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COOPERATION BY ASYMMETRIC AGENTS IN A JOINT PROJECT *

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Abstract

The object of study is cooperation in joint projects, where agents may have different desired sophistication levels for the project, and where some of the agents may have low budgets. In this context questions concerning the optimal realizable sophistication level and the distribution of the related costs among the participants are tackled. A related cooperative game, the enterprise game, and a non-cooperative game, the contribution game, are both helpful. It turns out that there is an interesting relation between the core of the convex enterprise game and the set of strong Nash equilibria of the contribution game. Special attention is paid to a rule inspired by the airport landing fee literature. For this rule the project is split up in a sequence of subprojects where the involved participants pay amounts which are, roughly speaking, equal, but not more than their budgets allow. The resulting payoff distribution turns out to be a core element of the related contribution game.

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1 Introduction

In situations where agents work together to achieve a joint project costs or surpluses frequently have to be allocated. In these cases formulating cost allocation problems as cost games may prove restrictive since these say nothing about the benefits that players receive. In particular costs allocated may exceed the benefits received and players might refuse to pay. Therefore, benefits should be incorporated into the analysis.

Unfortunately, benefits do not have the same objectivity as costs. Sometimes agents try to misreport their true benefits if this strategy results in lower cost contributions. We are confronted in this case with a demand revelation problem (See for example Young (1998)). However, we do not tackle this problem and hereafter we assume that profits are known and see them as budgets available for the project.

In this paper we deal with situations in which a group of agents aims to work together in achieving a project. However, the wishes of the agents with respect to the sophistication level of the project may differ, and also the benefits for players who wish for the same sophistication level. Notice that benefit is connected with willingness to contribute to the costs of some realization. Obvious questions in this context are:

Q.1 What will be an optimal sophistication level of the project for the whole group and for subgroups?

Q.2 What will be the contributions of the involved agents to the cost of the chosen realization?

Assuming that benefits are known, economic efficiency suggests that an optimal sophistication level is one that yields the greatest difference between benefits and costs. But there is still the question of the allocation of surpluses and, consequently, of the distribution of costs of the chosen project.

In what follows we illustrate some related economic examples already studied in literature.

Moulin (1994) considers that any group of potential users can jointly produce a non-rival and excludable public good with no congestion as long as they cover the costs of the largest amount of the good demanded. In that paper, where the serial cost method is analyzed, agents are allowed to consume different amounts of the public good, that is if agent i consumes y_i units of the good then any other consumer with the same or a lower demand can also consume but others may not. There may be reasons that justify this last restriction. Think for example of a group of people wanting to share a taxi home. If one person's demand is not going to be satisfied in its totality then he will not be interested in the service.

Some other well known economic studies about this type of public facilities are the following.

The construction of landing strips for the use of different types of aircraft discussed by Littlechild and Thompson (1977) may be considered one of the first analysis of cost sharing of a non-rival public good of this type. Also Young et al. (1982) study the cost allocation of a water distribution system in Sweden, while the cost allocations of the system of irrigation ditches in Montana has more recently been analyzed by Aadland and Kolpin (1998). A numerical example which ilustrates a water distribution system supplying a group of consumers with independent demands can also be found in Moulin (1988). In this example the cost structure depends on the number of agents to be served, and benefits to consumers from using this facility are also incorporated.

Now let us consider a typical economic situation to clarify the idea of sophistication level that we introduce in this paper.

Suppose two firms are located along a river. The first produces steel while the second operates a resort hotel somewhere downstream. Both use the river, but in different ways. The steal firm uses it as a sink for its waste, while the resort uses it to attract customers seeking water recreation. The establishing of a water treatment system to clean the river and how to share the costs of this treatment are important issues. Note that here the agents require different levels of cleanness of the water. i.e., different degrees of sophistication are required.

After introducing the model we also consider some proposals for sharing the surplus generated by projects of this type. Our favorite proposal is a constrained Baker-Thompson like rule inspired by the landing fee literature (Baker (1965), Thompson (1971), Littlechild and Owen (1973, 1977), Littlechild and Thompson (1977), Littlechild (1974, 1975), Potters and Sudhölter (1999)). In the paper by Littlechild and Owen (1977) benefits are also taken into account and no player contributes in costs more than his benefit.

The outline of the paper is as follows. In Section 2 we introduce our model of an enterprise situation, give some facts on optimal sophistication levels in connection to question Q.1. and define a (cooperative) enterprise game useful to tackle question Q.2. Section 3 deals with the convexity problem for such enterprise games. We give a suitable characterization of core elements which plays an important role in the paper; in particular it is useful in finding the extreme points of the core, starting with the introduction of the adjusted Bird rule (Bird (1976)). It concludes that enterprise games are convex games. In Section 4 we concentrate on the problem of sharing costs in joint enterprise situations with asymmetric agents. We consider classical game theoretic solutions and propose two new rules for allocating surpluses, inspired by airport fee literature. In Section 5 a (non-cooperative) contribution game related to a joint enterprise situation is introduced and non-cooperative allocations of surpluses are considered. We prove that strong Nash equilibria of contribution games correspond to core elements of the related enterprise games. Section 6 concludes the paper.

2 Joint enterprise situations and games

A joint enterprise situation (with asymmetric agents) will be in the following the tuple

$$J = \langle N,\ \Lambda,\ (d_i, b_i)_{i\in N},\ c\rangle$$

where $N = \{1, ..., n\}$ is the set of agents, $\Lambda = \{0, 1, ..., m\}$ is the set of sophistication levels; for each $i \in N$, $d_i \in \Lambda\backslash\{0\}$ is the demanded sophistication level and $b_i \in I\!R_{++}$ the benefit corresponding to any sophistication level $\lambda \geq d_i$, and $c : \Lambda \to I\!R$ is a strict increasing cost function with $c(0) = 0$, where $c(\lambda)$ indicates the cost of realizing a project with sophistication level $\lambda \in \Lambda$. W.l.o.g. we suppose that $1 = d_1 \leq ... \leq d_n = m$ and $d_{i+1} - d_i \in \{0, 1\}$ for all $i \in \{1, ..., n-1\}$. Level $\lambda = 0$ corresponds to a situation where there will be realized nothing. An agent $i \in N$ confronted with a chosen sophistication level $\lambda < d_i$ will have benefit 0, and hence he is not willing to contribute in the costs. For sophistication level $\lambda \geq d_i$ agent i wants to contribute at most b_i.
Let $R_i : \Lambda \to I\!R$ be the step function given by $R_i(\lambda) = b_i$ if $\lambda \geq d_i$ and $R_i(\lambda) = 0$ if $\lambda < d_i$. Then $R_i(\lambda)$ is the revenue for player i if sophistication level λ is chosen. Let $B_N : \Lambda \to I\!R$ be given by $B_N(\lambda) = \sum\limits_{i=1}^{n} R_i(\lambda) - c(\lambda)$ for each $\lambda \in \Lambda$. Then $B_N(\lambda)$ is the total net benefit obtainable by N if level λ is chosen. The maximal reward obtainable by N is given by

$$v(N) = \max\{B_N(\lambda) : \lambda \in \Lambda\}$$

and arg $\max\{B_N(\lambda) : \lambda \in \Lambda\}$ is *the set of optimal sophistication levels* guaranteeing $v(N)$. The largest element in this set is denoted by λ_N. So

$$\lambda_N = \max(\arg\max\{B_N(\lambda)\,|\lambda \in \Lambda\}).$$

Example 2.1

Let $N = \{1,2,3\}$, $\Lambda = \{0,1,2\}$, $d = (d_1, d_2, d_3) = (1,1,2)$, $(b_1, b_2, b_3) = (3,12,16)$ and $c(0) = 0$, $c(1) = 10$, $c(2) = 20$. Then $B_N(0) = 0$, $B_N(1) = R_1(1)+R_2(1)+R_3(1)-c(1) = 3+12+0-10 = 5$ and $B_N(2) = R_1(2)+R_2(2)+R_3(2)-c(2) = 3+12+16-20 = 11$. So $v(N) = \max\{0,5,11\} = 11$. Further $\lambda_N = 2$. In case λ_N is realized, player 1 can contribute in the cost $x_1 \in [0,3]$, player 2: $x_2 \in [0,12]$ and player 3: $x_3 \in [0,16]$. But it is not realistic to ask from players 1 and 2 together a cost contribution exceeding 10 because alone they can make a facility of their desired sophistication level $\lambda = 1$ with cost 10.

