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Discussion paper

Approximate fixed point theorems in Banach spaces with applications in game theory*

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Abstract

In this paper some new approximate fixed point theorems for multifunctions in Banach spaces are presented and a method is developed indicating how to use approximate fixed point theorems in proving the existence of approximate Nash equilibria for non-cooperative games.

Keywords: Approximate fixed point, approximate Nash equilibrium, Banach space, closed multifunction, upper semicontinuous multifunction.

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1. INTRODUCTION

In this paper we are interested in multifunctions $F : X \to\to X$ which possess fixed points or approximate fixed points. Fixed point theorems deal with sufficient conditions on X and F guaranteeing that there exists a fixed point, that is an $\hat{x} \in X$ with $\hat{x} \in F(\hat{x})$. There are many fixed point theorems known on topological spaces (Brouwer [5], Kakutani [8], Banach [3], ...) which have proved to be useful in many applied fields such as game theory, mathematical economics and the theory of quasi-variational inequalities (cf. Baiocchi and Capelo [2]). If X is a metric space, approximate fixed point theorems are theorems with conditions on X and F guaranteeing that, for each $\varepsilon > 0$, there is an ε-fixed point, i. e. an $x^* \in X$ with $d(x^*, F(x^*)) \leq \varepsilon$, where $d(x^*, F(x^*)) = \inf\{d(x^*, z) \mid z \in F(x^*)\}$. In Tijs, Torre and Brânzei [20], approximate fixed point theorems in the spirit of Brouwer, Kakutani and Banach were derived. In the first two theorems, in finite dimensional spaces, the compactness conditions used in the above quoted theorems were replace by boundedness conditions. In the third one, the completeness of the metric space (used in Banach's contraction theorem) was dropped.

In this paper we will present some new approximate fixed point theorems for multifunctions defined in Banach spaces. Weak and strong topologies play a role and bounded and unbounded regions are allowed.

The outline of the paper is as follows. In Section 2, first we present some approximate fixed point theorems for closed or upper semicontinuous (with respect to the weak or strong topologies) multifunctions on bounded or totally bounded convex regions. Then unbounded convex regions are considered and here the notion of tame multifunction plays a crucial role. Section 3 gives an outline of how to use approximate fixed point theorems to guarantee that non-cooperative games have approximate Nash equilibria and Section 4 concludes with some remarks.

2. NEW APPROXIMATE FIXED POINT THEOREMS

In this section, V will be a real Banach space and for $F : X \rightarrow\rightarrow X$ with $X \subseteq V$, the set $\{x \in V \mid d(x, F(x)) = \inf_{y \in F(x)} \parallel y - x \parallel \leq \varepsilon\}$ of the ε-fixed points of the multifunction F on X will be denoted by $FIX^{\varepsilon}(F)$.

First we present two theorems where the weak topology plays a role.

Theorem 2.1 *Let V be a reflexive real Banach space and let X be a non-empty bounded and convex subset of V. Assume that $F : X \rightarrow\rightarrow X$ is a weakly closed multifunction (that is a multifunction closed with respect to the weak topology) such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Suppose without loss of generality that $0 \in X$. Let $\alpha = \sup\{\parallel x \parallel \mid x \in X\}$. Take $\varepsilon > 0$ and $0 < \delta < 1$ such that $\delta\alpha \leq \varepsilon$. Let Y be the weakly compact and convex subset of X defined by $Y = (1 - \delta)\overline{X}$, where $\overline{X}$ is the closure of X. Define the multifunction $G : Y \rightarrow\rightarrow Y$ by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$. Then G is a weakly closed multifunction with non empty, convex and weakly compact values. But, with respect to the weak topology, V is an Hausdorff locally convex topological vector space, so, in view of Glicksberg's Theorem [7], G has at least one fixed point on Y. So there is an $x^* \in Y$ such that $x^* \in G(x^*) = (1 - \delta)F(x^*)$. Then there is a $z \in F(x^*)$ such that $x^* = (1 - \delta)z$, so $\parallel z - x^* \parallel = \delta \parallel z \parallel \leq \delta\alpha \leq \varepsilon$. Hence x^* is an ε-fixed point of F. 2

