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Discussion paper

Convex fuzzy games and participation monotonic allocation schemes*

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Abstract

In this paper the cone of convex cooperative fuzzy games is studied. As in the classical case of convex crisp games, these games have a large core and the fuzzy Shapley value is the barycenter of the core. Surprisingly, the core and the Weber set coincide

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as in the classical case but the coincidence of these sets for a fuzzy game does not imply automatically convexity as in the crisp case. Participation monotonic allocation schemes (pamas) are introduced and it turns out that each core element of a convex fuzzy game is pamas−extendable.

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1 Introduction

The basis of cooperative game theory was laid in the book of von Neumann and Morgenstern (1944). Since then several solution concepts for cooperative games have been proposed and several interesting subclasses of games have been introduced.

A highly interesting class of cooperative games is the class of convex games introduced by Shapley (1971). For this cone of games many solution concepts behave nicely and much is known about their interrelations. Convex games also arise naturally in connection with economic situations as sequencing (Curiel et al., 1989), bankruptcy (Curiel et al., 1988), and financing of public goods. Airport fee problems are related with concave games (Littlechild and Owen, 1973).

The theory of cooperative fuzzy games started with work of Aubin (1974, 1981) where the notions of fuzzy game and the core of a fuzzy game are introduced. In the meantime many solution concepts have been developed (cf. Butnariu, 1978; Butnariu and Klement, 1993; Molina and Tejada, 2002; Nishizaki and Sakawa, 2001; Sakawa and Nishizaki, 1994; Tsurumi et al., 2001).

The purpose of this paper is on one hand to present a detailed characterization of the class of convex fuzzy games, and on the other hand to study the solution concept of participation monotonic allocation scheme (pamas) for fuzzy games in connection with some solution concepts for these games. It turns out that convex fuzzy games form a convex subcone of the cone of cooperative fuzzy games with a pamas.

The outline of the paper is as follows. Sections 2 and 3 are introductory; they provide the necessary notions and facts for crisp games and fuzzy games, respectively. Section 4 deals with convex fuzzy games. Characterizing properties are discussed. Special attention is paid to the core and some other related solution concepts for convex games. In Section 5 we introduce participation monotonic allocation schemes for fuzzy games and prove that each core element of a convex fuzzy game can be extended to such a scheme. Section 6 concludes with some final remarks.

2 Cooperative crisp games

In the following N is a finite set of players (often $N = \{1, 2, \ldots, n\}$) and 2^N is the family of $2^{|N|}$ crisp subsets of N. A (crisp) *cooperative game* with player set N is a map $v : 2^N \to \Re$ with $v(\emptyset) = 0$. For $S \in 2^N$, $v(S)$ is called the *worth* of coalition S and it is interpreted as the amount of money (utility) the coalition can obtain, when the players in S work together.

The class of crisp games with player set N is denoted by G^N. We recall now some well−known facts from the theory of cooperative crisp games. The class G^N is a $\left(2^{|N|} - 1\right)$ −dimensional linear space. The family of unanimity games $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ is an interesting basis of this linear space, where

$$u_T(S) = \begin{cases} 1, & \text{if } S \supset T \\ 0, & \text{otherwise} \end{cases}.$$

Let $\Pi(N)$ be the set of linear orderings of N. Then for each $v \in G^N$ and each $\sigma \in \Pi(N)$ the marginal vector $m^\sigma(v) \in \Re^N$ is defined as follows: the $i-$th coordinate $m_i^\sigma(v)$ of $m^\sigma(v)$ is equal to

$$v(\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(k-1)\}) \text{ if } i = \sigma(k).$$

So $m_i^\sigma(v)$ is the marginal contribution of $i = \sigma(k)$ entering the coalition $\{\sigma(1), \sigma(2), \ldots, \sigma(k-1)\}$ of predecessors of i in the order σ.

The Shapley value (Shapley (1953)) $\phi(v)$ is equal to $\frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} m^\sigma(v)$ and the Weber set $W(v)$ (Weber, 1988) is the convex hull of the marginal vectors $conv\,\{m^\sigma(v) \mid \sigma \in \Pi(N)\}$.