To handle the cost sharing problem it is interesting to look at the cooperative game $\langle N, v\rangle$, where $v : 2^N \to I\!R$ is defined by $v(\emptyset) = 0$ and $v(S) = \max\{B_S(\lambda) : \lambda \in \Lambda\}$ where $B_S(\lambda) = \sum\limits_{i\in S} R_i(\lambda) - c(\lambda)$.

The amount $v(S)$ is the reward which can be generated by S when splitting off and realizing a sophistication level $\lambda_S = \max \operatorname{argmax}\{B_S(\lambda) : \lambda \in \Lambda\}$. Note that in this case the members in $N \backslash S$ cannot use the enterprise realized by S.

Example 2.2

Consider again the joint enterprise situation of Example 2.1. The corresponding enterprise game $\langle N, v\rangle$ is given by $N = \{1,2,3\}$, $v(\{1\}) = 0$, $v(\{2\}) = 2$, $v(\{3\}) = 0$, $v(\{1,2\}) = 5$, $v(\{1,3\}) = 0$, $v(\{2,3\}) = 8$, $v(\{1,2,3\}) = 11$. The sophistication levels of the coalitions are given by $\lambda_{\{1\}} = \lambda_{\{3\}} = \lambda_{\{1,3\}} = 0$, $\lambda_{\{2\}} = \lambda_{\{1,2\}} = 1$, $\lambda_{\{2,3\}} = \lambda_{\{1,2,3\}} = 3$.

For further use we conclude this section with some remarks.

Remark 2.1

If more players cooperate then a higher or equal optimal sophistication level is achieved.

Proof.

We have only to show that

$$\lambda_{S\cup\{k\}} \geq \lambda_S \text{ for all } S \in 2^N \text{ and } k \notin S.$$

Note that for $\lambda \leq \lambda_S$

$$B_{S\cup\{k\}}(\lambda) = B_S(\lambda) + R_k(\lambda) \leq B_S(\lambda_S) + R_k(\lambda_S) = B_{S\cup\{k\}}(\lambda_S),$$

where we use in the inequality the monotonicity of R_k. Hence $\lambda_{S\cup\{k\}} \geq \lambda_S$. □

Remark 2.2
The enterprise game $\langle N, v\rangle$ is a monotonic game and the marginal contribution of a player to any coalition does not exceed his benefit.
Proof. This follows from the inequalities

$$v(S) \leq v(S \cup \{k\}) \leq v(S) + b_k \text{ , for all } S \in 2^N,\ k \notin S.$$

To prove these inequalities, note that

$$B_S(\lambda) \leq B_{S\cup\{k\}}(\lambda) \leq b_k + B_S(\lambda).$$

Hence

$$\begin{aligned} \max_{\lambda\in\Lambda} B_S(\lambda) &\leq \max_{\lambda\in\Lambda} B_{S\cup\{k\}}(\lambda) \leq b_k + \max_{\lambda\in\Lambda} B_S(\lambda) \\ \text{or } v(S) &\leq v(S\cup\{k\}) \leq b_k + v(S). \end{aligned}$$

□

Remark 2.3
Suppose $\lambda_S \geq 1$ for $S \in 2^N$. Then there is at least one $i \in S$ with $d_i = \lambda_S$.

Proof.
That the set $\{i \in S : d_i = \lambda_S\}$ is non empty follows from

$$\begin{aligned} 0 &\leq B_S(\lambda_S) - B_S(\lambda_S - 1) = \sum_{\substack{i\in S \\ d_i=\lambda_S}} R_i(\lambda_S) \\ -(c(\lambda_S) - c(\lambda_S - 1)) &< \sum_{\substack{i\in S \\ d_i=\lambda_S}} R_i(\lambda_S). \end{aligned}$$

The first inequality follows from Remark 2.2 and the second inequality from the fact that c is strictly increasing.

□

Remark 2.4
Let $S \subset T$ and $\lambda_S = \lambda_T$. Then $v(T) = v(S) + \sum_{\substack{i\in T\setminus S \\ d_i\leq\lambda_T}} b_i$.

Proof.
$v(T) = \sum_{i\in T} R_i(\lambda_T) - c(\lambda_T) = \sum_{i\in S} R_i(\lambda_S) + \sum_{i\in T\setminus S} R_i(\lambda_T) - c(\lambda_S) = v(S) + \sum_{i\in T\setminus S} R_i(\lambda_T) =$
$v(S) + \sum_{\substack{i\in T\setminus S \\ d_i\leq\lambda_T}} b_i$.

□

Remark 2.5
Let $S \in 2^N$. Define $S' = \{i \in S : d_i \leq \lambda_S\}$. Then $\lambda_{S'} = \lambda_S$ and $v(S') = v(S)$.

Proof
Note that $\lambda_{S'} \leq \lambda_S$ by Remark 2.1. On the other hand, for $\lambda \leq \lambda_S$ we have $B_{S'}(\lambda) = B_S(\lambda) \leq B_S(\lambda_S) = B_{S'}(\lambda_S)$ implying that $\lambda_{S'} \geq \lambda_S$. So $\lambda_S = \lambda_{S'}$ and then $v(S) = v(S')$ by Remark 2.4 because $\{i \in S \backslash S' : d_i \leq \lambda_S\} = \emptyset$.

□

Remark 2.6
Let S and S' be as in Remark 2.5 and $S'' = \{i \in N : d_i \leq \lambda_S\}$. Then $\lambda_S = \lambda_{S''}$.

Proof
In view of Remark 2.3 we have $\lambda_{S''} \leq \lambda_S = \lambda_{S'}$. Since $S' \subset S''$ we have $\lambda_{S'} \leq \lambda_{S''}$. So $\lambda_{S''} = \lambda_S$. □

3 The core and the marginal vectors of an enterprise game

For a game $\langle N, v\rangle$ the core $C(v)$ is defined by

$$C(v) = \{x \in \mathbb{R}^N : \ x(N) = v(N),\, x(S) \geq v(S) \text{ for each } S \in 2^N\},$$

where $x(S) := \sum_{i \in S} x_i$. The core consists of efficient vectors x, where subsets $S \subset N$ have no incentive to split off because then they only can reach a payoff $v(S)$, which is not larger that $x(S)$.
For core elements of an enterprise game the following theorem gives a characterization which will be useful later.

Theorem 3.1
Let $\langle N, v\rangle$ be the enterprise game corresponding to the situation $\langle N, \Lambda, (d_i, b_i)_{i \in N}, c\rangle$ where $\Lambda = \{1, ..., m\}$. Let, for each $k \in \Lambda$, L_k be the set $\{i \in N : d_i \leq k\}$ of players with sophistication level at most k. Then the following two assertions are equivalent:

(i) $x \in C(v)$

(ii) $0 \leq x_i \leq b_i$ for each $i \in N$, $x(L_k) \geq v(L_k)$ for each $k \in \Lambda$, and $x(N) = v(N)$.

Proof

($(i) \Rightarrow (ii)$). Take $x \in C(v)$. Then $x_i \geq v(\{i\}) \geq 0$ for each $i \in N$. Further $x_i = x(N) - x(N \backslash \{i\}) \leq v(N) - v(N \backslash \{i\}) \leq b_i$, where the last inequality follows from Remark 2.2 with $N \backslash \{i\}$ in the role of S. The other two conditions are satisfied for the core element x.

($(ii) \Rightarrow (i)$). Suppose $x \in I\!R^N$ satisfies the conditions in (ii). Then $x(N) = v(N)$. Take $S \in 2^N \backslash \{\emptyset\}$. We have to prove that $x(S) \geq v(S)$. Define $S' = S \cap L_{\lambda_S}$ and $S'' = L_{\lambda_S}$. Then $S' \subset S''$, and in view of Remark 2.5 and Remark 2.6 we have $\lambda_{S'} = \lambda_{S''} = \lambda_S$ and $v(S') = v(S)$. Now $x(S) \geq x(S') = x(S'') - x(S'' \backslash S') \geq v(S'') - x(S'' \backslash S') = v(S') + b(S'' \backslash S') - x(S'' \backslash S') \geq v(S') = v(S)$, where we use in the first inequality that $x \geq 0$ and $S' \subset S$; in the second inequality that $S'' = L_{\lambda_S}$; in the next inequality Remark 2.4 and in the last inequality that $b \geq x$.