Theorem 2.2 *Let V be a reflexive and separable real Banach space and let X be a non-empty bounded and convex subset of V. Assume that $F : X \rightarrow\rightarrow X$ is a weakly upper semicontinuous multifunction (that is a multifunction upper semicontinuous with respect to the weak topology) such* *that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. As in the proof of Theorem 2.1, we assume that $0 \in X$ and $\alpha = \sup\{\| x \| \mid x \in X\}$. Take $\varepsilon > 0$, $0 < \delta < 1$ such that $\delta\alpha \leq \frac{\varepsilon}{2}$ and $Y = (1-\delta)\overline{X}$. Define the multifunction $G : Y \rightarrow\rightarrow Y$ by $G(x) = (1-\delta)\overline{F(x)}$ for all $x \in Y$. G is weakly upper semicontinuous. In fact, since V is a separable real Banach space and X is bounded, there exists a metric d_w on V such that the weak topology on X is induced by the metric d_w (see, for example, [6, Proposition 8.7]). Let $x \in Y$ and assume that A is a weakly open neighbourhood of $G(x)$. For $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \mid d_w(y, G(x)) < \sigma\}$. Since $G(x)$ is weakly compact, we have that $d_w(Y \backslash A, G(x)) = \inf\{d_w(y, z) \mid y \in Y \backslash A,\ z \in G(x)\} > 0$, where $Y \backslash A = \{y \in Y \mid y \notin A\}$. So, if $0 < \sigma' < \sigma < d_w(Y \backslash A, G(x))$, we have $G(x) \subset A_{\sigma'} \subset \{y \in Y \mid d_w(y, G(x)) \leq \sigma'\} \subset A_\sigma \subset A$. In view of the weakly upper semicontinuity of the multifunction $(1-\delta)F$, there exists an open neighbourhood I of x such that $(1-\delta)F(z) \subset A_{\sigma'}$ for all $z \in I$. Therefore $G(z) = (1-\delta)\overline{F(z)} \subseteq \{y \in Y \mid d_w(y, G(x)) \leq \sigma'\} \subset A$ for all $z \in I$. So G is a weakly upper semicontinuous multifunction at x. In the light of Proposition 4 pag. 72 in [1], G is also a weakly closed multifunction at x. Therefore, in view of Glicksberg's theorem, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$. Hence, there exists $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$, so $\| z - x^* \| = \delta \| z \| \leq \delta\alpha \leq \frac{\varepsilon}{2}$. Moreover, there is $z' \in F(x^*)$ such that $\| z' - z \| < \frac{\varepsilon}{2}$. Hence $\| z' - x^* \| < \varepsilon$, that is $x^* \in FIX^{\varepsilon}(F)$. 2

In the next theorem the strong topology is involved.

Theorem 2.3 *Let V be a real Banach space and let X be a non-empty, convex and totally bounded subset of V. Assume that $F : X \rightarrow\rightarrow X$ is a closed or upper semicontinuous multifunction such that $F(x)$ is a non-empty* *and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Assume without loss of generality that $0 \in X$. Take $\varepsilon > 0$ and $\eta > 0$. Since X is totally bounded there exists $m \in \mathbb{N}$ and $x_1, ..., x_m \in X$ such that $X \subseteq \cup_{i=1}^{m} \overset{o}{B}(x_i, \eta)$ (see, for example, [4]), where $\overset{o}{B}(x_i, \eta) = \{y \in V \mid \| y - x_i \| < \eta\}$. Moreover, let $h = \max\{\| x_i \| \mid i \in \{1, ..., m\}\}$. If $0 < \delta < 1$ the set $Y = (1 - \delta)\overline{X}$ is a non-empty, convex and totally bounded subset of V. Since Y is also closed, Y is complete and therefore compact.

