal	S_L	$\frac{c}{\sqrt{2\pi}}\frac{1}{y}\exp[-(c \ln y)^2/2]$	R^+	$c \in R^+$
Bounded range	S_B	$\frac{d}{\sqrt{2\pi}}\frac{1}{y(1-y)}\times$ $\exp[-\{c + d \ln (\frac{y}{1-y})\}^2/2]$	$[0, 1]$	$c \in R$, $d \in R^+$
Unbounded range	S_U	$\frac{d}{\sqrt{2\pi}}\frac{1}{\sqrt{1+y^2}}\times$ $\exp[-\{c + d \ln (y + \sqrt{1+y^2})\}^2/2]$	R	$c \in R, d \in R^+$

Our computer programme was applied for both systems of distributions to the observations of students 401-414. As a start, Table 6 shows the results for Case B with

respect to the Pearson system; remember that student 406 was deleted for this case. Presented are the estimated type of distribution (I-VII), the values of the parameters a, b, c (and d), and the estimated mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$.

Table 6. Results for Case B (age) of Method 4 (Pearson system)

Student	Type	a	b	c	d	$\hat{\mu}$	$\hat{\sigma}$
401	I	14.3095	74.1334	0.2136	0.3505	42.4	28.8
402	I	5.3496	75.7359	0.4297	0.5441	38.8	26.8
403	I	-42.0027	221.7407	7.0192	12.1825	39.1	23.8
404	II	5.0307	59.9387	0.9980		35.0	17.3
405	I	-71.4150	72.5905	0.2576	0.4061	43.2	27.4
406				*			
407	II	-25.9463	111.8926	3.2875		30.0	20.3
408	I	-64.4654	60.7221	0.3719	0.4510	37.0	22.4
409	IV	-66.6057	18.1109	2.3160	4.8767	33.0	44.9
410	II	0.0409	79.9182	0.9980		40.0	23.1
411	I	-59.9610	65.1226	0.3966	0.5065	31.4	23.4
412	I	24.9382	72.4344	0.4343	0.9775	47.2	21.5
413	I	22.3964	57.7354	0.7380	1.3303	43.0	15.8
414	I	-61.0291	63.7571	0.2643	0.4845	38.5	23.0

All observations point into the direction of a Beta 1 distribution, except student 409 (Arctan). In view of the fact that $\mu = 38.8$ and $\sigma = 22.8$ for Case B, the estimates in the last two columns seem quite sensible. From the distribution found, the values of the seven octiles as well as Q, R, S and T could be recalculated. The latter foursome was exact up to four decimal places; the three recalculated quantiles were very accurate as well, but the 'odd' octiles E_1, E_3, E_5, E_7 deviated somewhat more.

For the other three cases, however, not always estimates for μ and/or σ were found; Table 7 lists the eight observations showing this feature.

Table 7. Observations with non-existing $\hat{\mu}$ and/or $\hat{\sigma}$; Method 4 (Pearson system).

Case	Student	Type	a	b	c	d	$\hat{\mu}$	$\hat{\sigma}$
A	401	IV	2.832	0.9219	1.3428	3.2556	7.21	NaN
	406	IV	3.332	0.9219	1.3428	3.2556	7.71	NaN
	411	IV	4.6089	1.3590	0.9923	0.3427	NaN	NaN
C	401	IV	126.3657	129.5747	1.0391	0.2280	504	NaN
	407	VI	111.9905	315.6502	0.4220	1.4057	440	NaN
	408	IV	12.9916	51.8611	1.4232	2.0195	137	NaN
D	402	IV	44.8582	11.5816	1.0062	0.3608	379.8	NaN
	410	IV	58.8620	9.9750	1.1835	0.3600	68.6	NaN

All cases relate to type IV, except for the observations of student 407 in case C, where type VI is found. It is easy to check indeed, that the conditions for the kth moment to exist are:

$$\begin{cases} c > (k+1)/2, \text{ type IV} \\ d > k, \text{ type VI,} \end{cases}$$

in agreement with Table 7. Although the recalculated values of (Q, R, S, T) are again in full agreement with the observations, we have to conclude that Method 4 in combination with the Pearson systems fails in these eight cases.

For the Johnson system the problem of non-existence of moments can not occur. However, it is worth noting that often rather extreme estimates were obtained for the eight cases in Table 7. E.g. for student 411 in Case A, Method 4 (Johnson system) lead to $\hat{\sigma} = 13.88$ (for $\sigma = 2.70$), while student 401 for Case C gave $\hat{\sigma} = 830$ (for $\sigma = 100$).