□

Note that the reward sharing vector $x = (0, 5, 6)$ in Example 2.2 is a core element of $\langle N, v \rangle$ corresponding to the following contributions to the cost $20 = c(2)$ of the project: player 1 pays $b_1 = 3$, player 2 pays $12 - 5 = 7$ and player 3 pays $16 - 6 = 10 = c(2) - c(1)$. One can easily check that for this vector the inequalities in (ii) are satisfied. Now, note that $y = (4, 4, 3)$ is not in the core because $y_2 + y_3 = 7 < v(\{2, 3\})$. Also for $y = (4, 4, 3)$ all inequalities of (ii) are satisfied except $y_1 \leq b_1$.

Let $\Pi(N)$ be the set of $n!$ orderings of the player set N. For each $\sigma \in \Pi(N)$ we introduce two vectors: the cost vector $k^\sigma \in I\!R^N$ and the gain vector $g^\sigma \in I\!R^N$, for which the difference $g^\sigma - k^\sigma$ will turn out to be a core element of $\langle N, v \rangle$. The sum of the coordinates of k^σ equals the cost $c(m)$ of realizing the sophistication level $\lambda_N = m$ and $\sum_{k=1}^{n} g^\sigma_{\sigma(k)} = \sum_{i \in N} b_i$, the sum of the benefits if λ_N is chosen. Take $\sigma \in \Pi(N)$ and let $T_r = \{\sigma(1), \sigma(2), ..., \sigma(r)\}$, $l_r = \lambda_{T_r}$ for each $r \in \{1, ..., n\}$ and $T_0 = \emptyset$. Consider the situation where the grand coalition N forms by sequential joining of players: $\sigma(1)$ first, then $\sigma(2)$ etc. Then also the sophistication level gradually increases from 0 to m for the sets $\emptyset$, T_1, T_2,...,T_n. Players who enter and increase the sophistication level will take care of the cost increase and obtain at most their own benefit. Let us consider three possibilities when $\sigma(r)$ joins T_{r-1}:

(i) $l_r > l_{r-1}$. Then $d_{\sigma(r)} = l_r$. In this case $k^\sigma_{\sigma(r)} = c(l_r) - c(l_{r-1})$, $g^\sigma_{\sigma(r)} = \sum\{b_{\sigma(s)} : s \leq r,\ d_{\sigma(s)} \in (l_{r-1}, l_r]\}$, which is the sum of the own benefit $b_{\sigma(r)}$ and the benefits of the players who joined earlier but found in the coalition a sophistication level below their desired level.

(ii) $l_r = l_{r-1}$ and $d_{\sigma(r)} > l_r$. In this case $k^\sigma_{\sigma(r)} = 0$ $(= c(l_r) - c(l_{r-1}))$, and $g^\sigma_{\sigma(r)} = 0$.

(iii) $l_r = l_{r-1}$ and $d_{\sigma(r)} \leq l_r$. In this case $k^\sigma_{\sigma(r)} = 0$ $(= c(l_r) - c(l_{r-1}))$, and $g^\sigma_{\sigma(r)} = b_{\sigma(r)}$.

Summarizing, for each $r \in \{1, 2, ..., n\}$

(3.1) $k^\sigma_{\sigma(r)} = c(l_r) - c(l_{r-1})$ and

$$g^\sigma_{\sigma(r)} = \begin{cases} b_{\sigma(r)} + \sum_{\substack{s<r \\ d_{\sigma(s)} \in (l_{r-1}, l_r]}} b_{\sigma(s)} & \text{if } l_r > l_{r-1} \\ 0 & \text{if } l_r = l_{r-1} < d_{\sigma(r)} \\ b_{\sigma(r)} & \text{if } l_r = l_{r-1} \geq d_{\sigma(r)} \end{cases}$$

which can be summarized as follows

$$(3.2)\ g^\sigma_{\sigma(r)} = R_{\sigma(r)}(l_r) + \sum_{\substack{s<r \\ d_{\sigma(s)} \in (l_{r-1}, l_r]}} b_{\sigma(s)}, \text{ for all } r \in \{1, ..., n\},$$

where we define a sum over an empty set as 0.

The next theorem describes the relation between k^σ and g^σ and the marginal vector $m^\sigma(v) \in I\!R^N$ for the cooperative game $\langle N, v \rangle$, where the $\sigma(r) - th$ coordinate $m^\sigma_{\sigma(r)}(v)$ is given by

$$(3.3)\ m^\sigma_{\sigma(r)}(v) = v(\{\sigma(1), ..., \sigma(r)\}) - v(\{\sigma(1), ..., \sigma(r-1)\}).$$

Theorem 3.2

For each $\sigma \in \Pi(N)$ it holds that $m^\sigma(v) = g^\sigma - k^\sigma$.

Proof

Note that

$$(3.4) \quad v(\{\sigma(1), ..., \sigma(r)\}) = \sum_{k=1}^{r} R_{\sigma(k)}(l_r) - c(l_r) = -c(l_r) + R_{\sigma(r)}(l_r) + \sum_{\substack{k \leq r-1 \\ d_{\sigma(k)} \leq l_r}} b_{\sigma(k)},$$

$$(3.5) \quad v(\{\sigma(1), \ldots, \sigma(r-1)\}) = -c(l_{r-1}) + \sum_{\substack{k \leq r-1 \\ d_{\sigma(k)} \leq l_r}} b_{\sigma(k)}.$$

From (3.3), (3.4) and (3.5) we obtain

$m^\sigma_{\sigma(r)}(v) = -(c(l_r) - c(l_{r-1})) + R_{\sigma(r)}(l_r) + \sum_{\substack{k \leq r-1 \\ l_{r-1} < d_{\sigma(k)} \leq l_r}} b_{\sigma(k)}$ and using now (3.1) and (3.2) we obtain $m^\sigma_{\sigma(r)}(v) = -k^\sigma_{\sigma(r)} + g^\sigma_{\sigma(r)}$.

□

Remark 3.1

In some economic situations including airport situations one can start with a project of low level of sophistication and let the sophistication level increase with growing population of users. The adjustment of cost by going from λ to λ' should be described by $c(\lambda') - c(\lambda)$ (and no extra cost). In such a case a nice economic interpretation can be given to k^σ and g^σ.

In the following we want to prove that for the enterprise games all marginal vectors are in the core. The way we do it is as follows. First we show in Proposition 3.1 that $m^{\sigma_0}(v)$ is in the core where, $\sigma_0 = (1, 2, \dots, n)$. We call this vector the Bird allocation (Bird (1976)) and denote it by $Bi(v)$. Then we use the fact that each $\sigma \in \Pi(N)$ can be obtained from σ_0 by neighbor switching in which one neighbor pair (i, j) with $i < j$ is involved. Lemma 3.1 is then the key for theorem together with our characterization of core elements in Theorem 3.1. First we give an example.

Example 3.1

Consider the game of Example 2.2. The Bird allocation is given by $Bi(v) = (0, 5, 6)$ and we have already noted that $Bi(v) \in C(v)$. By switching the neighbors 2 and 3 in $\sigma_0 = (1, 2, 3)$ one arrives at $\sigma_1 = (1, 3, 2)$ and $m^{\sigma_1} = (0, 11, 0)$ is also a core element. The ordering $\sigma_2 = (3, 1, 2)$ can be obtained from σ_1 by the neighbor switch of 1 and 3. Also $m^{\sigma_2} = (0, 11, 0)$ is a core element as well as $m^{(3,2,1)} = (3, 8, 0)$ etc.

We can consider the directed graph with the elements of $\Pi(\{1, 2, 3\})$ as nodes and arc between two orderings σ and τ if and only if τ can be obtained from σ by a neighbor switch of i and j where $i < j$.

Note that for such σ and τ connected in the graph, we have $m^\sigma_1 \leq m^\tau_1$, $m^\sigma_1 + m^\sigma_2 \leq m^\tau_1 + m^\tau_2$, $m^\sigma_1 + m^\sigma_2 + m^\sigma_3 = m^\tau_1 + m^\tau_2 + m^\tau_3$. For example, for $\sigma = (2, 3, 1)$ and $\tau = (3, 2, 1)$ obtainable by switching 2 and 3 we have $3 \leq m^\sigma_1 \leq m^\tau_1 = 3$, $5 = m^\sigma_1 + m^\sigma_2 \leq m^\tau_1 + m^\tau_2 = 11$.

Proposition 3.1

$Bi(v) \in C(v)$ for any enterprise game $\langle N, v \rangle$.

Proof

In view of Theorem 3.1 this follows from the fact that for each $i \in N$, $Bi_i(v) = v(\{1,\dots,i\}) - v(\{1,\dots,i-1\}) \in [0, b_i]$ by Remark 2.2; $\sum_{i=1}^{n} Bi_i(v) = v(N)$ and $\sum_{i=1}^{k} Bi_i(v) = \sum_{i=1}^{k}(v(\{1,...,i\}) - v(\{1,...,i-1\})) = v(\{1,...,k\}) - v(\emptyset) = v(\{1,...,k\})$ for each $k \in N$.