- If we assume that F is a closed multifunction and we take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \varepsilon$, then the multifunction $G : Y \to\to Y$ defined by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$ is closed. This implies by Glicksberg's theorem that G possesses a fixed point x^*. Then there is a $z \in F(x^*)$ such that $x^* = (1 - \delta)z$. Because $X \subseteq \cup_{i=1}^{m} \overset{o}{B}(x_i, \eta)$, there exists an $r \in \{1, ..., m\}$ such that $z \in \overset{o}{B}(x_r, \eta)$. So $\| x^* - z \| = \delta \| z \| \leq \delta(\| z - x_r \| + \| x_r \|) < \delta(\eta + h) \leq \varepsilon$. Hence $x^* \in FIX^{\varepsilon}(F)$.
- Assume now that F is an upper semicontinuous multifunction. We take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \frac{\varepsilon}{2}$. Let $G : Y \to\to Y$ defined by $G(x) = (1 - \delta)\overline{F(x)}$ for all $x \in Y$. We claim that G is upper semicontinuous. In fact, let $x \in Y$ and assume that A is an open neighbourhood of $G(x)$. If $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \mid \inf_{z \in G(x)} \| z - y \| < \sigma\}$. Since $G(x)$ is compact, we have that $d(Y \backslash A, G(x)) = \inf\{\| y - z \| \mid y \in Y \backslash A,\ z \in G(x)\} > 0$. So, if $0 < \sigma' < \sigma < d(Y \backslash A, G(x))$, we have $G(x) \subset A_{\sigma'} \subset \overline{A_{\sigma'}} = \{y \in Y \mid \inf_{z \in G(x)} \| z - y \| \leq \sigma'\} \subset A_\sigma \subset A$. In view of the upper semicontinuity of the multifunction $(1 - \delta)F$, there exists an open neighbourhood I of x such that $(1 - \delta)F(z) \subset A_{\sigma'}$ for all $z \in I$. Therefore $G(z) = (1 - \delta)\overline{F(z)} \subseteq \overline{A_{\sigma'}} \subset A$ for all $z \in I$. So G is an upper semicontinuous multifunction at x and is also a closed multifunction at x. Therefore, in view of Glicksberg's theorem, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$ and $z \in \overline{F(x^*)}$ such that $x^* = (1 - \delta)z$. Since $X \subseteq \cup_{i=1}^{m} \overset{o}{B}(x_i, \eta)$, there exists an $s \in \{1, ..., m\}$ such that $z \in \overset{o}{B}(x_s, \eta)$, so

$\| z - x^* \| = \delta \| z \| \leq \delta(\| z - x_s \| + \| x_s \|) < \delta(\eta + h) \leq \frac{\varepsilon}{2}$. But there exists a $z' \in F(x^*)$ such that $\| z' - z \| < \frac{\varepsilon}{2}$, so $\| z' - x^* \| < \varepsilon$, that is $x^* \in FIX^\varepsilon(F)$. 2

The next theorems deal with the existence of approximate fixed points for multifunctions on convex regions which are not necessarily bounded. Useful here is the notion of a *tame multifunction*, which we introduce in

Definition 2.1 Let U be a normed space and $X \subseteq U$ with $0 \in X$. A multifunction $F : X \to\to X$ is called a *tame multifunction* if, for each $\varepsilon > 0$, there is an $R > 0$ such that for each $x \in B(0, R) \cap X$ the set $F(x) \cap B(0, R+\varepsilon)$ is non-empty, where $B(0, R) = \{z \in U \mid \| z \| \leq R\}$.

Example 2.1 For a normed linear space U the translation $T : U \longrightarrow U$ given by $T(x) = x + a$, where $a \in U \backslash \{0\}$, is not tame and T has for small $\varepsilon > 0$ no ε-fixed points.

Example 2.2 The map $F : [0, \infty[\to\to [0, \infty[$ defined by

$$F(x) = [x + (x+1)^{-1}, \infty[\quad \text{for all} \quad x \in [0, \infty[$$

is a tame multifunction and F has ε-fixed points for each $\varepsilon > 0$.