The core of a game $v \in G^N$ (Gillies, 1953) is the convex set

$$C(v) = \left\{ x \in \Re^N \mid \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\},$$

consisting of efficient vectors with sum of the coordinates equal to $v(N)$ and with the property that no coalition S can obtain more than $\sum_{i\in S} x_i$ in splitting off.

It is well$-$known (Weber, 1988) that $C(v) \subset W(v)$ for each game $v \in G^N$.

Further, for convex games the Shapley value is the barycenter of the core (i.e. the average of the $n!$ marginal vectors which are precisely the extreme points of the core).

Here a game $v \in G^N$ is called *convex* if it satisfies one of the following equivalent conditions (cf. Shapley, 1971; Ichiishi, 1981; Curiel, 1997):

Supermodularity property: for each $S, T \in 2^N$

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T); \tag{1}$$

Increasing marginal contribution property for players: for each $S, T \in 2^N$ with $S \subset T$ and for each $i \in N \setminus T$

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T); \tag{2}$$

Increasing marginal contribution property for coalitions: for each $S, T, U \in 2^N$ with $S \subset T \subset N \setminus U$

$$v(S \cup U) - v(S) \leq v(T \cup U) - v(T); \tag{3}$$

Stable marginal vector property: for each $\sigma \in \Pi(N)$

$$\text{the marginal vector } m^\sigma(v) \text{ is a core element.} \tag{4}$$

Note that (4) and the result of Weber (1988) imply that $C(v) = W(v)$ for convex crisp games.

3 Cooperative fuzzy games

Given a finite set N of players, a *fuzzy coalition* is a vector $s \in [0,1]^N$. The $i-$th coordinate s_i of s is called the *participation level* of player i in the fuzzy coalition s. Instead of $[0,1]^N$ we will also write $\mathcal{F}^N$ for the set of fuzzy coalitions. A crisp coalition $S \in 2^N$ corresponds in a canonical way with the fuzzy coalition e^S, where $e^S \in \mathcal{F}^N$ is the vector with $\left(e^S\right)_i = 1$ if $i \in S$, and $\left(e^S\right)_i = 0$ if $i \in N \setminus S$. The fuzzy coalition e^S corresponds to the situation where the players in S fully cooperate (i.e. with participation level 1) and the players outside S are not involved at all (i.e. they have participation level 0). We denote by e^i the fuzzy coalition corresponding to the crisp coalition $S = \{i\}$ (and also the $i-$th standard basis vector in $\Re^N$). The fuzzy coalition e^N is called the grand coalition, and the fuzzy coalition (the $n-$dimensional vector) $(0, 0, \ldots, 0)$ corresponds to the empty crisp coalition. We can identify the fuzzy coalitions with points in the hypercube $[0, 1]^N$ and the crisp coalitions with the $2^{|N|}$ extreme points (vertices) of this hypercube.

A *fuzzy game* with player set N is a map $v : \mathcal{F}^N \to \Re$ with the property $v(0) = 0$. The map v assigns to each fuzzy coalition a number, telling what such a coalition can achieve in cooperation. In the following the set of fuzzy games with player set N will be denoted by FG^N. Note that FG^N is an infinite dimensional linear space.

Of course, the theory of cooperative crisp games is an inspiration source for the development of the theory of cooperative fuzzy games. Here operators from $FG^N \to G^N$ and from $G^N \to FG^N$ play a role (cf. Owen, 1972; Weiß, 1998). In the following we consider only the multilinear operator $ml : G^N \to FG^N$ (Owen, 1972) and the crisp operator $cr : FG^N \to G^N$. Here for a crisp game $v \in G^N$, the multilinear extension $ml(v) \in FG^N$ is defined by

$$ml(v)(s) = \sum_{S \in 2^N \setminus \{\emptyset\}} \left(\prod_{i \in S} s_i \prod_{i \in N \setminus S} (1 - s_i) \right) v(S), \text{ for each } s \in \mathcal{F}^N.$$

For a fuzzy game $v \in FG^N$, the crisp game $cr(v) \in G^N$ is given by $cr(v)(S) = v(e^S)$ for each $S \in 2^N$.