$\square$

Lemma 3.1

Suppose $\sigma \in \Pi(N)$ with $\sigma(k) < \sigma(k+1)$ and $m^{\sigma}(v) \in C(v)$. Let $\tau \in \Pi(N)$ be the ordering with $\tau(k) = \sigma(k+1)$, $\tau(k+1) = \sigma(k)$ and $\tau(r) = \sigma(r)$ for each $r \in N \backslash \{k, k+1\}$. Then $m^{\tau}(v) \in C(v)$.

Proof

Put $\sigma(k) = i$, $\sigma(k+1) = j$. Note that

(1) $m^{\sigma}_{\sigma(r)}(v) = m^{\sigma}_{\tau(r)}(v)$ for each $r \in N \backslash \{k, k+1\}$,

(2) $m^{\sigma}_i(v) + m^{\sigma}_j(v) = m^{\tau}_i(v) + m^{\tau}_j(v)$, because the left side and the right side of (2) are both equal to $v(\{\sigma(1), ..., \sigma(k), \sigma(k+1)\})v(\{\sigma(1), ..., \sigma(k-1)\})$. In view of (2) and Theorem 3.1 if is sufficient to prove that

(3) $m^{\sigma}_i(v) \leq m^{\tau}_i(v)$ or $m^{\sigma}_j(v) \geq m^{\tau}_j(v)$ because then $\sum_{s=1}^{k} m^{\tau}_s(v) \geq \sum_{s=1}^{k} m^{\sigma}_s(v) \geq v(\{1,...,k\})$ for each $k \in N$.

To prove (3) let $S = \{\sigma(1), \sigma(2), ..., \sigma(k-1)\}$. Consider the following three cases: $\lambda_S = \lambda_{S\cup\{j\}} < d_j$, $\lambda_S = \lambda_{S\cup\{j\}} \geq d_j$ and $\lambda_S \leq \lambda_{S\cup\{j\}} = d_j$.
If $\lambda_S = \lambda_{S\cup\{j\}} < d_j$, then $m^{\tau}_j(v) = v(S \cup \{j\}) - v(S) = 0 \leq m^{\sigma}_j(v)$.
If $\lambda_S = \lambda_{S\cup\{j\}} \geq d_j$, then $d_j \geq d_i$ implies that $m^{\sigma}_i(v) = m^{\tau}_i(v) = b_i$.
If $\lambda_S \leq \lambda_{S\cup\{j\}} = d_j$ then $m^{\tau}_i(v) = v(S \cup \{i, j\}) - v(S \cup \{j\}) = b_i \geq m^{\sigma}_i(v)$.
where the second equality follows from $d_i \leq d_j = \lambda_{S\cup\{j\}}$.

$\square$

Now we come to the main result of this section.

Theorem 3.3

Let $\langle N, v\rangle$ be an enterprise game. Then

(i) For each $\sigma \in \Pi(N)$ it holds that $m^{\sigma} \in C(v)$

(ii) $\langle N, v\rangle$ is a convex game

(iii) The Shapley value $\phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v)$ is a core element.

Proof

(i) follows from Lemma 3.1 and Proposition 3.1, because each σ can be reached from σ_0 by neighbor switches where a player with a larger index comes earlier, and $m^{\sigma_0}(v) \in C(v)$.

(ii) follows from the well-known fact (Ichiishi (1983)) that a game is convex if and only if all marginal vectors are in the core.

(iii) is a direct consequence of (i) and the fact that $C(v)$ is convex.

□

This theorem implies many interesting properties for solutions.

4 Cost allocation rules for joint projects

Of course, there are many ways to allocate the costs in a joint project. We describe some interesting possibilities.

One way is to apply on the corresponding cooperative games a solution concept ψ from cooperative game theory. If $\langle N, v\rangle$ corresponds to $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$, then $b_i - \psi_i(v)$ is the cost which player $i \in N$ contributes. If we consider the Shapley value (Shapley (1953)), then $\phi(v)$ is a central core element of $\langle N, v\rangle$, which follows from the fact that $\langle N, v\rangle$ is a convex game. (Shapley (1971)). See also Theorem 3.3.

For convex games the τ-value (Tijs (1981)) is also attractive. For such games the calculation of the τ-value is easy: the k-th coordinate of $\tau(v)$ is then given by

$$\tau_k(v) = \alpha v^*(\{k\}) + (1-\alpha)v(\{k\}),$$

where $v^*(\{k\}) = v(N) - v(N\backslash\{k\})$, and where $\alpha \in [0,1]$ is such that $\sum_{k=1}^{n} \tau_k(v) = v(N)$. However, for $n \geqslant 5$ the τ-value may be an element outside the core (Driessen, Tijs (1985)) as Example 4.1 shows. A separate paper (Branzei et al. (2002)) is devoted to the nucleolus (Schmeidler (1969)) of enterprise games.

Our allocation rule β, which we propose now, is inspired by the airport fee literature (Baker (1965), Thompson (1971); Littlechild and Thompson (1977)).

The idea of Baker and Thompson for the cost allocation of airport strips is that only users of a piece of the strip contribute to the cost of that piece and each of the users contributes an equal part. Translating this to our problem $J = \langle N, \Lambda, (d_i, b_i)_{i \in N}, c\rangle$ with sophistication level λ_N, each cost $c(r) - c(r-1)$ for a level increase from $r-1$ to r with $r \leqslant \lambda_N$ is divided equally among the set $H_r = \{i \in N : r \leqslant d_i \leqslant \lambda_N\}$ of agents who are interested in this sophistication level. This leads to the reward vector $\beta(J) \in \mathbb{R}^n$ with

$$\beta_i(J) = \begin{cases} b_i - \sum_{r=1}^{\lambda_N} \delta_i^r(J) & \text{for } i \in L_{\lambda_N} \\ 0 & \text{for } i \in N \text{ with } d_i > \lambda_N \end{cases}$$

where $\delta^r(J) \in \mathbb{R}^n$ is such that $\delta_i^r(J) = |H_r|^{-1}(c(r) - c(r-1))$ for $i \in H_r$ and $\delta_i^r(J) = 0$ otherwise.

In case we have a joint enterprise J with large benefits, Theorem 4.1 shows that the Shapley value of the corresponding cooperative game is easy to calculate and coincides with $\beta(J)$. In the proof we use dual unanimity games $\langle N, u_S^*\rangle$ for $S \subset N$ with $u_S^*(T) = 1$ if $S \cap T \neq \emptyset$ and $u_S^*(T) = 0$ otherwise. Further $\langle N, u_i\rangle$ is the game with $u_i(T) = 1$ if $i \in T$ and $u_i(T) = 0$ otherwise, so $u_i = u_{\{i\}}^*$. It is well-known that the Shapley value $\phi(u_S^*)$ is equal to $|S|^{-1}e^S$, where $e^S \in \mathbb{R}^n$ is the characteristic vector of S, with $e_i^S = 1$ if $i \in S$, $e_i^S = 0$ if $i \notin S$.

Theorem 4.1 (Joint projects with large benefits).
Suppose that for a joint enterprise situation $J = \langle N, \Lambda, (d_i, b_i)_{i \in N}, c\rangle$ with $N = \{1, 2, \ldots, n\}$ and $\Lambda = \{1, 2, \ldots, m\}$ we have

$$c(d_i) \leqslant b_i \text{ for each } i \in N \qquad \text{(large benefit condition).}$$

Then for the corresponding game $\langle N, v\rangle$ we have:

(i) $v = \sum_{i=1}^{n} b_i u_i - \sum_{r=1}^{m} (c(r) - c(r-1)) u_{M_r}^*$, where $M_r = \{i \in N : d_i \geqslant r\}$

(ii) $\phi(v) = b - \sum_{r=1}^{m} |M_r|^{-1}(c(r) - c(r-1)) e^{M_r}$

(iii) $\beta(J) = \phi(v)$.