Example 2.3 Each $F : X \to\to X$, where X is a bounded subset of a normed space U and $F(x)$ is non-empty for all $x \in X$ is a tame multifunction.

Theorem 2.4 *Let X be a convex subset, containing 0, of a reflexive real Banach space. Assume that $F : X \to\to X$ is a tame and weakly closed multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^\varepsilon(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Take $\varepsilon > 0$ and $R > 0$ such that $F(x) \cap B(0, R + \frac{\varepsilon}{2}) \neq \emptyset$ for each

$x \in B(0,R) \cap X$. Let $C = B(0,R) \cap X$. C is a non-empty, bounded and convex set. Then $G : C \to\to C$, defined by

$$G(x) = R(R + \frac{\varepsilon}{2})^{-1} F(x) \cap B(0, R + \frac{\varepsilon}{2}) \quad \text{for all} \quad x \in C$$

satisfies the conditions of Theorem 2.1. Hence there is $x^* \in FIX^{\frac{\varepsilon}{4}}(G)$ such that $d(x^*, G(x^*)) \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$ and there exists $x' \in G(x^*)$ such that $\| x' - x^* \| < \frac{\varepsilon}{2}$. Moreover there exists an element $z \in F(x^*)$ such that $z = R^{-1}(R + \frac{\varepsilon}{2})x'$. This implies that

$$\| z - x^* \| \leq \| R^{-1}(R + \frac{\varepsilon}{2})x' - x' \| + \| x' - x^* \| < \frac{\varepsilon}{2} R^{-1} \| x' \| + \frac{\varepsilon}{2} \leq \varepsilon$$

So $x^* \in FIX^{\varepsilon}(F)$. 2

Theorem 2.5 *Let X be a convex subset, containing 0, of a reflexive and separable real Banach space. Assume that $F : X \to\to X$ is a tame and weakly upper semicontinuous multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Using the same arguments of the proof of Theorem 2.4, we can show that the multifunction G defined on $B(0,R) \cap X$ by

$$G(x) = R(R + \frac{\varepsilon}{2})^{-1} F(x) \cap B(0, R + \frac{\varepsilon}{2})$$

satisfy the conditions of Theorem 2.2 and the conclusion follows as in Theorem 2.4. 2

3. APPROXIMATE NASH EQUILIBRIA FOR STRATEGIC GAMES

In Nash [10], Nash-equilibria for n-person non-cooperative games were introduced. Further using Kakutani's fixed point theorem it was shown that mixed extensions of finite n-person games non-cooperative games possess at least one Nash equilibrium. With the aid of best response multifunctions for each player the aggregate best response multifunction on the Cartesian product of the strategy spaces was constructed and the fixed points of this multifunction coincide with the Nash equilibria of the game.

Of course, for many non-cooperative games Nash equilibria do not exist. Interesting are games for which still ε-Nash equilibria exist for each $\varepsilon > 0$. Here a strategy profile is called an ε-Nash equilibrium if unilateral deviation of one of the players does not increase his payoff with more than ε. One can try to derive the existence of approximate equilibrium points following the next scheme:

(i) develop ε-fixed point theorems and find conditions on strategy spaces and payoff functions of the game such that the aggregate ε-best response multifunction satisfies conditions in an ε-fixed point theorem;

(ii) add extra conditions on the payoff-functions, guaranteeing that points in the cartesian product of the strategy spaces nearby each other have payoffs sufficiently nearby.

We will derive in this section a key proposition, which gives the possibility to find various approximate equilibrium theorems.