5 Accuracy of mean and variance estimation

Based on the results of the previous section, it was decided to leave the eight observations in Table 7 out of the subsequent analysis of Method 4 (Pearson system); they will be indicated by the symbol - . Table 8A-D give the estimates μ and $\hat{\sigma}$ for the Cases A-D for all remaining observations; per column the *lowest* and **highest** estimates are indicated.

Table 8A. Estimates of mean (4.50) and standard deviation (2.70) for Case A (exam).

Method i	1	2	3	4P	4J	1	2	3	4P	4J
Student			$\hat{\mu}$					$\hat{\sigma}$		
i01	5.27	5.00	5.27	-	6.57	1.70	2.32	1.62	-	6.30
i02	5.09	4.99	4.82	5.00	5.00	1.98	2.09	2.02	2.68	2.62
i03	5.63	**6.06**	**5.80**	5.50	5.50	1.89	2.18	*1.49*	*1.33*	*1.33*
i04	5.09	5.63	5.02	4.64	*4.67*	1.68	1.82	1.93	2.05	2.05
i05	5.30	5.03	5.02	5.41	5.42	1.51	1.98	1.71	2.62	2.58
i06	5.34	*4.80*	5.30	-	**7.07**	1.88	2.34	1.80	-	6.30
i07	5.62	5.23	5.00	4.96	5.04	1.95	1.98	2.00	2.43	2.33
i08	5.17	5.77	5.02	4.74	4.74	1.74	*1.76*	2.12	1.84	1.86
i09	5.19	4.83	5.32	5.47	5.43	2.10	**2.38**	1.90	1.86	1.79
i10	*4.87*	5.75	5.48	*4.26*	5.50	**2.13**	2.29	2.00	4.38	2.45
i11	**5.77**	5.85	*4.26*	-	6.51	1.92	2.13	1.87	-	**13.88**
i12	5.69	5.40	5.56	**6.20**	6.13	1.24	1.80	1.94	**4.49**	2.68
i13	5.27	5.35	5.33	5.56	5.50	*1.22*	2.13	**2.35**	1.86	1.78
i14	5.00	5.15	5.70	5.85	5.83	1.82	2.13	1.76	1.73	1.74

Table 8B. Estimates of mean (38.8) and standard deviation (22.8) for Case B (age).

Method i	1	2	3	4P	4J	1	2	3	4P	4J
Student			$\hat{\mu}$					$\hat{\sigma}$		
i01	44.1	41.7	41.1	42.4	41.9	19.8	23.8	25.4	**28.8**	**28.5**
i02	40.4	40.9	40.3	38.8	38.6	19.8	26.7	23.0	26.8	27.0
i03	37.9	42.7	34.8	39.1	39.0	22.6	*18.3*	20.4	23.8	23.8
i04	*34.0*	41.8	39.5	35.0	35.0	16.9	28.2	24.1	17.3	17.5
i05	40.9	40.9	42.1	43.2	42.9	22.2	22.9	19.0	27.4	27.5
i06	34.6	42.3	40.0	*	*	23.6	25.2	*17.4*	*	*
i07	39.4	38.6	*32.9*	*30.0*	*30.0*	23.6	26.5	19.8	20.3	20.4
i08	43.4	*37.7*	45.6	37.0	37.1	26.2	**28.4**	**26.4**	22.4	22.6
i09	**48.6**	39.0	41.9	33.0	33.7	**24.6**	20.0	19.2	44.9	26.8
i10	39.1	**49.2**	44.2	40.0	40.0	*16.8*	25.1	23.7	23.1	23.3
i11	37.6	43.8	36.6	31.4	31.5	23.3	25.0	19.1	23.4	23.6
i12	36.6	39.2	**48.9**	**47.2**	**46.7**	21.6	24.8	19.4	21.5	21.1
i13	43.0	39.9	40.3	43.0	42.8	21.9	22.1	24.2	*15.8*	*15.7*
i14	40.7	47.8	43.4	38.5	38.5	23.1	19.7	21.9	23.0	23.0

Table 8C. Estimates of mean (200) and standard deviation (20.0) for Case C (life time).