Proof

(i) Take $S \in 2^N \backslash \{\emptyset\}$. First we show that in view of the large benefit condition it is optimal for S to realize a sophistication level $\lambda_S = \max\{d_i : i \in S\}$. This follows because for $\lambda > \lambda_S$: $B_S(\lambda) = \sum_{i\in S} R_i(\lambda) - c(\lambda) = \sum_{i\in S} R_i(\lambda_S) - c(\lambda) < \sum_{i\in S} R_i(\lambda_S) - c(\lambda_S) = B_S(\lambda_S)$, and for $\lambda < \lambda_S$, by taking an $i(S) \in S$ with $d_{i(S)} = \lambda_S$, we obtain $\sum_{i\in S} R_i(\lambda_S) - \sum_{i\in S} R_i(\lambda) \geqslant b_{i(S)} \geqslant c(d_{i(S)}) \geqslant c(d_{i(S)}) - c(\lambda)$, where the large benefit condition is applied in the second inequality; so $B_S(\lambda_S) = \sum_{i\in S} R_i(\lambda_S) - c(d_{i(S)}) \geqslant \sum_{i\in S} R_i(\lambda) - c(\lambda) = B_S(\lambda)$. Hence, we have proved that $\lambda_S = \max\{d_i : i \in S\}$. This implies that $v(S) = B_S(\lambda_S) = \sum_{i\in S} b_i - c(\lambda_S)$. To prove (i) in the theorem, note that for each $S \subset N$: $\sum_{i\in S} b_i = \sum_{i\in N} b_i u_i(S)$. Further $M_r \cap S \neq \emptyset$ iff $r \leqslant \lambda_S$ iff $u^*_{M_r}(S) \neq 0$. So, $c(\lambda_S) = \sum_{r=1}^{\lambda_S}(c(r) - c(r-1)) = \sum_{r=1}^{m}(c(r) - c(r-1))u^*_{M_r}(S)$.

(ii) It follows from (i) and the additivity of the Shapley value ϕ that $\phi(v) = \sum_{i=1}^{n} b_i\phi(u_i) - \sum_{r=1}^{m}(c(r) - c(r-1))\phi(u^*_{M_r}) = b - \sum_{r=1}^{m} |M_r|^{-1}(c(r) - c(r-1))e^{M_r}$.

(iii) For each $i \in N$: $\phi_i(v) = b_i - \sum_{r=1}^{m} |M_r|^{-1}(c(r) - c(r-1))e_i^{M_r} = b_i - \sum_{r=1}^{d_i} |M_r|^{-1}(c(r) - c(r-1)) = \beta_i(J)$, because $\lambda_N = m$, $M_r = H_r$. □

Example 4.1

Consider the joint enterprise situation $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$, where $N = \{1, 2, 3, 4, 5\}$, $\Lambda = \{0, 1, 2, \}$, $d_i = 1$, $b_i = 20$ for $i \in \{1, 2, 3\}$, $d_i = 2$, $b_i = 110$ for $i \in \{4, 5\}$ and $c(0) = 0$, $c(1) = 10$, $c(2) = 90$. This is a situation with large benefits because $c(d_i) \leqslant b_i$ for each $i \in N$, implying that $\lambda_S = \max\{d_i : i \in S\}$ for each $S \subset N$. By Theorem 4.1, for the corresponding game $\langle N, v\rangle$ we have $v = \sum_{i=1}^{5} b_i u_i - c(1)u^*_N - (c(2) - c(1))u^*_{\{4,5\}} = 20(u_1 + u_2 + u_3) + 110(u_4 + u_5) - 10u^*_N - 80u^*_{\{4,5\}}$. From this it follows that the Shapley value $\phi(v)$ is equal to (18,18,18,68,68). The Baker-Thompson like allocation $\beta(J)$ we obtain from $b - \delta^1(J) - \delta^2(J) = (b_1, b_2, b_3, b_4, b_5) - \frac{10}{5}e^N - \frac{80}{2}e^{\{4,5\}} = (18, 18, 18, 68, 68)$ and it is equal to $\phi(v)$, which is in accordance with Theorem 4.1. To calculate the τ-value, note that $v^*(\{k\}) = v(N) - v(N\backslash\{k\}) = (\sum_{i=1}^{5} b_i - c(2)) - (\sum_{i\in N\backslash\{k\}} b_i - c(2)) = b_k$, and $v(\{i\}) = b_i - c(i) = 20 - 10$ if $i \in \{1, 2, 3\}$ and $v(\{i\}) = 110 - 90 = 20$ if $i \in \{4, 5\}$. Hence $\tau(v) = \alpha(20, 20, 20, 110, 110) + (1-\alpha)(10, 10, 10, 20, 20)$ with $\alpha = \frac{3}{7}$, i.e. $\tau(v) = (15\frac{5}{7}, 15\frac{5}{7}, 15\frac{5}{7}, 71\frac{3}{7}, 71\frac{3}{7})$. Note that $\sum_{i=1}^{3} \tau_i(v) < 50 = v(\{1, 2, 3\})$, so $\tau(v) \notin C(v)$. The nucleolus $nu(v)$ is equal to $(17\frac{1}{2}, 17\frac{1}{2}, 17\frac{1}{2}, 68\frac{3}{4}, 68\frac{3}{4})$.

Example 4.2

Consider the joint enterprise situation $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$, where $N = \{1, 2\}$, $\Lambda = \{0, 1, 2\}$, $d_1 = 1$, $b_1 = 3$, $d_2 = 2$, $b_2 = 25$, $c(0) = 0$, $c(1) = 10$, $c(2) = 22$.

Then player 1 has a small benefit: $b_1 = 3 < c(1) = 10$. For the corresponding game $\langle N, v\rangle$

we have $v(\{1\}) = 0$, $v(\{2\}) = 3$ and $v(\{1,2\}) = 6$, so $v = 3u_{\{2\}} + 3u_{\{1,2\}}$. The Shapley value $\phi(v) = (1\frac{1}{2}, 4\frac{1}{2})$ is unequal to the allocation $\beta(J) = (3,25) - (5,5) - (0,12) = (-2,8)$. Note that $\beta_1(J) = -2 < 0$, so $\beta(J) \notin C(v)$, and $\beta(J)$ is not an attractive allocation. The allocation rule β^c, which we introduce soon, takes into account that player 1 with benefit 3 cannot contribute $\frac{1}{2}b_1 = 5$ in the costs $c(1) - c(0)$.

In this subsection we introduce a new allocation rule β^c, based still on the Baker-Thompson principle that only players contribute to the cost $c(r) - c(r-1)$ of a raise in sophistication level from r to $r+1$, who are interested in such a raise, and where, roughly speaking, these contributions are as equal as possible but never larger than the benefit which a player can obtain from such raises. The contribution to level raises will be determined sequentially in n steps starting with the highest level raise, then the second highest level raise, etc. In each step the benefits will be adjusted taking into account the contribution in costs in earlier steps. To avoid technical obstacles we suppose in this subsection that the sophistication level of the grand coalition is equal to $\max(\Lambda)$, or $\lambda_N = m$ if $\Lambda = \{1, 2, \ldots, m\}$. Note that this does not harm the generality because in a joint enterprise situation where $\lambda_N < \max(\Lambda)$ the players with $\lambda > \lambda_N$ play a dummy role and can be removed from the problem.

Note further that

$$(4.1)\ \lambda_N = m \iff B_N(m) \geqslant B_N(k-1) \text{ for all } k \in \Lambda$$
$$\iff \sum_{i \in M_k} b_i \geqslant c(m) - c(k-1)$$

(where $M_k = \{i \in N : d_i \geqslant k\}$).
So, $\lambda_N = m$ iff for each $k \in \Lambda$ the players in M_k can cover the costs $c(m) - c(k-1)$ of a raise in sophistication level from $k-1$ to m.
To give a smooth formal introduction of β^c it is convenient to introduce the notions of feasible simple cost sharing problem and of constrained equal cost sharing vectors for such a problem. Let us call a quadruplet $\langle N, b, c, S \rangle$ a *simple cost sharing problem* if $N = \{1, 2, \ldots, n\}$, $\emptyset \neq S \subset N$, $b \in \mathbb{R}^n_+$, $c \in \mathbb{R}_+$. It corresponds to a situation, where a cost c has to be covered by a non-empty subset S of the player set N, where one has to take into account not to exceed the available budgets, where b_i is the budget of player i. Such a simple cost sharing problem $\langle N, b, c, S \rangle$ is called *feasible* if $\sum_{i \in S} b_i \geqslant c$, i.e. if the total budget of the players in S is sufficient to cover the involved cost.
Note that for a feasible cost sharing problem $\langle N, b, c, S \rangle$ there is a unique real number $\alpha \in [0, \max_{i \in S} b_i]$, such that $\sum_{i \in S} \min\{b_i, \alpha\} = c$. The *constrained equal cost sharing*

vector is then the vector $\varepsilon(N,b,c,S) \in I\!R^n$ with $\varepsilon_i(N,b,c,S) = 0$ for $i \in N\backslash S$, and $\varepsilon_i(N,b,c,S) = min\{b_i, \alpha\}$ for $i \in S$. This vector corresponds to a cost sharing of c by members of S only, where the players in S with budget higher than α pay α and the others spend their whole budget to cover the cost.

