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# Optimal Auctions with Financial Externalities[¤]

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Abstract

We construct optimal auctions when bidders face …nancial externalities. In a Coasean World, in which the seller cannot prevent a perfect resale market, nor withhold the object, the lowest-price all-pay auction is optimal. In a Myersonean World, in which the seller can both prevent resale after the auction, and fully commit to not selling the object, an optimal two-stage mechanism is derived. In the …rst stage, bidders are asked to pay an entry fee. In the second stage, bidders play the lowest-price all-pay auction with a reserve price. In both worlds, the expected revenue is increasing in the …nancial externality, and each bidder's expected utility is independent of the …nancial externality.

Keywords: Optimal auctions, …nancial externalities, lowest-price all-pay auction, Coasean World, Myersonean World.

JEL classi…cation: D44

## 1 Introduction

We will consider the problem of a seller who wishes to sell one indivisible object in an optimal auction in an environment with …nancial externalities. An optimal auction is a feasible auction mechanism that maximizes the seller's expected revenue. To get an idea about the environment, imagine that two …rms bid for a license to increase their capacity in the market in which they compete. When …nancial markets are assumed not to work perfectly, the winner is able to invest less, the higher the price it pays in the auction. This is an advantage to the losing …rm, so that the losing …rm's utility depends on the payments made in the auction by the winner. Throughout the paper, we will refer to the

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e¤ect of other bidders' payments on a bidder's utility as a …nancial externality.[1] Especially in high stake auctions, like the UMTS auctions that took place in Europe in 2000 and 2001, …nancial externalities may in‡uence bidding behavior (Maasland and Onderstal, 2002; Börgers and Dustmann, 2001).

Myerson (1981) initiates research on optimal auctions in an environment without …nancial externalities.[2,3] He derives three important results. The …rst is the celebrated Revenue-Equivalence Theorem, which states that the expected utility of both the bidders and the seller is completely determined by the allocation rule of the feasible auction mechanism and the utilities of the lowest types. We refer to this result as the Weak Revenue-Equivalence Theorem. Second, with symmetric bidders, all standard auctions yield the same expected revenue. We refer to this result as the First Strong Revenue-Equivalence Theorem. Third, with symmetric bidders, all standard auctions are optimal when the seller imposes the same, optimal reserve price. We refer to this result as the Second Strong Revenue-Equivalence Theorem.[4]

With asymmetric bidders, under a regularity condition, Myerson shows that the optimal auction assigns the object to the bidder with the highest marginal revenue, provided that the highest marginal revenue is nonnegative. In case all bidders have a negative marginal revenue, the seller keeps the object. Moreover, the utilities of the lowest types in an optimal auction are equal to zero. For this result, Myerson assumes that (1) the seller can prevent resale of the object after the auction, and (2) he can fully commit to not selling the object. The …rst assumption is made, as the seller may need to misassign the object, i.e., assign it to a bidder who does not have the highest value for it. The second assumption is made, as the seller may optimally withhold the object when only low valued bidders participate. When these assumptions hold, we will speak of a Myersonean World.

Ausubel and Cramton (1999) argue that sometimes the assumption of a Myersonean World is not realistic, and study optimal auctions in a setting in which (1) the seller cannot prevent the object changing hands in a perfect resale market,[5] and (2) he cannot commit to keeping the object. We will refer to this setting as a Double Coasean World, as the …rst assumption is related to the Coase Theorem (Coase, 1960), and the second to the Coase Conjecture (Coase, 1972). Haile (1999) proves that, with symmetric bidders, equilibrium bidding in standard auctions does not change when bidders are o¤ered a re-

[1] In our companion paper (Maasland and Onderstal, 2002), we study equilibrium behavior in …rst-price and second-price sealed-bid auctions in an environment with …nancial externalities. Theories of equilibrium bidding in related environments can be found in Engelbrecht-Wiggans (1994), and in Bulow et al. (1999).

[2] Independently and simultaneously, Riley and Samuelson (1981) derive similar results.

[3] Myerson (1981) was followed by, among others, Engelbrecht-Wiggans (1988), Cremer and McLean (1985, 1988), Maskin and Riley (1989), McAfee and Reny (1992), and Bulow and Klemperer (1996).

[4] Myerson does not mention this result explicitly, but it follows from his study. Riley and Samuelson (1981) explicitly derive the result in an independent private values model.

[5] In a perfect resale market, the object, when being sold in the auction, always ends up in the hands of the bidder with the highest value.

sale market opportunity after the auction. With this result, the Third Strong Revenue-Equivalence Theorem can be derived: In a Double Coasean World, with symmetric bidders, all standard auctions (without reserve price) are optimal.