Method i	1	2	3	4P	4J	1	2	3	4P	4J
Student			$\hat{\mu}$					$\hat{\sigma}$		
i01	202	220	168	-	219	159	192	108	-	**830**
i02	153	*104*	213	174	172	68	85	164	105	104
i03	147	198	227	**250**	246	170	137	*82*	195	190
i04	183	172	208	146	148	127	*47*	149	76	73
i05	193	199	248	211	206	123	139	127	159	154
i06	214	217	219	212	208	168	158	163	145	139
i07	*88*	214	**276**	-	**293**	61	124	**186**	-	328
i08	215	141	210	-	121	146	84	155	-	174
i09	152	189	213	*	*	*58*	150	153	*	*
i10	203	261	*117*	*102*	*100*	148	**243**	96	*69*	*66*
i11	186	185	211	215	210	126	159	127	155	148
i12	97	**265**	207	241	231	78	222	113	**228**	201
i13	213	199	198	113	107	179	139	151	195	109
i14	**221**	133	206	202	198	**190**	116	149	137	134

Table 8D. Estimates of mean (61.5) and standard dev.(13.0) for Case D (amount A).

Method i	1	2	3	4P	4J	1	2	3	4P	4J
Student			$\hat{\mu}$					$\hat{\sigma}$		
i01	*	61.5	48.4	58.0	57.3	*	17.5	23.6	**32.6**	32.1
i02	*	*	59.8	-	60.6	*	*	19.0	-	**106.1**
i03	57.6	57.5	**63.4**	**69.2**	**68.6**	22.1	22.2	*11.4*	22.0	21.4
i04	**65.0**	**64.9**	58.7	65.5	65.9	15.7	11.3	19.5	23.7	23.4
i05	61.1	60.5	*34.8*	61.4	61.3	13.1	14.4	18.9	*10.5*	*10.5*
i06	*	*38.0*	59.9	63.1	63.2	*	23.4	21.6	18.7	21.4
i07	58.2	52.5	*	*	*	**28.1**	22.2	*	*	*
i08	*48.7*	62.0	62.3	*43.3*	*43.8*	25.6	**34.3**	20.4	23.9	23.6
i09	59.9	43.1	*	*	*	*6.8*	26.3	*	*	*
i10	63.2	48.2	58.1	-	65.3	12.4	21.2	**26.4**	-	32.4
i11	62.3	62.1	58.9	61.5	61.7	14.0	*6.6*	17.8	11.9	11.7
i12	60.4	59.3	*	65.6	65.1	17.8	19.1	*	17.5	17.7
i13	59.0	57.9	61.2	50.0	49.9	20.7	18.3	21.5	14.0	14.2
i14	58.5	58.4	60.7	49.8	50.0	20.3	28.0	20.1	31.3	31.3

To compare the accuracy of the different methods, the MAD and MSE are calculated per column. The values of these accuracy measures for μ and σ are presented separately in Tables 9A-B.

Table 9A. Accuracy of estimating μ.

	MAD					MSE			
Case	A	B	C	D	Case	A	B	C	D
Method					Method				
1	0.81	*3.12*	33.5	*3.17*	1	0.72	*15.74*	2291	*20.6*
2	0.85	3.21	32.7	**6.38**	2	0.87	19.32	1945	**93.2**
3	*0.74*	3.77	*25.4*	5.08	3	*0.64*	20.56	*1278*	84.2
4P	0.78	**3.84**	39.5	6.25	4P	0.84	**23.54**	2541	70.9
4J	**1.14**	3.67	**43.5**	5.57	4J	**1.76**	21.51	**3114**	58.3

Table 9B. Accuracy of estimating σ

	MAD					MSE			
Case	A	B	C	D	Case	A	B	C	D
Method					Method				
1	0.93	*2.03*	48.0	*6.09*	1	0.94	*7.71*	2757	*59.0*
2	*0.60*	2.83	54.6	8.61	2	*0.40*	10.56	4416	102.3
3	0.81	2.60	*40.5*	7.30	3	0.69	8.72	*2194*	61.8
4P	0.85	**4.26**	57.5	8.32	4P	1.05	**50.01**	4609	109.3
4J	**1.78**	2.91	**113.3**	**16.44**	4J	**11.19**	13.63	**47590**	**844.4**

Method 1 is best twice, in estimiating both μ and σ; Methods 2 and 3 are best once each. Methods 1 and 3 are never worst. A slightly more sophisticated comparison is obtained by ranking per column from 1 (highest accuracy) to 5 (lowest). Adding these rankings over the four cases leads to Table 10.

Table 10. Total rankings of estimation methods.