Example 4.3
Let $N = \{1,2,3,4\}$, $S = \{3,4\}$, $b(1) = (2,6,0,18)$, $b(2) = (2,6,5,25)$, $c(1) = 10$ and $c(2) = 12$. Then $\varepsilon(N,b(2),c(2),S) = (0,0,5,7)$ and $\varepsilon(N,b(1),c(1),N) = (2,4,0,4)$.

Let $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$ be a joint enterprise situation with $\Lambda = \{1,2,\ldots,m\}$ and $\lambda_N = m$. Consider the sequence of m feasible (Lemma 4.1) and interrelated simple cost sharing problems $P_m, P_{m-1}, \ldots, P_1$ with $P(m) = \langle N, b(m), c(m) - c(m-1), M_m\rangle$, $P_{m-1} = \langle N, b(m-1), c(m-1) - c(m-2),\ M_{m-1}\rangle, \ldots, P_2 = \langle N, b(2), c(2) - c(1), M_2\rangle$, $P_1 = \langle N, b(1), c(1) - c(0), M_1\rangle$, where $b(m) = b$, and for $r < m : b(r) = b(r+1) - \varepsilon(P_{r+1})$. Then $\beta^c(J) = b - \sum\limits_{r=1}^{m} \varepsilon(P_r)$.
So, β^c assigns to J a vector where the i-th coordinate is equal to the benefit b_i minus the contributions of i in costs in the m simple cost sharing problems.
Important is the feasibility of each problem P_r, because otherwise we cannot define $\varepsilon(P_r)$. This feasibility is proved in

Lemma 4.1
Let $J, P_1, P_2, \ldots, P_m$ be as above. The $P_1, P_2, \ldots, P_m$ are feasible simple cost sharing problems.

Proof
The proof is by backward induction. First note that P_m is feasible because $\lambda_N = m$ implies that $\sum\limits_{i\in H_m} b_i \geqslant c(m) - c(m-1)$. For each $k \in \{1,\ldots,m-1\}$ for which $P_{k+1},\ldots,P_m$ are feasible, we have to prove that P_k is feasible. Take such a k.
Note that

(a) $b(k) = b(m) - \sum\limits_{r=k+1}^{m} \varepsilon(P_r)$

(b) $\sum\limits_{i\in M_k} b_i(m) \geqslant c(m) - c(k-1)$

where (b) follows from the fact that $m = \lambda_N$. Then

$$\begin{aligned}\sum_{i\in M_k} b_i(k) &\geqslant c(m) - c(k-1) - \sum_{r=k+1}^{m} \sum_{i\in M_k} \varepsilon_i(P_r)\\ &= c(m) - c(k-1) - \sum_{r=k+1}^{m} \sum_{i\in N} \varepsilon_i(P_r)\\ &= c(m) - c(k-1) - \sum_{r=k+1}^{m} (c(r) - c(r-1)) = c(k) - c(k-1).\end{aligned}$$

Hence, P_k is feasible.

□

Example 4.4

Let $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$, where $N = \{1,2,3,4\}$, $\Lambda = \{1,2\}$, $b_1 = 2$, $b_2 = 6$, $b_3 = 5$, $b_4 = 25$, $d_1 = d_2 = 1$, $d_3 = d_4 = 2$, $c(1) = 10$ and $c(2) = 22$.

Then $P_2 = \langle N, (2,6,5,25), 12, \{3,4\}\rangle$, $\varepsilon(P_2) = (0,0,5,7)$, and $P_1 = \langle N, (2,6,5,25) - (0,0,5,7), 10, \{1,2\}\rangle$, $\varepsilon(P_1) = (2,4,0,4)$. Hence $\beta^c(J) = b - \varepsilon(P_2) - \varepsilon(P_1) = (0,2,0,14)$. Note that $\beta^c(J)$ is a core element of the cooperative game corresponding to J.

While $\beta(J)$ was not necessarily a core element of the related cooperative game, $\beta^c(J)$ is always a member of the core as the next theorem shows.

Theorem 4.2

Let $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$ be a joint enterprise situation with $\lambda_N = m = \max(\Lambda)$ and let $\langle N, v\rangle$ be the corresponding cooperative game. Then

(i) $\beta^c(J) \in C(v)$

(ii) if $b_i \geqslant c(d_i)$ for all $i \in N$, then $\beta^c(J) = \beta(J) = \phi(v)$.

Proof

(i) We prove (i) using Theorem 3.1, so we have to prove for $x = \beta^c(J)$: (a) $x(N) = v(N)$, (b) $0 \leqslant x_i \leqslant b_i$ for each $i \in N$ and (c) $x(L_k) \geqslant v(L_k)$ for each $k \in \Lambda$.

(a) $\beta^c(J) = b - \sum_{r=1}^{m} \varepsilon(P_r)$ and $\sum_{i\in N} \varepsilon_i(P_r) = c(r) - c(r-1)$, so $\sum_{i\in N} \beta_i^c(J) = \sum_{i\in N} b_i - \sum_{r=1}^{m}(c(r) - c(r-1)) = \sum_{i\in N} b_i - c(\lambda_N) = v(N)$.

(b) From $b_i = b_i(m) \geqslant b_i(m-1) \geqslant \ldots \geqslant b_i(1) \geqslant \beta_i^c(J) \geqslant 0$ follows that $\beta_i^c(J) \in [0, b_i]$ for each $i \in N$.

(c) Take $k \in \{1, 2, \ldots, m\}$. Let $\bar{L}_k = \{i \in L_k : d_i \leqslant \lambda_{L_k}\}$. Then $v(L_k) = \sum_{i \in \bar{L}_k} b_i - c(\lambda_{L_k})$. Now $\sum_{i \in L_k} \beta_i^c(J) \geqslant \sum_{i \in \bar{L}_k} \beta_i^c(J) = \sum_{i \in \bar{L}_k} (b_i - \sum_{r=1}^{\lambda_{L_k}} \varepsilon_i(P_r))$
$\geqslant \sum_{i \in \bar{L}_k} b_i - \sum_{r=1}^{\lambda_{L_k}} \sum_{i \in N} \varepsilon_i(P_r) = \sum_{i \in \bar{L}_k} b_i - \sum_{r=1}^{\lambda_{L_k}} (c(r) - c(r-1)) =$
$\sum_{i \in \bar{L}_k} b_i - c(\lambda_{L_k}) = v(L_k)$.

(ii) In case of high benefits we have $\varepsilon(P_r) = |M_r|^{-1}(c(r) - c(r-1))$ for each $r \in \{1, 2, \ldots, m\}$. So $\beta_i^c(L) = b_i - \sum_{r=1}^{m} \varepsilon_i(P_r) = b_i - \sum_{r=1}^{d_i} |M_r|^{-1}(c(r) - c(r-1)) = \beta_i(L)$. Further, by Theorem 4.1, $\beta(L) = \phi(v)$.

□

We like to conclude this section with the remark that in our opinion the β^c rule is an attractive allocation scheme for joint projects. It is based on sound economic principles and it leads to a stable reward allocation, which, moreover, equals the Shapley value of the related cooperative game in case the benefits for the players are high.