In this paper, we modify Myerson's model by allowing for …nancial externalities, given by an exogenous parameter '. We assume a model with independent private signals. The model has independent private values models and pure common value models as special cases. With symmetric bidders, this model is a special case of the a¢liated private signals model of Milgrom and Weber (1982). We will show that with …nancial externalities, the Weak Revenue-Equivalence Theorem remains valid. Also the conditions for optimality remain the same as in Myerson.

Our companion paper (Maasland and Onderstal, 2002) shows that the strong revenue-equivalence results are not valid when bidders are confronted with …nancial externalities. The First Strong Revenue-Equivalence Theorem does not hold as in the case of two bidders, the …rst-price sealed-bid auction yields less expected revenue than the second-price sealed-bid auction. The driving force behind this result is that the expected utility of the lowest type in the …rst-price auction is higher than the expected utility of the lowest type in the second-price auction. The Second Strong Revenue-Equivalence Theorem does not hold for two reasons. First, a standard auction with reserve price gives the lowest type strictly positive expected utility because of the payments by others. Second, the …rst-price sealed-bid auction and the second-price sealed-bid auction may not have equilibria in which active bidders submit bids according to a function that is strictly increasing in their type, so that the winner of the object is not always the bidder with the highest marginal revenue. The Third Strong Revenue-Equivalence Theorem fails to hold as in both the …rst-price and the second-price sealed-bid auction, the lowest type gets a strictly positive expected utility.

In the remainder of the paper, we will show the optimality of the lowest-price all-pay auction when we take a symmetry assumption. (Goeree and Turner, 2002, derive a similar result in a related environment.) In Section 4, we solve the seller's problem in a Double Coasean World. We start with this setting, as it is more straightforward to …nd an optimal auction here than in a Myersonean World. We derive that the lowest-price all-pay auction is optimal, as the lowest type gets zero expected utility. In Section 5, we …nd a two-stage feasible auction mechanism which solves the seller's problem in a Myersonean World. In the …rst stage, all bidders pay an entry fee, in order to make sure that the lowest type gets zero expected utility. If at least one of the bidders indicate not to be willing to accept the entry fee, the seller keeps the object, and no payments are made. Otherwise, in the second stage, the lowest-price all-pay auction with a reserve price is played. The optimality of the lowest-price all-pay auction with a reserve price follows from the observation that, if it assigns the object, it always assigns the object to the bidder with the highest marginal revenue. In both worlds, in an optimal auction, the highest possible expected revenue is strictly increasing in ', and a bidder's expected utility is independent of '.

# 2 The model

Consider a seller, who wishes to sell one indivisible object to one out of $n$ risk neutral bidders, numbered $1, 2, \ldots, n$. The seller aims at finding a feasible auction mechanism which gives him the highest possible expected revenue. We assume that the seller does not attach any value to the object. Each bidder $i$ receives a one-dimensional private signal $t_i$. (We also say that "bidder $i$ is of type $t_i$".) $t_i$ is drawn, independently from all the other private signals, from a distribution function $F_i$. $F_i$ has support on the interval $[\underline{t}_i, \bar{t}_i]$, and continuous density $f_i$ with $f_i(t_i) > 0$, for every $t_i \in [\underline{t}_i, \bar{t}_i]$. Define the sets

$$T \equiv [\underline{t}_1, \bar{t}_1] \times \ldots \times [\underline{t}_n, \bar{t}_n],$$

and

$$T_{-i} \equiv \times_{j \neq i} [\underline{t}_j, \bar{t}_j],$$

with typical elements $t \equiv (t_1, \ldots, t_n)$ and $t_{-i} \equiv (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ respectively. Let

$$g(t) \equiv \prod_j f_j(t_j)$$

be the joint density of $t$, and let

$$g_{-i}(t_{-i}) \equiv \prod_{j \neq i} f_j(t_j)$$

be the joint density of $t_{-i}$.

The value of the object for a bidder is defined as a function of her own signal, and the signals of all the other bidders. Denote by $v_i(t)$ the value for bidder $i$ given that the vector of types is $t$. We make the following assumptions on the functions $v_i$.[6]

Value Differentiability: $v_i$ is differentiable in all its arguments, for all $i, t$.
Value Monotonicity: $v_i(t) \geq 0$; $\frac{\partial v_i(t)}{\partial t_i} > 0$; and $\frac{\partial v_i(t)}{\partial t_j} \geq 0$, for all $i, j, t$.

Value Differentiability ensures the existence of each bidder's marginal revenue (which will be defined later). Value Monotonicity indicates that all bidders are serious, and that bidders' values are strictly increasing in their own signal, and weakly increasing in the signals of the others.

[6] Myerson (1981) uses the following value functions:

$$v_i(t) \equiv t_i + \sum_{j \neq i} e_j(t_j);$$

where $e_j$ is the revision effect function related to bidder $j$, with $e_j : [\underline{t}_j, \bar{t}_j] \to \mathbb{R}$. These value functions are not necessarily included in our model.