	μ		σ	
Method	MAD	MSE	MAD	MSE
1	8	7	8	7
2	13	13	11	10
3	8	9	7	7
4P	15	15	15	17
4J	16	16	19	19

It is clear that Method 4 performs worst - even after the exlusion of the eight problem cases for 4P. Methods 1 and 3 do not show much difference and both outperform Method 2. So, the preliminary conclusion of our experiment is that the

$$\begin{cases} \text{Extended Pearson-Turkey, 3-parameter} \\ \text{Sebridge-Lan-Ho, 7-parameter} \end{cases}$$

methods are most suitable for the practical elicitation of experts' knowledge.

6 Discussion

Four different methods were compared empirically for obtaining from experts estimates for mean and standard deviation; they all ask the expert only quantiles. The comparison was based on an experiment in which 54 students were confronted with four cases, each applying one of the four methods. The main conclusion of this -limited- experiment is that the

Extended Pearson-Turkey, 3 parameter

Sebridge-Lan-Ho, 7-parameter

methods seem to outperform the

Extended Swanson-Megill, 3-parameter

method and is superior to the octile-based method

Moors-Schuld-Mathijssen, 7-parameter

The latter conclusion is disappointing - at least for the first author who was a main developer. It was expected that this Method 4 could be superior, mainly for the following reason. A quartile is in fact the median of the lower or upper half of the distribution, and an 'odd' octile the median of a quarter; hence the determination of the seven octiles boils to the repeated estimation of medians. The exact relation between distributions within the Pearson or Johnson system and the derived quartet (Q, R, S, T) was considered to be a second advantage: no approximation formulae for estimating μ and σ from the quantiles were needed. Hower, Method 4 was not even superior with respect to the accuracy of quantile estimation. The non-existence of moments for a set of distributions within the Pearson system proved to be further handicap.

Of course, it would be premature to dismiss methods on the base of this limited experience. Further experimentation will be needed.

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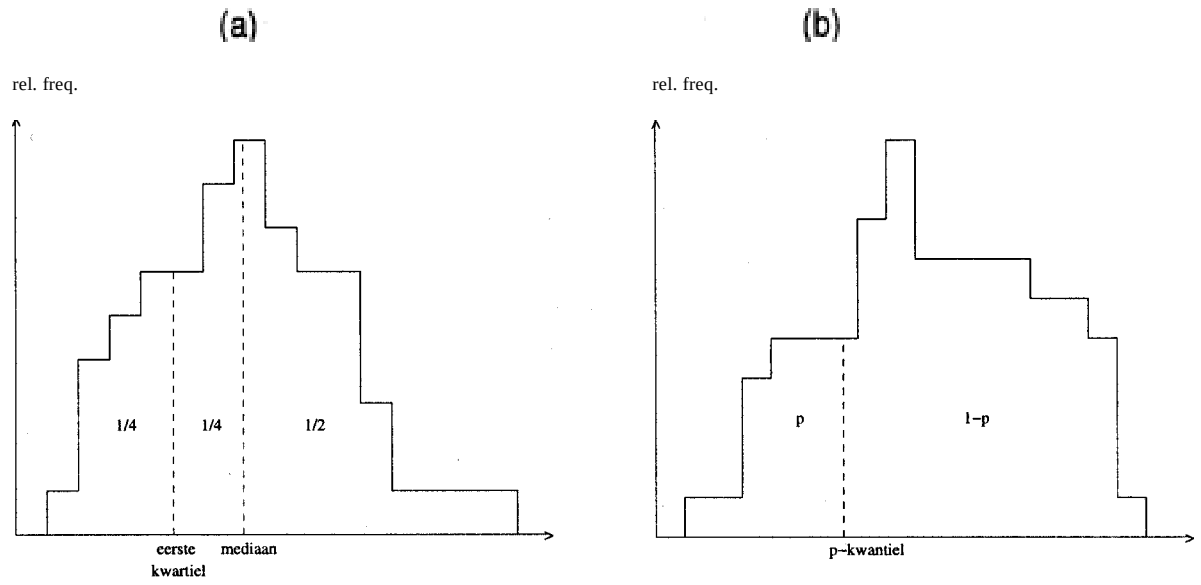
Appendix 1: Instruction Method 1 (in Dutch)

STATISTISCH EXPERIMENT

Bij dit experiment wordt je gevraagd enkele *kwantielen* to bepalen. Daarom leggen we eerst nog even uit wat kwantielen zijn.

We beginnen met twee voorbeelden. De *mediaan* verdeelt bij een frequentieverdeling alle waarden in twee gelijke helften: er liggen dus evenveel waarden boven de mediaan als eronder. Het *eerste kwartiel* verdeelt de waarnemingen in één kwart 'lage' waarden en drie kwart 'hoge' waarden. Zie Figuur 1a.