Example 1 *For the crisp unanimity game u_T the multilinear extension is given by $ml\left(u_T\right)(s) = \prod_{i \in T} s_i$ (cf. Weiß, 1998) and $cr\left(ml\left(u_T\right)\right) = u_T$. Note that for the games $v, w \in FG^{\{1,2\}}$, where $v(s_1, s_2) = s_1 \left(s_2\right)^2$ and $w(s_1, s_2) = s_1 \sqrt{s_2}$ for each $s \in \mathcal{F}^{\{1,2\}}$, we have $cr(v) = cr(w)$.*

In general the composition $cr \circ ml : G^N \to G^N$ is the identity map on G^N. But $ml \circ cr : FG^N \to FG^N$ is not the identity map on FG^N if $|N| \geq 2$.

The *core* of a fuzzy game v (Aubin, 1974) is defined by

$$C(v) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{F}^N \right\}.$$

So $x \in C(v)$ can be seen as a distribution of the value of the grand coalition e^N, where for each fuzzy coalition s, the total payoff is not smaller than $v(s)$, if each player $i \in N$ with participation level s_i is paid $s_i x_i$.

Remark 1 *The core $C(cr(v))$ of the crisp game corresponding to v includes $C(v)$: $C(v) \subset C(cr(v))$. Later we will see that for convex fuzzy games the two cores coincide.*

Clearly, the core $C(v)$ of a fuzzy game v is a closed convex subset of $\Re^N$ for each $v \in FG^N$. Of course, the core may be empty as Example 3 shows or can consist of one point as in Example 2.

Example 2 *Consider the fuzzy three−person game v with $v(s_1, s_2, s_3) = \min\{s_1 + s_2, s_3\}$ for each $s = (s_1, s_2, s_3) \in \mathcal{F}^{\{1,2,3\}}$. One can think of a situation where players 1, 2, 3 have one unit of goods A, A and B, respectively, where A and B are complementary goods, and where combining a fraction α of a unit of A and of B leads to a gain α. Then in the grand coalition good B is scarce which is reflected in the fact that the core consists of one point $(0, 0, 1)$, corresponding to the situation where all gains go to player 3 who possesses the scarce good.*

Example 3 *Take the two−person unanimity fuzzy game $u_{(\frac{1}{2},\frac{1}{2})}$ where the value is 1 for all coalitions where the participation levels are at least $\frac{1}{2}$ and 0* *otherwise:* $u_{\left(\frac{1}{2},\frac{1}{2}\right)} = 1$ *if* $s_1 \geq \frac{1}{2}$, $s_2 \geq \frac{1}{2}$ *and* $u_{\left(\frac{1}{2},\frac{1}{2}\right)} = 0$ *otherwise. The core* $C\left(u_{\left(\frac{1}{2}\cdot\frac{1}{2}\right)}\right)$ *is empty because for a core element* x *it should hold* $x_1 + x_2 = u_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(e^{\{1,2\}}\right) = 1$ *and also* $\frac{1}{2}x_1+\frac{1}{2}x_2 \geq u_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{1}{2},\frac{1}{2}\right) = 1$, *which is impossible.*

Let us now introduce for a fuzzy game v the *marginal vectors* $m^\sigma(v)$ for each $\sigma \in \Pi(N)$, the *fuzzy Shapley value* $\phi(v)$ and the *fuzzy Weber set* $W(v)$ as follows:

(i) $m^\sigma(v) = m^\sigma(cr(v))$ for each $\sigma \in \Pi(N)$;

(ii) $\phi(v) = \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} m^\sigma(v)$;

(iii) $W(v) = conv\left\{m^\sigma(v) \mid \sigma \in \Pi(N)\right\}$.

Note that $\phi(v) = \phi(cr(v))$, $W(v) = W(cr(v))$. Note further that for $i = \sigma(k)$, the $i-$th coordinate $m_i^\sigma(v)$ of the marginal vector is given by

$$m_i^\sigma(v) = v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right).$$

One can identify a $\sigma \in \Pi(N)$ with an $n-$step walk along the edges of the hypercube of fuzzy coalitions starting in 0 and ending in e^N by passing the vertices $e^{\sigma(1)}, e^{\sigma(1)} + e^{\sigma(2)}, \ldots, \sum_{r=1}^{n-1} e^{\sigma(r)}$. The vector $m^\sigma(v)$ records the changes in value from vertex to vertex.