In Sections 4 and 5, we make the following extra assumption.

Symmetry: $F_i = F_j$ for all $i, j$, and $v_i(\dots, t_i, \dots, t_j, \dots) = v_j(\dots, t_j, \dots, t_i, \dots)$ for all $t_i, t_j, i, j$.

Symmetry may be crucial for the existence of efficient equilibria in standard auctions.[7] Value Differentiability, Value Monotonicity, and Symmetry together ensure that the bidder with the highest signal is also the bidder with the highest value, so that these assumptions imply that the seller assigns the object efficiently if and only if the bidder with the highest signal gets it.

When Symmetry holds, let $F \equiv F_1 = \dots = F_n$, $f \equiv f_1 = \dots = f_n$, $\underline{t} \equiv \underline{t}_1 = \dots = \underline{t}_n$, and $\bar{t} \equiv \bar{t}_1 = \dots = \bar{t}_n$. Also, we will let $F^{[1]}$ and $f^{[1]}$ ($F^{[n-1]}$ and $f^{[n-1]}$) denote the cumulative distribution function and density function of $\max_{j \neq 1} t_j$ ($\min_{j \neq 1} t_j$). Finally, let us define $v(y, z)$ as the expected value that bidder $i$ assigns to the object, given that her signal is $y$, and that the highest signal of all the other bidders is equal to $z$.

$$v(y, z) \equiv E_{t_{-i}}[v_i(t) \mid t_i = y, \ \max_{j \neq i} t_j = z].$$

With Symmetry, this model is a special case of the affiliated signals model of Milgrom and Weber (1982).

Throughout the paper, we use the following definition of bidder $i$'s marginal revenue.

$$MR_i(t) \equiv v_i(t) - \frac{1 - F_i(t_i)}{f_i(t_i)} \frac{\partial v_i(t)}{\partial t_i}.$$

This expression can be derived, like in Bulow and Roberts (1989) (for independent private values) and Bulow and Klemperer (1996) (for independent private signals), from the monopolist's problem in third-degree price discrimination. This can be done by constructing bidder $i$'s demand curve from her value function and signal distribution function, and differentiate the related monopolist's profit function with respect to quantity. When Symmetry is satisfied, let $MR(t) \equiv MR_1(t) = \dots = MR_n(t)$. We make the following assumption on $MR_i$.

MR Monotonicity: $MR_i(t)$ is strictly increasing in $t_i$ for all $i, t$.

MR Monotonicity is equivalent to the assumption made in standard microeconomic theory that the monopolist's demand curve is downward-sloping.

The bidders are risk-neutral expected utility maximizers. In order to incorporate the financial externalities, we insert an exogenous nonnegative parameter

[7] Klemperer (1998) shows that a slight asymmetry in value functions may have dramatic effects on bidding behavior in the English auction in a(n almost) common value setting. Although efficiency is not an issue with (almost) common values, the result shows the importance of symmetry in value functions. Maskin and Riley (2000) study the effect of asymmetric distributions on bidding behavior in the first-price and the second-price sealed-bid auction, and show that the equilibrium of the first-price auction is inefficient.

$\varphi$ into the bidders' utility functions. This parameter indicates each bidder's interest in the others' payments. The utility of bidder $i$ is

$$v_i - x_i + \varphi \sum_{j \neq i} x_j$$

when $i$ wins the object, and

$$- x_i + \varphi \sum_{j \neq i} x_j$$

when $i$ does not win the object, with $x_j$ the payment by bidder $j$ to the seller. We assume $\varphi \in [0; \frac{1}{n-1})$.[8]

## 3 Weak revenue equivalence

Using the Revelation Principle of Myerson (1981), we may assume, without loss of generality, that the seller only considers feasible auction mechanisms in the class of feasible direct revelation mechanisms.[9] Let $(p; x)$ be a feasible direct revelation mechanism, with

$$p : T \to [0; 1]^n;$$

where

$$\sum_j p_j(t) \leq 1,$$

and

$$x : T \to \mathbb{R}^n:$$

We interpret $p_i(t)$ as the probability that bidder $i$ wins, and $x_i(t)$ as the expected payments by $i$ to the seller when $t$ is announced.

Bidder $i$'s utility of $(p; x)$ given $t$ is given by

$$v_i(t)p_i(t) - x_i(t) + \varphi \sum_{j \neq i} x_j(t);$$

so that bidder $i$'s interim utility of $(p; x)$ can be written as

[8] In case $\varphi \in [0; \frac{1}{n-1})$, a bidder's interest in his own payments is larger than his interest in the payments by the other bidders. In footnote 13, we will discuss the consequences of allowing $\varphi$ to be larger than $\frac{1}{n-1}$.