First we recall some definitions. An *n-person strategic game* is a tuple $\Gamma = \langle X_1, ..., X_n, u_1, ..., u_n \rangle$ where for each player $i \in N = \{1, ..., n\}$ X_i is the set of strategies and $u_i : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ is the payoff function. If players $1, ..., n$ choose strategies $x_1, ..., x_n$, then $u_1(x_1, ..., x_n), ..., u_n(x_1, ..., x_n)$ are the resulting payoffs for the players $1, ..., n$ respectively. Let $\varepsilon > 0$. Then we say that $(x_i^*)_{i \in N} \in \prod_{i \in N} X_i$ is an *ε-Nash equilibrium* if

$$u_i(x_i, x_{-i}^*) \leq u_i(x^*) + \varepsilon \text{ for all } x_i \in X_i \text{ and for all } i \in N.$$

Here x_{-i}^* is a shorthand for $(x_j^*)_{j \in N \setminus \{i\}}$ and we will denote by $NE^\varepsilon(\Gamma)$ the set of ε-Nash equilibria. Note that for an $x^* \in NE^\varepsilon(\Gamma)$, a unilateral deviation by a player does not improve the payoff with more than ε. Useful will be for each $i \in N$ the ε*-best response multifunction* $B_i^\varepsilon : \prod_{j \in N \setminus \{i\}} X_j \to\to X_i$ defined by

$$B_i^\varepsilon(x_{-i}) = \{x_i \in X_i \mid u_i(x_i, x_{-i}) \geq \sup_{t_i \in X_i} u_i(t_i, x_{-i}) - \varepsilon\}$$

and the *aggregate* ε*-best response multifunction* $B^\varepsilon : X \to\to X$, defined by

$$B^\varepsilon(x) = \prod_{i \in N} B_i^\varepsilon(x_{-i}).$$

Obviously, if $x^* \in B^\varepsilon(x^*)$, then $x^* \in NE^\varepsilon(\Gamma)$ and conversely. So if B^ε has a fixed point, then we have an ε-Nash equilibrium. If we do not know whether B^ε has a fixed point but we know that B^ε has δ-fixed points for each $\delta > 0$, then this leads under extra continuity conditions to the existence of approximate Nash equilibria for the game as we will see.

The next result we call the key proposition because this proposition opens the door to obtain different ε-equilibrium point theorems, using as inspiration source the existing literature on Nash equilibrium point theorems. Many of them contain collections of sufficient conditions on strategy space and payoff functions, guaranteeing that the aggregate best response multifunction has a fixed point. To guarantee the existence of ε-fixed points one has to modify, often in an obvious way, the conditions guaranteeing the existence of δ-fixed points for the aggregate ε-best response multifunction and to replace the condition (iii) in the key proposition by the obtained conditions.

KEY PROPOSITION *Let* $\Gamma = \langle X_1, ..., X_n, u_1, ..., u_n \rangle$ *be an* n*-person strategic game with the following three properties:*

(i) for each $i \in N = \{1, ..., n\}$*, the strategy space* X_i *is endowed with a metric* d_i*;*

(ii) the payoff functions $u_1, ..., u_n$ *are uniform continuous functions on* $X =$

$\prod_{i=1}^{n} X_i$, where X is endowed with the metric d, defined by

$$d(x,y) = \sum_{i=1} d_i(x_i, y_i) \quad \textit{for all } x, y \in X;$$

(iii) for each $\varepsilon > 0$ and $\delta > 0$, the aggregate ε-best response multifunction B^ε possesses at least one δ-fixed point, i. e. $FIX^\delta(B^\varepsilon) \neq \emptyset$.
Then, $NE^\varepsilon(\Gamma) \neq \emptyset$ for each $\varepsilon > 0$.

Proof. Take $\varepsilon > 0$. Because of (ii) we can find $\eta > 0$ such that for all $x, x' \in X$ with $d(x,x') < \eta$ we have $| u_i(x) - u_i(x') | < \frac{1}{2}\varepsilon$ for each $i \in N$. We will prove that

$$x^* \in FIX^{\frac{1}{2}\eta}(B^{\frac{1}{2}\varepsilon}) \Longrightarrow x^* \in NE^\varepsilon(\Gamma).$$