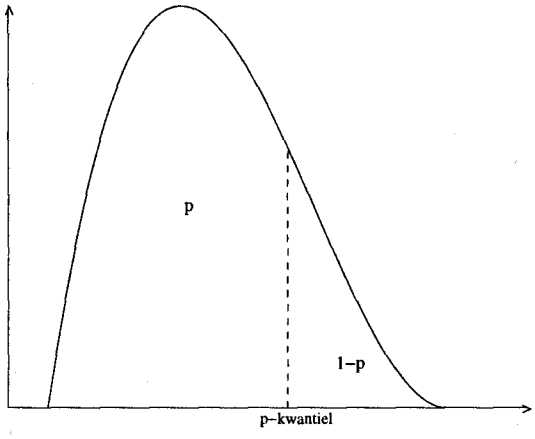
Figuur 1. Kwantielen bij frequentieverdelingen.

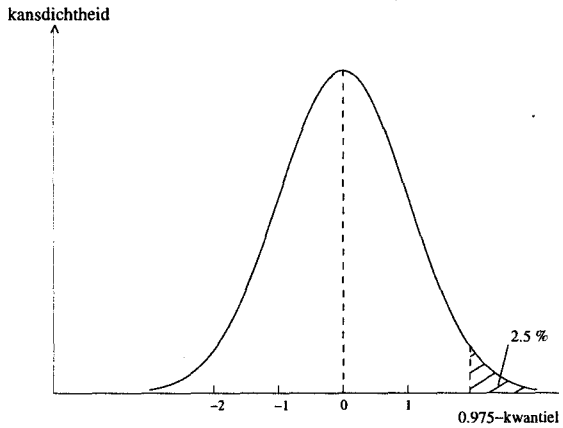


Algemeen verdeelt een *p*-kwantiel de waarden in een fractie *p* van 'lage' waarden en een fractie 1-p 'hoge' waarden; zie Figuur 1b. (De mediaan is dus tegelijk ook het 0.5-kwantiel en het eerste kwartiel is tevens het 0.25 kwantiel.)

Hetzelfde geldt eigenlijk bij de kansverdeling van een toevalsvariabele X; alleen wordt het woord 'fractie' door 'kans' vervangen. Met een kans p is nu de toevalsvariabele kleiner dan het p-kwantiel. Oppervlakten onder de kansdichtheid geven kansen aan. Figuur 2 laat een algemeen plaatje zien en als voorbeeld het bekende 0.975-kwantiel 1.96 van de standaardnormale verdeling.

Figuur 2. Kwantielen bij kansverdelingen



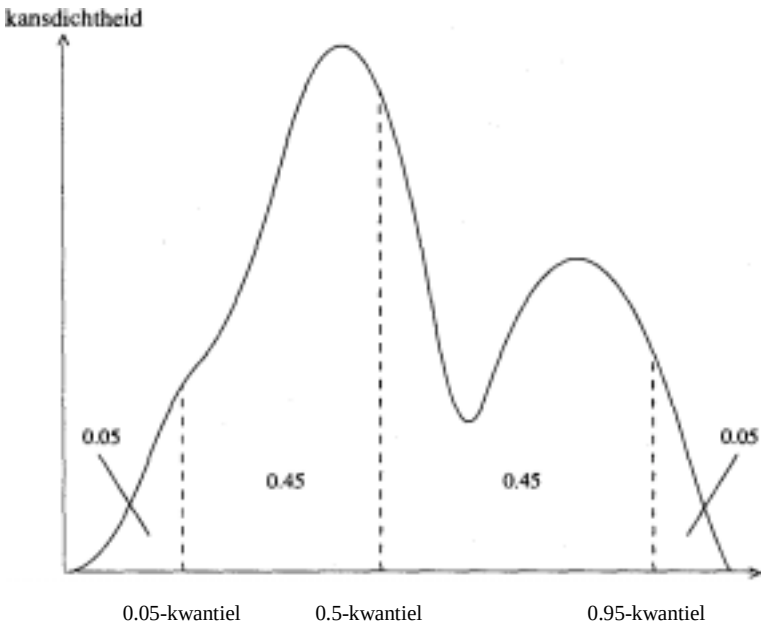


Wat jou bij dit experiment gevraagd wordt is om in vier verschillende gevallen aan te geven wat naar jouw mening de waarden zijn van drie kwantielen. Het gaat om de p-kwantielen met voor p de volgende waarden:

0.05 0.5 0.95.