[9] A feasible direct revelation mechanism is an auction mechanism in which each bidder is asked to announce his type, which satisfies individual rationality conditions, incentive compatibility conditions, and straightforward restrictions on the allocation rule.

$$U_i(p;x;t_i) \equiv \int_{T_{-i}} [v_i(t)p_i(t) - x_i(t) + \alpha \sum_{j \neq i} x_j(t)] g_{-i}(t_{-i}) dt_{-i}, \tag{1}$$

with $dt_{-i} \equiv dt_1 \ldots dt_{i-1} dt_{i+1} \ldots dt_n$.

Similarly, the seller's expected utility of $(p;x)$ is

$$U_0(p;x) \equiv \int_T \sum_{i=1} x_i(t) g(t) dt;$$

with $dt \equiv dt_1 \ldots dt_n$.

The following two lemmas will be used to solve the seller's problem.

**Lemma 1** Let $(p;x)$ be a feasible direct revelation mechanism. Then the interim utility of $(p;x)$ for bidder $i$ is given by

$$U_i(p;x;t_i) = U_i(p;x;\underline{t}_i) + \int_{\underline{t}_i}^{t_i} w_i(s_i) ds_i, \tag{2}$$

with

$$w_i(t_i) \equiv E_{t_{-i}} \{ p_i(t) \frac{\partial v_i(t)}{\partial t_i} \}.$$

*Proof.* Incentive compatibility implies

$$U_i(p;x;s_i) \geq U_i(p;x;t_i) + E_{t_{-i}} \{ p_i(t)(v_i(s_i;t_{-i}) - v_i(t)) \}$$

for all $s_i$, $t_i$ and $t_{-i}$, or, equivalently

$$\frac{\partial U_i(p;x;t_i)}{\partial t_i} = E_{t_{-i}} \{ p_i(t) \frac{\partial v_i(t)}{\partial t_i} \} = w_i(t_i), \tag{3}$$

at all points where $p_i$ is differentiable in $t_i$ (by Value Differentiability, $v_i$ is differentiable at any $t_i$). By integration of (3), we get (2). ∎

**Lemma 2** Let $(p;x)$ be a feasible direct revelation mechanism. The seller's expected revenue from $(p;x)$ is given by

$$U_0(p;x) = \frac{E_t \{ \sum_{i=1} MR_i(t) p_i(t) \} - \sum_{i=1} U_i(p;x;\underline{t}_i)}{1 - \alpha(n-1)}. \tag{4}$$

*Proof.* Define

$$X_i \equiv \int_T x_i(t)g(t)dt, \tag{5}$$

$$V_i \equiv \int_T v_i(t)p_i(t)g(t)dt, \tag{6}$$

and

$$Y_i \equiv \int_{T_i} U_i(p;x;t_i)f_i(t_i)dt_i. \tag{7}$$

By (1), we have, for all $i$,

$$Y_i = V_i - X_i + \varphi \sum_{j \neq i} X_j. \tag{8}$$

Adding the equations in (8) over $i$ and rearranging terms implies that the seller's expected revenue from a feasible direct revelation mechanism $(p;x)$ is given by

$$U_0(p;x) = \sum_{i=1} X_i = \frac{\sum_{i=1} V_i - \sum_{i=1} Y_i}{1 - \varphi(n-1)}. \tag{9}$$

Taking the expectation of (2) over $t_i$ and using integration by parts, we obtain

$$E_{t_i}\{U_i(p;x;t_i)\} = U_i(p;x;\underline{t}_i) + E_{t_i}\{\frac{1 - F_i(t_i)}{f_i(t_i)} w_i(t_i)\},$$

with

$$w_i(t_i) \equiv E_{t_{-i}}\{p_i(t)\frac{\partial v_i(t)}{\partial t_i}\},$$

so that (4) follows with (9) and (5)-(7). ■

From Lemmas 1 and 2, it immediately follows that the Weak Revenue-Equivalence Theorem remains valid with financial externalities.

**Corollary 3** Both the seller's and the bidders' expected utility from any feasible auction mechanism is completely determined by the probability function $p$ and the utilities of the lowest types $U_i(p;x;\underline{t}_i)$ for all $i$ related to its equivalent feasible direct revelation mechanism $(p;x)$.

Observe from Lemmas 1 and 2 respectively that, provided that the expected utility of the lowest type remains zero when $\varphi$ is varied, a bidder's interim utility does not depend on $\varphi$, whereas the seller's expected revenue is increasing in $\varphi$. An intuition for the first observation is the following. Suppose that bidders, instead of receiving financial externalities, obtain a fraction $\varphi$ of what the other bidders pay in the auction. Then Myerson (1981) shows that the interim utility of a bidder does not depend on $\varphi$. From a bidder's perspective, these two situations are equivalent, and the observation follows immediately. The intuition for the second observation follows from the first. Fix the payments of all bidders. Then a bidder's expected utility increases with $\varphi$. Therefore, to make sure that a bidder's interim utility does not depend on $\varphi$, her expected payment must increase as well.