5 Non-cooperative contribution games for joint enterprise situations

In this section we describe a strategic (non-cooperative) approach to a joint enterprise situation, which can be useful to solve the problem of choosing a suitable sophistication level as well as the cost sharing problem connected with the realized sophistication level. In this approach the members involved in the joint enterprise decide independently what they will contribute to each of the possible m level increases, from 0 to 1, from 1 to 2, …, from $m-1$ to m, where $\Lambda = \{1, 2, ..., m\}$. They deliver the contribution vector describing their wishes to the central planner and also the corresponding amount of money. Hence, a strategy of player $i \in N$ can be identified with the contribution vector $u^i = (u_1^i, u_2^i, ..., u_m^i) \in \mathbb{R}_+^m$, where u_λ^i is the amount of money which player i hands in as a contribution to raise the sophistication level from $\lambda - 1$ to λ. Suppose players 1, 2, …, n have decided to the strategies (the contribution vectors) $u^1, \ldots, u^n$. If the joint contribution $\sum_{i \in N} u_\lambda^i$ for the raise from $\lambda - 1$ to λ is smaller than the cost $c(\lambda) - c(\lambda - 1)$, then this raise is not realized and also not the higher raises: from λ to $\lambda + 1, \ldots$ Players never get money back, neither if the joint contribution to a raise is insufficient nor if the total contribution exceeds the cost of the raise. Given a strategy profile $u = (u^1, \ldots, u^n)$ of contribution vectors, the sophistication level which will be realized is given by

$$\lambda(u) = max\{\lambda \in \Lambda \mid \sum_{i\in N} u^i_\beta \geq c(\beta) - c(\beta - 1) \text{ for all } \beta \leq \lambda\}.$$

The corresponding payoff for player i, given contribution profile u, is described by

$$(5.1)\ \Pi_i(u) = R_i(\lambda(u)) - \sum_{\lambda\in\Lambda} u^i_\lambda, \qquad \text{for } i \in N$$

i.e. the payoff for player i corresponding to the realized sophistication level $\lambda(u)$ minus the contributed costs. Formally this leads to the non-cooperative game $\Gamma(J)$, corresponding to the joint enterprise situation $J = \langle N, \Lambda, (d_i, b_i)_{i\in N}, c\rangle$ with $N = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2, \ldots, m\}$, with $\Gamma(J) = \langle N, S_1, S_2, \ldots, S_n, \Pi_1, \Pi_2, \ldots, \Pi_n\rangle$, where N is the set of players; for each $i \in N$ the strategy set S_i equals $I\!R^m_+$, the set of possible contribution vectors; and the payoff function Π_i is given by (5.1) for each $u \in (I\!R^m_+)^n$. The game $\Gamma(J)$ is called the contribution game associated to J. In the following we are interested in Nash equilibria (NE) and strong Nash equilibria (SNE) of the game $\Gamma(J)$. A Nash equilibrium for $\Gamma(J)$ is a strategy profile $(u^i)_{i\in N}$, where unilateral deviation of a player does not pay. A strong Nash equilibrium for $\Gamma(J)$ is a strategy profile u such that no coalition S can deviate and obtain a payoff at least as large as $\Pi_i(u)$ for each of its members and more for at least one of its members. So $u \in (I\!R^m_+)^n$ is a SNE if there is no $S \subset N$ with a strategy profile $\bar{u}^S = (\bar{u}_i)_{i\in S}$ such that $\Pi_i(\bar{u}^S, u^{N\setminus S}) \geq \Pi_i(u^S, u^{N\setminus S}) = \Pi_i(u)$ for each $i \in S$ and where at least one inequality is strict. The objective of the rest of this section is to show that for each SNE of $\Gamma(J)$ the corresponding payoffs to the players form a core element of the corresponding cooperative enterprise game v; and, conversely, that each core element of the enterprise game is achieved via payoffs related to at least one SNE of $\Gamma(J)$. To obtain these results we need three lemmas. But first we give an example.

Example 5.1. Let $J = \langle N, \Lambda, (d_i,\ b_i)_{i\in N},\ c\rangle$ be the joint enterprise situation where $N = \{1, 2, 3\}, \Lambda = \{0, 1, 2\}, d = (1, 1, 2), b = (4, 12, 10), c(0) = 0, c(1) = 10$ and $c(2) = 20$. For the contribution game $\Gamma(J)$ the strong Nash equilibria $u = ((3, 0), (7, 0), (0, 0))$ and $\tilde{u} = ((4, 0), (6, 0), (0, 10))$ correspond to the core elements $(4-3, 12-7, 0-0) = (1, 5, 0)$ and $(0, 6, 0)$, respectively, of the enterprise game $\langle N, v\rangle$ with $v(\emptyset) = v(\{1\}) = v(\{3\}) = v(\{1, 3\}) = 0,\ v(\{2\}) = v(\{2, 3\}) = 2$, and $v(\{1, 2\}) = v(\{1, 2, 3\}) = 6$. Given the core element $(1, 5, 0)$ above (which corresponded to the SNE u) another strong Nash equilibrium $((3, 0), (7, 0), (0, 10))$ is found if we use the method described in the proof of

Proposition 5.2.

Lemma 5.1 *Let* $u = (u^i)_{i\in N}$ *be a Nash equilibrium of the contribution game* $\Gamma(J)$*. Then for every* $i \in N$ *we have*

(i) $0 \le \Pi_i(u) \le b_i$*, for each* $i \in N$

(ii) $d_i < \lambda$ *implies* $u^i_\lambda = 0$

(iii) $\lambda(u) < d_i$ *implies* $u^i = (0, 0, \ldots, 0)$

(iv) $\lambda(u) < \lambda$ *implies* $u^i_\lambda = 0$.

Proof

(i) It is obvious from (5.1) that $\Pi_i(u) \le b_i$ for each $i \in N$. Suppose that there exists a player $i \in N$ such that $\Pi_i(u) < 0$. Then player i could unilaterally deviate by choosing the strategy $\tilde{u}^i = (0, \ldots, 0)$ and he will receive at least 0 independently of what the other players do, because:

$$\Pi_i(u^{-i}, \tilde{u}^i)) = R_i(\lambda(u^{-i}, \tilde{u}^i)) - \sum_{\lambda\in\Lambda} \tilde{u}^i_\lambda = \begin{cases} b_i, & \text{if } \lambda\big(u^{-i}, \tilde{u}^i\big) \ge d_i; \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda(u^{-i}, \tilde{u}^i) \le \lambda(u)$. Then $\Pi_i(u^{-i}, \tilde{u}^i) \ge \Pi_i(u)$, that is u is not a Nash equilibrium.
(ii) Note that if $u^i_\lambda > 0$ for some $\lambda > d_i$, then player i could unilaterally deviate by choosing the strategy $\tilde{u}^i$ defined by

$$\tilde{u}^i_\alpha = \begin{cases} u^i_\alpha & \text{if } \alpha \ne \lambda \\ 0 & \text{if } \alpha = \lambda. \end{cases}$$

Then $\lambda(u) = \lambda(u^{-i}, \tilde{u}^i)$, so $\Pi_k(u^{-i}, \tilde{u}^i) = \Pi_k(u)$ for each $k \in N\backslash\{i\}$ and $\Pi_i(u^{-i}, \tilde{u}^i) = R_i(\lambda(u)) - \sum_{\alpha\in\Lambda} \tilde{u}^i_\alpha = \Pi_i(u) + u^i_\alpha > \Pi_i(u)$. This is in contradiction with the assumption that u is a Nash equilibrium.
(iii) Suppose $\lambda(u) < d_i$ and $u^i_\lambda > 0$ for some $\lambda \in \Lambda$. Then $\Pi_i(u) < 0$ and this is a contradiction with (i). So $\lambda(u) < d_i$ implies $u^i_\lambda = 0$.
(iv) Suppose $\lambda > \lambda(u)$ and $u^i_\lambda > 0$. Take $\tilde{u}^i = (\tilde{u}^i_\alpha)_{\alpha\in\Lambda}$ as in (ii) and player i improves.

□

Lemma 5.2 *Let* $u = (u^i)_{i\in N}$ *be a Nash equilibrium of the contribution game* $\Gamma(J)$*. Then it holds that:*

(i) $\sum_{i\in N} u^i_\lambda \leq c(\lambda) - c(\lambda-1)$, for every $\lambda \in \Lambda$, *with equality if* $\lambda \leq \lambda(u)$*;*

(ii) $\sum_{\alpha\leq\lambda} \sum_{i\in N} u^i_\alpha \leq c(\lambda)$, *for every* $\lambda \in \Lambda$, *with equality if* $\lambda \leq \lambda(u)$.

Proof
(i) Let $\lambda \in \Lambda$ and $k = \sum_{i\in N} u^i_\lambda - (c(\lambda) - c(\lambda-1))$. If $k > 0$ there exists (at least) one player $i \in N$ for which $u^i_\lambda > 0$, and then for this player it would exist a profitable deviation, namely the strategy $\tilde{u}^i$ defined by

$$\tilde{u}^i_\alpha = \begin{cases} u^i_\alpha & \text{if } \alpha \neq \lambda \\ \max\{0, u^i_\lambda - k\} & \text{if } \alpha = \lambda. \end{cases}$$

The equality sign for $\lambda \leq \lambda(u)$ follows from the definition of $\lambda(u)$.
(ii) It is a straightforward consequence of (i).