Figuur 3 laat voor één kansdichtheid deze drie kwantielen nog eens zien.

Figuur 3. De drie gevraagde kwantielen.



GEVAL 1.

De eerste vraag gaat over het Statistiektentamen van 7 januari - waar je dadelijk jouw eigen cijfer van hoort. We willen weten hoe goed je de frequentieverdeling van de cijfers van alle deelnemers kunt schatten. Je hoeft alleen de drie gevraagde kwantielen in te vullen op het antwoordformulier. Het gaat over de oorspronkelijke niet-afgeronde cijfers. Twee voorbeelden: als je denkt dat 40% van de deelnemers lager scoort dan 4.6, dan geldt voor jou in Geval 1:

0.4-kwantiel = 4.6.

Als je verwacht dat 15% hoger scoort dan 7.8 vul je in

0.85-kwantiel = 7.8.

(Let er op: deze twee kwantielen worden niet gevraagd!)

GEVAL 2.

De tweede vraag gaat over de leeftijdsverdeling van de Nederlandse vrouwen op 1 januari 1996. Vul de kwantielen in in hele jaren. Vermoed je dat 30% van de vrouwen toen ouder was dan 58, dan geldt voor jou

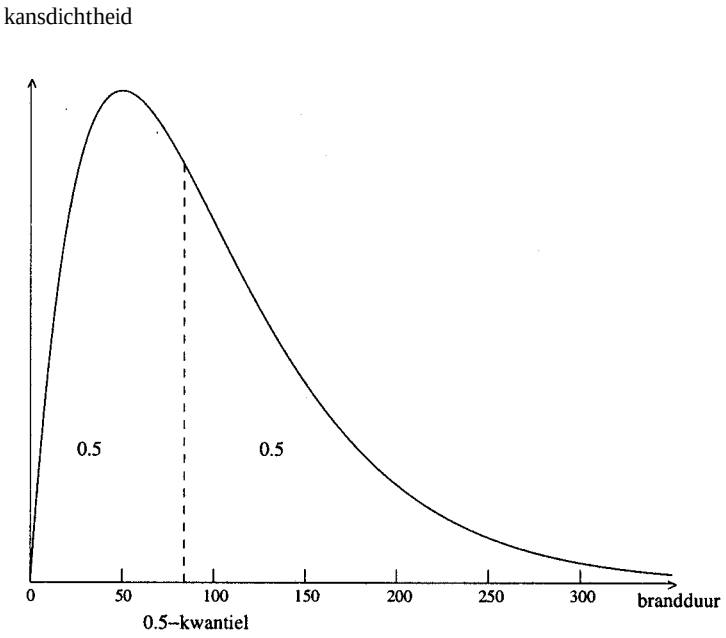
0.7-kwantiel = 58.

Vul de drie kwantielen weer in op het antwoordformulier.

GEVAL 3.

Deze (en de laatste) vraag gaat over kansverdelingen. De brandduur van een bepaalde soort gloeilampen hangt of van het toeval. Figuur 4 toont de kansdichtheid van de brandduur van zo'n gloeilamp; ter verduidelijking is de mediaan ingetekend.

Figuur 4. Kansdichtheid brandduur gloeilamp.



De vraag gaat nu over de totale brandduur van **twee** van deze gloeilampen samen. Elke van deze twee lampen heeft de kansverdeling uit Figuur 4; de twee brandduren zijn onafhankelijk. Vul op het antwoordformulier de waarden in die de drie kwantielen van deze totale brandduur volgens jou hebben.

GEVAL 4.

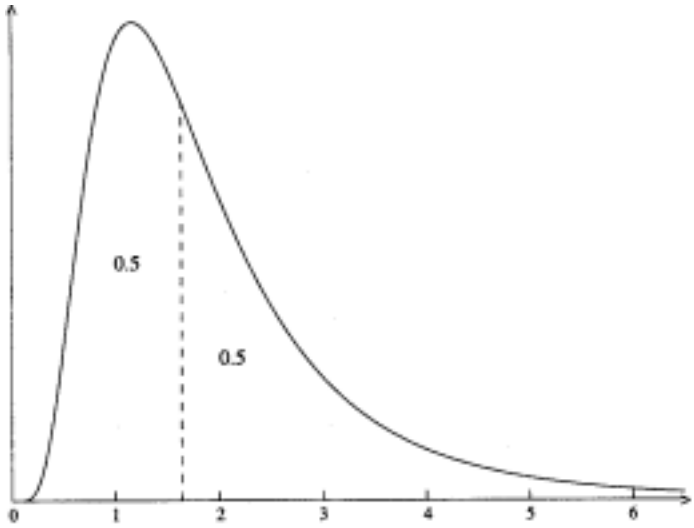
Twee personen, A en B, mogen *f100,-* verdelen. De verhouding tussen het bedrag dat A krijgt en het bedrag van B wordt geloot: dit *verhoudingsgetal*