From Lemma 2, interesting insights can be drawn with respect to optimal auctions. Observe that in the expression for the seller's expected revenue, a key role is played by the marginal revenues of the bidders. Suppose that the seller finds a feasible auction mechanism that assigns the object to the bidder with the highest marginal revenue, provided that the marginal revenue is nonnegative, and that leaves the object in the hands of the seller if the highest marginal revenue is negative. Suppose also that this feasible auction mechanism gives the lowest types zero expected utility. Then, under MR Monotonicity,[10] with the individual rationality constraints $U_i(p, x, \underline{t}_i) \geq 0$, this feasible auction mechanism is optimal. In Section 5, we will discuss this observation in more detail, and we will show how the seller can construct an optimal auction in an environment with financial externalities.

## 4 The Double Coasean World

For the remainder of the analysis, we assume that Symmetry holds. Consider the lowest-price all-pay auction, which has the following rules. All bidders simultaneously and independently announce a bid to the seller. The bidder who announces the highest bid gets the object, with ties being broken among the highest bidders with equal probability. Each bidder has to pay the lowest submitted bid. We will show now that in a Double Coasean World, the lowest-price all-pay auction is optimal.

Recall that a Double Coasean World is a situation in which (1) the seller cannot prevent a perfect resale market, and (2) the seller cannot withhold the object. These assumptions impose two extra restrictions on the seller's problem, namely

$$\text{for all } t \text{ and } i,\ p_i(t) > 0 \text{ only if } t_i = \max_j t_j \tag{10}$$

and

$$\text{for all } t,\ \sum_i p_i(t) = 1 \tag{11}$$

[10] This assumption is needed for incentive compatibility considerations. See Myerson (1981) for a further discussion on the consequences of relaxing this assumption.

respectively. In fact, these restrictions fix $p_i(t)$ (apart from the zero mass events $t_i = t_j$ for some $i$ and $j$) in such a way that the object is always assigned to the bidder with the highest signal.

As restrictions (10) and (11) fix the allocation rule $p$, by Lemma 2, a sufficient condition for the optimality of a feasible auction mechanism is that the lowest types expect zero utility (from the auction plus resale market). The lowest-price all-pay auction is a natural candidate to fulfill this requirement. To see this, suppose that in equilibrium, the auction is efficient, and that a bidder with the lowest type considers to bid $b$. Then, as the equilibrium is efficient, all the other bidders have to pay $b$. The expected utility of the lowest type equals $-b + (n-1)\varphi b$, which is strictly negative for all $b > 0$ when $\varphi \in [0; \frac{1}{n-1})$. Therefore, the lowest type prefers to bid zero, so that she obtains zero expected utility, as when she is present, each bidder pays zero in the auction.

Proposition 4 characterizes the symmetric equilibrium for the lowest-price all-pay auction. By a standard argument, the equilibrium bid function must be strictly increasing and continuous. Let $U(t; s)$ be the utility for a bidder with signal $t$ who behaves as if having signal $s$, whereas the other bidders play according to the equilibrium bid function. A necessary equilibrium condition is that

$$\frac{\partial U(t; s)}{\partial s} = 0$$

at $s = t$. From this condition, a differential equation can be derived, from which the equilibrium bid function is uniquely determined (at least if we restrict our attention to differentiable bid functions). Observe that indeed the lowest type bids zero, that the equilibrium is efficient, and that bids increase with $\varphi$.[11]

**Proposition 4** Suppose that all bidders submit a bid according to the following bid function.

$$B(t) = \frac{1}{(1 - \varphi(n-1))} \int_{\underline{t}}^{t} \frac{v(y; y) f^{[1]}(y)}{1 - F^{[n-1]}(y)} dy.$$

Then $B$ constitutes the unique symmetric differentiable Bayesian Nash equilibrium of the lowest-price all-pay auction. The outcome of this auction is efficient.

**Proof.** The following observations imply that a symmetric equilibrium bid function must be strictly increasing and continuous. First, a higher type of a bidder cannot submit a lower bid than a lower type of the same bidder. (If the low type gets the same expected surplus from strategies with two different

[11] In case of a uniform signal distribution on the interval $[0; 1]$, independent private values, and two bidders, the unique symmetric differentiable Bayesian Nash equilibrium of the lowest-price all-pay auction is established by

$$B(t) = \frac{1}{1 - \varphi}[-t - \log(1 - t)].$$

probabilities of being the winner of the object, the high type strictly prefers the strategy with the highest probability of winning, so that the high type will not submit a lower bid than the low type.) Second, $B(t)$ cannot be constant on an interval $[t'; t'']$. (By bidding slightly higher, $t''$ can largely improve its probability of winning, while only marginally influencing the payments by her and the other bidders.) Third, $B(t)$ cannot be discontinuous at any $t$. (Suppose that $B(t)$ makes a jump from $\underline{b}$ to $\overline{b}$ at $t^*$. A type just above $t^*$ has an incentive to deviate to $\underline{b}$. Doing so, she is able to substantially decrease the expected auction price, while just slightly decreasing the probability of winning the object.)