□

Lemma 5.3. *Let* u *be a strong Nash equilibrium of* $\Gamma(J)$ *and let* $\beta \in \Lambda\backslash\{0, 1, 2, \ldots, \lambda(u)\}$, $T_\beta = \{i \in N : \lambda(u) < d_i \leq \beta\}$. *Then*

(i) $c(\beta) - c(\lambda(u)) \geq \sum_{i\in T_\beta} b_i$

(ii) $c(\lambda_N) - c(\lambda(u)) = \sum_{i\in T_{\lambda_N}} b_i$ *if* $\lambda_N > \lambda(u)$.

Proof (i) Suppose, for a moment, that for some $\beta \in (\lambda(u), m]$

$$(5.2)\ c(\beta) - c(\lambda(u)) < \sum_{i\in T_\beta} b_i.$$

We show that then u cannot be a SNE of $\Gamma(J)$. Given the inequality (5.2) it is possible to find a matrix $(v^i_\alpha)_{i\in T_\beta,\ \alpha\in(\lambda(u),\ \beta]}$ such that

(5.3)	$v^i_\alpha \geq 0,$	for all $i \in T_\beta,\ \alpha \in (\lambda(u),\ \beta]$
(5.4)	$\sum_{i\in T_\beta} v^i_\alpha = c(\alpha) - c(\alpha-1),$	for all $\alpha \in (\lambda(u), \beta]$
(5.5)	$\sum_{\alpha=\lambda(u)+1}^{\beta} v^i_\alpha \leq b_i,$	for all $i \in T_\beta$.

To find such a matrix one can use e.g. an algorithm from the theory of transportation, since the above problem can be seen as a simple transportation situation, where the players $i \in T_\beta$ are suppliers with supply b_i and where the levels $\alpha \in (\lambda(u), \beta]$ are demand points with demand $c(\alpha) - c(\alpha-1)$. By (5.2) the total supply exceeds the total demand, so all demanders can be completely satisfied and there is (at least) one supplier i^* who cannot get rid of his total supply, i.e.

$$(5.6) \qquad b_{i^*} - \sum_{\alpha \in (\lambda(u), \beta]} v_\alpha^{i^*} > 0.$$

Consider now the strategy profile $(u_{N \setminus T_\beta}, \bar{u}_{T_\beta}) = ((u^i)_{i \in N \setminus T_\beta}, (\bar{u}^i)_{i \in T_\beta})$ where the players in T_β deviate from u as follows: $\bar{u}_\alpha^i = v_\alpha^i$ if $i \in T_\beta, \alpha \in (\lambda(u), \beta]$ and $\bar{u}_\alpha^i = 0$, otherwise. Then, by (5.4), $\lambda(u_{N \setminus T_\beta}, \bar{u}_{T_\beta}) \geq \beta$ and $\Pi_i(u_{N \setminus T_\beta}, \bar{u}_{T_\beta}) \geq \Pi_i(u) = 0$ for all $i \in T_\beta$ and with strict inequality for i^*, by (5.6). So $\bar{u}_{T_\beta}$ is an improvement, u is not a SNE.
(ii) In case $\beta = \lambda_N > \lambda(u)$, we obtain from (i) that $c(\lambda_N) - c(\lambda(u)) \geq \sum_{i \in T_{\lambda_N}} b_i$. The converse inequality follows from (4.1) because $\beta = \lambda_N$. □

Proposition 5.1 *Let $\Gamma(J)$ and $\langle N, v \rangle$ be the contribution game and the enterprise game, respectively, corresponding to the joint enterprise situation J. Let u be a strong Nash equilibrium of $\Gamma(J)$ and $z \in I\!R^n$ the vector with $z_i = \Pi_i(u)$ for each $i \in N$. Then $z \in C(v)$.*

Proof To prove that $z \in C(v)$ we will use the characterization of core elements in Theorem 3.1.
(i) Note that $0 \leq z_i \leq b_i$ for each $i \in N$ follows from Lemma 5.1.(i).
(ii)Take $k \in \Lambda$. Note that it follows from Lemma 5.3.(i) that there is an optimal level k' for L_k with $k' \leq \lambda(u)$ and hence

$$(5.7) \; v(L_k) = v(L_{k'}) = \sum_{i \in L_{k'}} b_i - c(k').$$

Using respectively Lemma 5.1.(i), the definition of z_i, Lemma 5.1.(ii) and the fact that $u \in (I\!R^n_+)^m$ we obtain $\sum_{i \in L_k} z_i \geq \sum_{i \in L'_k} z_i = \sum_{i \in L_{k'}} (b_i - \sum_{\lambda \in \Lambda} u_\lambda^i) = \sum_{i \in L'_k} (b_i - \sum_{\lambda \leq k'} u_\lambda^i) \geq \sum_{i \in L'_k} b_i - \sum_{\lambda \leq k'} \sum_{i \in N} u_\lambda^i)$. Using Lemma 5.2.(ii) the right hand term is equal to $\sum_{i \in L_{k'}} b_i - c(k') = v(L_k)$. So $\sum_{i \in L_k} z_i \geq v(L_k)$ for each $k \in \Lambda$.
(iii) The proof of the efficiency $\sum_{i \in N} z_i = v(N)$ runs along similar lines as (ii) after remarking that in view of Lemma 5.3. the level $\lambda(u)$ is an optimal sophistication level for the grand coalition N. □

Proposition 5.2 Let $x = (x_1, \ldots, x_n)$ be a payoff vector in the core of v. Then there exists a profile of strategies u, such that $\Pi_i(u) = x_i$ for every $i \in N$.

Proof

Define inductively u_α^i in the following way:

$u^n_m = \min\{c(m) - c(m-1), b_n - x_n\}$,
$u^n_\alpha = \min\{c(\alpha) - c(\alpha - 1), (b_n - x_n - \sum_{\beta>\alpha} u^n_\beta)_+\}$ if $\alpha < m$.
And if $i < n$ define
$u^i_\alpha = 0$ if $\alpha > d_i$,
$u^i_\alpha = \min\{c(\alpha) - c(\alpha-1) - \sum_{j>i} u^j_\alpha, (b_i - x_i - \sum_{\beta>\alpha} u^i_\beta)_+\}$ otherwise.
We show that u is a SNE. First we prove that $\sum_{i\in N} u^i_a = c(\alpha) - c(\alpha - 1)$, for every $\alpha \in \Lambda$. By definition it is clear that $\sum_{i\in N} u^i_a \leq c(\alpha) - c(\alpha - 1)$. Assume that this inequality was strict. In this case $\sum_{\alpha\in\Lambda} u^i_\alpha \geq b_i - x_i$, so we would have

$$c(m) > \sum_{\alpha\in\Lambda}\sum_{i\in N} u^i_\alpha \geq \sum_{i\in N}(b_i - x_i) = c(m),$$

where the equality follows by taking into account that $\sum_{i\in N} x_i = v(N) = \sum_{i\in N} b_i - c(m)$. And this would be in contradiction with our assumption that u is a SNE.
Now if a player i changes his strategy by paying strictly less, it has to be done in a level lower than d_i and then this level will not be realized and he will not obtain a higher payoff. And if he pays more in a level, his benefit will be reduced in this amount. □

Combining Propositions 5.1 and 5.2 we obtain

Theorem 1 *Every strong Nash equilibrium (SNE) of the contribution game corresponds to one element in the core of the related enterprise game, and conversely, each core element of the enterprise game corresponds to payoffs of at least one SNE.*

6 Concluding Remarks

We related to each joint enterprise situation a cooperative enterprise game and a strategic contribution game. The characterization in Theorem 3.1 of core elements of the enterprise game played a crucial role at several parts of this paper. To mention three places: in the proof of the convexity of the enterprise game; in the description of the relation between core elements of the enterprise game and strong Nash equilibrium payoffs of the contribution game; further to prove that the rule β^c leads to core elements of the enterprise game (Theorem 4.2).
Let us mention some issues for possible further research.

- In this paper there is a natural linear order on the set $\Lambda = \{1, 2, \ldots, m\}$ of sophistication levels. It will be interesting to consider joint projects where Λ is a partially ordered set. In case this partially ordered set leads to a tree many of the results in this paper can be extended.

- In this paper the reward function $R_i : \Lambda \to I\!R$ is a step function. It would be interesting to consider weakenings of this condition and see what are the consequences.

- In Section 4 the rule β^c was constructed by considering a sequence of m interrelated simple cost sharing problems and using the constrained equal cost sharing rule for these problems. Of course, other rules for such simple cost sharing problems can also be used and can generate in a similar way interesting stable rules for joint enterprise situations.

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