$$V = \frac{\text{bedrag A}}{\text{bedrag B}}$$

is dus een toevalsvariabele, waarvan de kansdichtheid is weergegeven in Figuur 5. Te zien is dat met een kans van 0.5 het verhoudingsgetal V groter is dan 1.64.

Figuur 5. Kansdichtheid verhoudingsgetal *V*.

kansdichtheid



Het bedrag dat A in handen krijgt is ook van het toeval afhankelijk. Is *v* = 4 bijvoorbeeld de lotingsuitkomst, dan krijgt A vier keer zoveel als B: A ontvangt dus f80,-. Bij *v* = 1/3 krijgt juist B driemaal zoveel als A, zodat A slechts f25,- krijgt. Algemeen geldt:

$$\text{bedrag A} = \frac{\text{v}}{\text{v}+1}\, f100,-$$

De vraag gaat over de kansverdeling van het bedrag dat A ontvangt. Vul de naar jouw mening juiste waarden van de drie kwantielen in op het antwoordformulier.

Dat was alles! Hartelijk dank voor je medewerking en hopelijk voor jou was je zò goed dat je een boekenbon wint.

ANTWOORDFORMULIER

Naam:

Administratienummer:

Geval	1 (tentamen)	2 (leeftijd)	3 (brandduur)	4 (bedrag A)
0.05-kwantiel				
0.50-kwantiel				
0.95-kwantiel				

7 Appendix 2: Observed quantiles

Method 1

Student	A (exam)			B (age)			C (life time)			D (amount A)		
101	2	5.6	7.4	4	50	64	28	160	520		*	
102	2	5	8.5	4	43	68	30	160	250		*	
103	2	6	8	4	36	78	20	80	500	15	62	85
104	2.5	5	8	10	32	65	20	160	425	33	70	80
105	3.2	5.1	8.1	6	40	79	48	165	435	37.5	62.5	80
106	2.7	5.1	8.8	5	30	80	30	170	550		*	
107	2	5.9	8.3	5	37	82	4	80	200	5	63	95
108	2.1	5.3	7.8	1	43	87	32	186	495	8	48	92
109	1.9	5.1	8.8	9	48	90	60	150	250	47	61	69
110	1.5	4.8	8.5	10	40	65	50	160	500	40	65	80
111	2.4	5.9	8.7	4.3	35	80	17.2	166	423	38.2	63	84
112	4	5.5	8	6	34	76	3	80	250	25	65	80
113	3.4	5.2	7.4	3.1	45.6	74.2	16	166	570	18.7	63.4	84.1
114	2	5	8	4	40	80	16	170	600	20	62	85

Method 2

Student	A (exam)			B (age)			C (life time)			D (amount A)		
201	2	5	8	9	45	70	20	160	500	40	60	85
202	2	5.5	7.3	7	40	76	11	82	225		*	
203	3.1	6.3	8.7	21	40	68	60	150	400	25	65	80
204	3.8	5	8.3	9	37	81	110	176	230	48	70	75
205	2.5	5	7.6	12	40	71	50	160	400	41	62	78
206	2	4.5	8	10	42	75	60	160	450	10	35	70
207	2.7	5.2	7.8	7	35	75	100	160	400	20	60	75
208	3.3	6.1	7.8	3	35	76	19	166	230	10	80	90
209	2	4.5	8.1	10	45	60	18	160	400	25	26.5	83.3
210	2.4	6.5	8.1	14	54	78	24	170	620	24	44	78
211	3	6	8.5	9	48	73	50	105	425	53	63	70
212	3.3	5.1	7.9	8	38	72	70	160	600	33	66	82
213	2.5	5.5	8	20	30	73	50	160	400	33	60	80
214	2.5	5	8	20	52	70	10	100	300	20	62	92