We proceed assuming a strictly increasing and differentiable equilibrium bid function. The probability of having the lowest bid for a bidder with signal $t$ is equal to $1 - F^{[n-1]}(t)$. If $x$ is the auction price, then, in terms of utility, each bidder loses $(1 - \varphi(n-1))x$: Define

$$\tilde{B}(s) \equiv (1 - \varphi(n-1))B(s),$$

and $U(t; s)$ as the expected utility of a bidder with type $t$ who misrepresents herself as type $s$ given that the other bidders report truthfully. Then,

$$U(t; s) = \int_{\underline{t}}^{s} v(t; y)dF^{[1]}(y) - \left[1 - F^{[n-1]}(s)\right]\tilde{B}(s) - \int_{\underline{t}}^{s} \tilde{B}(y)dF^{[n-1]}(y).$$

The first term of the RHS refers to the value of the object when the highest bid is submitted. The second term refers to the payments made in case the lowest bid is submitted, and the third term refers to the expected payments in case another bidder submits a lower bid. The FOC of the equilibrium is given by

$$v(t; t)f^{[1]}(t) + f^{[n-1]}(t)\tilde{B}(t) - \left[1 - F^{[n-1]}(t)\right]\tilde{B}'(t) - f^{[n-1]}\tilde{B}(t) = 0. \quad (12)$$

With some manipulation, we get

$$\tilde{B}(t) = \tilde{B}(\underline{t}) + \int_{\underline{t}}^{t} \frac{v(y; y)f^{[1]}(y)}{1 - F^{[n-1]}(y)}dy$$

or, equivalently

$$B(t) = \frac{B(\underline{t})}{1 - \varphi(n-1)} + \frac{1}{1 - \varphi(n-1)} \int_{\underline{t}}^{t} \frac{v(y; y)f^{[1]}(y)}{1 - F^{[n-1]}(y)}dy.$$

The only best response of a bidder with signal $\underline{t}$, given that the outcome of the auction is efficient, is to bid zero, so that $B(\underline{t}) = 0$. The SOC is fulfilled, as

$$\text{sign}\left(\frac{\partial U(t; s)}{\partial s}\right) = \text{sign}\left(\frac{\partial U(t; s)}{\partial s} - \frac{\partial U(s; s)}{\partial s}\right) = \text{sign}(v(t; s) - v(s; s)) = \text{sign}(t - s):$$

An immediate consequence of the fact that $v(y; y) > 0$ for all $y > \underline{t}$ (by Value Monotonicity) is that the bid function $B(t)$ is strictly increasing in $t$, which is the assumption we started with. ■

In Proposition 5, we establish that the presence of a perfect resale market has no influence on equilibrium behavior. This result follows from our companion paper, where we derive that any Bayesian Nash equilibrium of any auction (without resale market) which leads to an efficient assignment of the object, is also a Bayesian Nash equilibrium when the same auction is followed by a resale market where the same bidders participate. As $B$ constitutes an efficient Bayesian Nash equilibrium, the proposition must be true.

**Proposition 5** The bid function $B$ described in Proposition 4 establishes a Bayesian Nash equilibrium of the lowest-price all-pay auction when this auction is followed by a (perfect) resale market with the same bidders participating.

The optimality of the lowest-price all-pay auction immediately follows.[12]

**Proposition 6** Consider a Double Coasean World. Suppose that in the lowest-price all-pay auction, bidders play according to the equilibrium bid function given in Proposition 4. Then the lowest-price all-pay auction is optimal.

**Proof.** The equilibrium bid function of the lowest-price all-pay auction given in Proposition 4 is an efficient Bayesian Nash equilibrium, in which the expected utility of the lowest type is zero. By Proposition 5, this is still an equilibrium when the auction is followed by a resale market, so that the expected utility of the lowest type remains zero. Then, by Lemma 2, with restrictions (10) and (11), the lowest-price all-pay auction is optimal. ■

**Corollary 7** Consider a Double Coasean World. Then the highest possible expected revenue is strictly increasing in $\varphi$. In an optimal auction, a bidder's expected utility is independent of $\varphi$.