Take $x^* \in FIX^{\frac{1}{2}\eta}(B^{\frac{1}{2}\varepsilon})$, which is possible by (iii). Then there is an $\hat{x} \in B^{\frac{1}{2}\varepsilon}(x^*)$ such that $d(x^*, \hat{x}) < \eta$. Then for each $i \in N$: $d((x_i^*, x_{-i}^*), (\hat{x}_i, x_{-i}^*)) < \eta$. This implies that

$$u_i(x_i^*, x_{-i}^*) \geq u_i(\hat{x}_i, x_{-i}^*) - \frac{1}{2}\varepsilon \text{ for all } i \in N. \tag{1}$$

Further $\hat{x} \in B^{\frac{1}{2}\varepsilon}(x^*)$ implies

$$u_i(\hat{x}_i, x_{-i}^*) \geq \sup_{t_i \in X_i} u_i(t_i, x_{-i}^*) - \frac{1}{2}\varepsilon \text{ for all } i \in N. \tag{2}$$

Combining (1) and (2) we obtain

$$u_i(x_i^*, x_{-i}^*) \geq \sup_{t_i \in X_i} u_i(t_i, x_{-i}^*) - \varepsilon \text{ for all } i \in N, \tag{3}$$

and this is equivalent to $x^* \in NE^\varepsilon(\Gamma)$. 2

It will be clear that using the key proposition many approximate Nash equilibrium theorems can be obtained. We restrict ourselves here in giving two examples.

Example 3.1 (Games on the open unit square). Let $\langle]0,1[,]0,1[, u_1, u_2\rangle$ be a game with uniform continuous payoff functions u_1 and u_2. Suppose that u_1 is concave in the first coordinate and u_2 is concave in the second coordinate. Then for each $\varepsilon > 0$, the game has an ε-Nash equilibrium point.

Example 3.2 (Completely mixed approximate Nash equilibria for finite games). Let A and B be $m \times n$-matrices of real numbers. Consider the two-person game $\langle \overset{o}{\Delta}_m, \overset{o}{\Delta}_n, u_1, u_2\rangle$, where:

$$\overset{o}{\Delta}_m = \{p \in \mathbb{R}^m \mid p_i > 0 \text{ for each } i \in \{1, ..., m\}, \sum_{i=1}^{m} p_i = 1\},$$

$$\overset{o}{\Delta}_n = \{q \in \mathbb{R}^n \mid q_j > 0 \text{ for each } j \in \{1, ..., n\}, \sum_{j=1}^{n} q_j = 1\},$$

$$u_1(p,q) = p^T A q,\ \ u_2(p,q) = p^T B q \text{ for all } p \in \overset{o}{\Delta}_m,\ q \in \overset{o}{\Delta}_n.$$

Then for each $\varepsilon > 0$ this game has an ε-Nash equilibrium. Such an ε-Nash equilibrium is called *completely mixed*, because both players use each of their pure strategies with a positive probability.

4. CONCLUDING REMARKS

In Section 2 we developed new approximate fixed point theorems in infinite dimensional spaces. It seems important to find more sophisticated approximate fixed point theorems, especially for multifunctions on unbounded sets. Also finding new applications in economic theory and in the study of well-posed fixed point problems (Lemaire, Salem and Revalsky [9]) could be interesting. In Section 3 we indicated, via the key proposition, how approximate fixed point theorems can play a role in non-cooperative game theory to prove the existence of approximate Nash equilibria. For a survey of techniques to prove the existence of (ε-) Nash equilibria see Tijs [19]. For approximate equilibrium theorems using approximations of games with smaller subgames see Lucchetti, Patrone and Tijs [13]. Also we want to refer to Lignola [10] for the existence of Nash equilibria for games with non compact strategy sets and to Lignola and Morgan [11] for convergence of Nash equilibria. The importance of ε-Nash equilibria is also motivated by well-posedness for Nash equilibria (cf. Lignola and Morgan [12], Margiocco, Patrone and Pusillo Chicco [16]), convergence properties of approximate Nash equilibria (cf. Morgan and Raucci [17]) and approximate solutions for hierarchical games (cf. Mallozzi and Morgan [14] and Mallozzi and Morgan [15] for approximate mixed strategies).

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