Method 3

Student	A (exam)							B (age)						
301	2.5	3	4	5.2	6.5	7.5	9	1	4	20	40	60	80	93
302	1	2	3.2	5	6	7.75	9	1	10	20	40	55	76	90
303	2	4.3	5	5.6	6.6	7.7	9.5	2	5	15	34	55	63	76
304	1.2	3	3.5	4.5	6.5	8.1	9.3	2	8	19	37	57	75	98
305	1	2.5	3.8	5.5	6	7	8.5	4	15	30	40	58	68	80
306	1	3	3.5	5.8	6.6	7.5	8.8	4	12	25	43	57	60	71
307	1	2	4	5	6	8	9	2	10	18	25	50	65	80
308	1	2	3.5	5	6.5	8	9.5	0.1	0.5	25	45	75	80	90
309	1.5	2.3	4.4	5.3	6.8	7.4	9.7	8	15	27	43	52	71	83
310	0.5	3	4.5	5.5	6.5	8	10	3	8	25	45	62	80	88
311	1	2	2.5	4.5	5	7	9	3	12	25	35	45	65	85
312	1	3	4	5.5	7.5	8	9	5	20	40	50	60	73	90
313	1.4	1.8	3	5.5	7.3	8.9	9.5	1	8	20	36	60	80	89
314	1.5	3.5	4	6	7	8	9	5	10	25	45	60	75	85

Method 3

Student	C (life time)							D (amount A)						
301	10	40	70	160	240	300	500	5	15	25	50	70	80	95
302	8	40	100	160	300	430	800	13	30	50	62	75	80	96
303	80	120	160	200	320	360	370	30	45	61	65	70	77	80
304	15	40	90	160	300	460	609	20	25	45	62	75	83	92
305	70	90	160	230	300	450	600	10	15	20	30	45	60	100
306	3	20	90	174	320	500	620	1	33	47	60	75	90	95
307	20	60	150	200	400	600	750				*			
308	20	50	80	160	300	450	700	10	33	50	62	80	90	92
309	10	40	80	170	300	500	580				*			
310	2	20	40	80	160	300	380	1	15	40	60	83	92	99
311	50	75	100	170	300	400	600	20	30	50	62	69	80	95
312	50	100	150	180	200	400	600				*			
313	5	20	60	170	280	460	580	15	28	42	65	80	90	94
314	10	50	90	160	280	450	650	2	25	50	70	75	80	87

Method 4

Student	A (exam)							B (age)						
401	2	4	4.5	5	5.5	7	8	10	15	25	30	55	73	80
402	2	3.5	4	5	5.5	6.5	7.5	6	12	23	35	56	65	80
403	3.5	4.5	5	5.5	6.2	6.5	7	15	22	30	38	45	55	70
404	2	3	4.5	5	6	6.5	7	15	20	25	35	40	50	60
405	2.3	4.3	4.6	6	7	7.2	8.5	6	15	30	53	60	70	75
406	3	4.5	5	5.5	6	7.5	9				*			
407	2	3.5	4.5	5.8	6.5	7	8	10	15	20	30	35	45	60
408	2.5	3	4	4.8	6	6.5	7.5	8	14	20	40	50	60	70
409	3	4	4.5	5	6	6.5	7.5	15	25	35	40	45	50	60
410	1	3	4.5	5.5	6	7	8	10	20	25	40	55	60	80
411	1.5	4	4.8	5.2	5.7	7	8.5	3	8	15	35	45	55	68
412	3.5	4.5	5	5.5	6	7	8	17	28	35	40	52	63	70
413	3.5	4	4.5	4.8	5.1	6.7	7.2	22	29	37	40	48	55	60
414	3	4.1	4.6	5.5	7.1	7.6	8	9	17	33	48	55	60	67

Method 4

Student	C (life time)							D (amount A)						
401	20	50	100	160	200	300	600	10	30	35	50	60	80	90
402	50	80	100	150	172	253	300	20	40	45	50	55	65	80
403	40	80	120	160	300	400	550	45	50	55	60	70	85	97
404	60	100	125	170	190	210	250	30	50	60	70	80	85	90
405	60	100	134	178	220	280	380	48	52	55	60	67	70	78
406	60	90	120	160	240	300	420	35	50	57	62	70	75	81
407	65	120	140	160	210	300	500				*			
408	20	43	75	80	102	145	210	15	25	45	50	60	65	70
409				*							*			
410	38	47	62	83	105	140	192	40	55	56	62	66	71	77.5
411	60	90	150	170	220	300	380	45	53	58	65	68	72	75
412	44	80	125	160	220	320	460	40	53	57	62	68.8	74.4	79.2
413	30	50	60	80	100	130	190	31	35	40	50	60	65	69
414	50	80	120	160	240	300	400	10	20	40	60	67	80	85