**Proof.** Follows immediately from Lemmas 1 and 2, Propositions 4-6, and the fact that the lowest-price all-pay auction is efficient with zero expected utility for the lowest type. ■

[12] In the light of Myerson and Satterthwaite (1983), the assumption of a perfect resale market seems rather strong. However, if MR Monotonicity holds, the assumption of a perfect resale market can be relaxed to allow for any type of resale market. In our companion paper we show that auctions with efficient equilibria still have an equilibrium with efficient bidding in case of a resale market. Therefore, when MR Monotonicity is satisfied, Lemma 2 implies that every efficient auction with zero utility for the lowest type (which includes the lowest-price all-pay auction) is optimal under the restriction that the seller cannot keep the object.

## 5 The Myersonean World

Consider a Myersonean World. As said, Lemma 2 implies that a feasible auction mechanism is optimal when it yields zero expected utility for the lowest type, leaves the object in the hands of the seller when all marginal revenues are negative, and assigns the object to the bidder with the highest marginal revenue otherwise. Consider two-stage auction mechanism $\Gamma$. In the first stage of $\Gamma$, the bidders are asked whether or not to participate. If at least one of the bidders refuses to participate, the game ends, and the seller keeps the object. Otherwise, each bidder pays the seller the same entree fee, which we denote by $\Phi$. Then the bidders enter the second stage, and play the lowest-price all-pay auction with reserve price $R$. Each bidder follows the strategy to choose "participate" in the first stage, and to play according to a Bayesian Nash equilibrium in the second stage.

The lowest-price all-pay auction with a reserve price $R$ has the following rules. Each bidder either submits a bid of at least $R$, or abstains from bidding. If all bidders abstain, the object remains in the hands of the seller, otherwise it will be sold to the bidder with the highest bid. In the case of ties, the winner is chosen from the highest bidders with equal probability. All bidders who submitted a bid pay the auction price, which is equal to the lowest submitted bid in case all bidders submit a bid, and equal to $R$ otherwise.

Proposition 8 shows that the lowest-price all-pay auction has an equilibrium in which, up to a threshold type $\hat{t}$, bidders do not submit a bid, and all bidders with a type $t \geq \hat{t}$ bid $h(t, \hat{t})$, with

$$h(t, \hat{t}) \equiv R + \frac{1}{(1 - \varphi(n-1))} \int_{\hat{t}}^{t} \frac{v(y, y) f^{[1]}(y)}{1 - F^{[n-1]}(y)} dy.$$

We derive $h$ using the same differential equation as for the lowest-price all-pay auction without a reserve price, with boundary condition $h(\hat{t}, \hat{t}) = R$. Observe that $h(t, \hat{t})$ is strictly increasing in both $t$ and $\varphi$. In equilibrium, a type $\hat{t}$ is indifferent between bidding $R$ and submitting no bid.

**Proposition 8** Let $B^R(t)$, the bid of a bidder with signal $t$, be given by

$$B^R(t) \equiv \begin{cases} h(t, \hat{t}) & \text{for } t \geq \hat{t} \\ \text{"no bid"} & \text{for } t < \hat{t}, \end{cases}$$

where $\hat{t}$ is the unique solution to

$$\int_{\underline{t}}^{\hat{t}} v(\hat{t}, y) dF^{[1]}(y) = R. \quad (13)$$

Then $B^R$ constitutes a symmetric Bayesian Nash equilibrium of the lowest-price all-pay auction with a reserve price $R$.

Proof. Assume that a threshold type $\hat{t}$ exists such that in equilibrium, all types $t < \hat{t}$ abstain from bidding, and all types $t \geq \hat{t}$ bid according to $h$. It is straightforwardly verified that $h(\cdot, \hat{t})$ satisfies (12) with the boundary condition $h(\hat{t}, \hat{t}) = R$. In equilibrium, $\hat{t}$ must be indifferent between not bidding and bidding $R$. Hence

$$\varphi R N(R) = -R + \varphi R N(R) + \int_{\underline{t}}^{\hat{t}} v(\hat{t}, y) dF^{[1]}(y); \tag{14}$$

where $N(R)$ is the expected number of the other bidders who submit a bid. (14) is equivalent to (13). Since $v$ is strictly increasing in its first argument (by Value Monotonicity), (13) has a unique solution for $\hat{t}$. It is then standard to check that no type has an incentive to deviate from the equilibrium. ■

Proposition 9 shows that when MR Monotonicity is satisfied, $\Gamma$ is optimal if the entry fee is given by (15).[13] In an optimal auction, the seller's revenue is strictly increasing in $\varphi$, and a bidder's expected utility does not depend on $\varphi$:

Proposition 9 Consider a Myersonean World. Suppose that MR Monotonicity is satisfied. Let the entry fee in $\Gamma$ be given by

$$\Phi \equiv \frac{\underline{u}}{1 - \varphi(n-1)}, \tag{15}$$

where $\underline{u}$ is the expected utility of the lowest type in the lowest-price all-pay auction when the equilibrium of Proposition 8 is played. Also, suppose that the reserve price $R$ is such that for the threshold type $\hat{t}$ $MR(\hat{t}) = 0$ holds, that all bidders choose "participate" in equilibrium, and that bidders play according to the equilibrium given in Proposition 8. Then $\Gamma$ is optimal.

Proof. According to the equilibrium defined in Proposition 8, all types above $\hat{t}$ submit a bid according to a strictly increasing bid function. All types below $\hat{t}$ abstain from bidding. Let $p^*$ be the allocation rule of the feasible direct revelation mechanism related to the lowest-price all-pay auction with the specified reserve price and the given equilibrium. Then, by MR Monotonicity, $p^*$ maximizes $E_t\{\sum_{i=1} MR_i(t)p_i(t)\}$ over all feasible direct revelation mechanisms $(p, x)$. Moreover, by definition of $\Phi$, the expected utility of bidder $i$'s lowest type equals zero over both stages of $\Gamma$, as

$$\underline{u} + \varphi \sum_{j \neq i} \Phi - \Phi = 0.$$

[13] The assumption $\varphi \in [0, \frac{1}{n-1})$ is crucial for Proposition 9. If $\varphi > \frac{1}{n-1}$, the seller can establish an arbitrarily high revenue by a take-it-or-leave-it offer to all bidders, in which he asks an arbitrarily high entry fee $\Phi$. The take-it-or-leave-it offer is such that only if every bidder accepts to pay the fee, the seller collects the payments. It is a dominant strategy for every bidder to accept the mechanism, since participation gives a utility larger than zero ($\Phi[-1 + \varphi(n-1)] > 0$).

The given strategies constitute a Bayesian Nash equilibrium, and when these are played, ¡ maximizes (4). Therefore, ¡ is optimal. ■

**Corollary 10** Consider a Myersonean World. Suppose that MR Monotonicity is satisfied. Then the highest possible expected revenue is strictly increasing in '. In an optimal auction, a bidder's expected utility is independent of '.

**Proof.** Follows immediately from Lemmas 1 and 2, and Propositions 8 and 9. ■

# 6 Concluding remarks

In this paper, we have investigated optimal auctions with financial externalities. We have established the optimality of the lowest-price all-pay auction in this environment. In a Double Coasean World, the lowest-price all-pay auction itself is optimal. In a Myersonean World, we have found an optimal two-stage auction mechanism in which each bidder pays an entry fee, and plays the lowest-price all-pay auction with a reserve price.

Goeree and Turner (2001) study optimal auctions in an environment that is related to ours. In Goeree and Turner's model, bidders receive (potentially different) shares of the seller's revenue. The seller's net revenue is optimized under the restriction that the seller cannot withhold the object. Goeree and Turner define an auction, called the all-pay-all auction, in which each bidder's payment is a weighted sum of all bids, which depends on all bidders' shares in the seller's revenue. Goeree and Turner show that with symmetric bidders, the all-pay-all auction is optimal. Moreover, with equal shares, Goeree and Turner prove the optimality of the lowest-price all-pay auction in their environment.

So far it's unclear whether there exists an auction in our environment (perhaps having the same structure as the all-pay-all auction), which is optimal when we allow for asymmetric financial externalities. A major advantage of the lowest-price all-pay auction over the all-pay-all auction is that the rules of the lowest-price all-pay auction are context independent, in contrast to the rules of the all-pay-all auction. The rules of both auction games do not depend on the distribution function $F$ or the value functions $v$. However, the rules of the all-pay-all auction do depend on the bidders' shares of the seller's revenue, whereas the rules of the lowest-price all-pay auction do not.

Jehiel et al. (1996) study optimal auctions in environments with allocative externalities, i.e., environments in which a loser's utility depends on the identity of the winner (not on how much she pays). They derive the optimality of a feasible auction mechanism which is similar to two-stage feasible auction mechanism ¡. In the first stage of this feasible auction mechanism, bidders are asked whether to participate or not. In the second stage, depending on which bidders participate, the object remains in the hands of the seller, or is allocated to one of the bidders. Each participating bidder receives (pays) money from (to) the seller.

It remains an open question whether the lowest-price all-pay auction performs well in practice. The auction seems to be very sensitive for collusion. Moreover, apart from the efficient equilibrium, the lowest-price all-pay auction also has highly inefficient equilibria in the case of two bidders. It is easily verified that there is an equilibrium in which one bidder submits a very high bid, and the other bids zero.[14] An experimental study may put some light on this matter.

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[14] In case of three bidders there is no equilibrium in which one bidder bids very high and the other bidders bid zero, because in such an equilibrium one of the low bidders has an incentive to overbid the high bidder.

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