The result of Weber (1988) that the core of a crisp game is included in the Weber set of the game can be extended for fuzzy games as we see in

Proposition 1 *Let* $v \in FG^N$*. Then* $C(v) \subset W(v)$*.*

Proof. According to Remark 1 we have $C(v) \subset C(cr(v))$. Weber proved that $C(cr(v)) \subset W(cr(v))$. Since $W(cr(v)) = W(v)$ we obtain $C(v) \subset W(v)$. ■

Inspired by Owen (1972) one can define the *diagonal value* $\delta(v)$ for a C^1-fuzzy game (i.e. a game v which is differentiable with continuous derivatives) as follows: for each $i \in N$ the $i-$th coordinate of $\delta(v)$ is given by

$$\delta_i(v) = \int_0^1 D_i v(t, t, \dots, t) dt,$$

where D_i is the partial derivative of v with respect to the $i-$th coordinate. Owen (1972) proved that for each crisp game $v \in G^N$:

$$\phi_i(v) = \delta_i(ml(v)) \text{ for each } i \in N.$$

The next example shows that for a fuzzy game v, $\delta(v)$ and $\phi(cr(v))$ may differ.

Example 4 *Let* $v \in FG^{\{1,2\}}$ *with* $v(s_1, s_2) = s_1(s_2)^2$ *for* $(s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. *Then*

$$m^{(1,2)} = (v(1,0) - v(0,0), v(1,1) - v(1,0)) = (0,1),$$

$$m^{(2,1)} = (v(1,1) - v(0,1), v(0,1) - v(0,0)) = (1,0),$$

so

$$\phi(v) = \frac{1}{2}((0,1) + (1,0)) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Further

$$D_1 v(s_1, s_2) = (s_2)^2,\ D_2 v(s_1, s_2) = 2 s_1 s_2;$$

so

$$\delta_1(v) = \int_0^1 t^2 dt = \frac{1}{3},\ \delta_2(v) = \int_0^1 2t^2 dt = \frac{2}{3}.$$

Hence

$$\delta(v) = \left(\frac{1}{3}, \frac{2}{3}\right) \neq \left(\frac{1}{2}, \frac{1}{2}\right) = \phi(v).$$

4 The cone of convex fuzzy games

Let N be a finite set and let $v : [0,1]^N \to \Re$ be a real−valued function on $[0,1]^N$. Then

(i) v is called a *supermodular function on* $[0,1]^N$ if

$$v(s \vee t) + v(s \wedge t) \geq v(s) + v(t) \text{ for all } s, t \in [0,1]^N, \tag{5}$$

where $s \vee t$ and $s \wedge t$ are those elements of $[0,1]^N$ with the $i-$th coordinate equal to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$, respectively;

(ii) v is called a *coordinate−wise convex* function if for each $i \in N$ and each $s^{-i} \in [0,1]^{N \setminus \{i\}}$ the function $g_{s^{-i}} : [0,1] \to \Re$ with $g_{s^{-i}}(t) = v(s^{-i} \parallel t)$ is a convex function. Here $(s^{-i} \parallel t)$ is the element in $[0,1]^N$ with $(s^{-i} \parallel t)_j = s_j$ for each $j \in N \setminus \{i\}$ and $(s^{-i} \parallel t)_i = t$.

Now we are ready for

Definition 1 *Let $v \in FG^N$. Then v is called a* ***convex fuzzy game*** *if the function $v : [0,1]^N \to \Re$ is a supermodular and a coordinate-wise convex function on $[0,1]^N$.*

Remark 2 *Convex fuzzy games form a convex cone.*

Remark 3 *For a weaker definition of a convex fuzzy game see Tsurumi et al. (2001), who use only the supermodularity property.*

Some properties of convex fuzzy games are given in the next propositions.

Proposition 2 *Suppose $v \in FG^N$ is a convex game. Then the game $cr(v)$ is a convex crisp game.*

Proof. We will prove that $cr(v)$ satisfies the supermodularity property (1). Take $S, T \in 2^N$ and apply the supermodularity property (5) with $e^S, e^T, e^{S\cup T}, e^{S\cap T}$ in the roles of $s, t, s \vee t, s \wedge t$, respectively, and we obtain

$$cr(v)\,(S \cup T) + cr(v)(S \cap T) \geq cr(v)(S) + cr(v)(T).$$

■

The next property for convex fuzzy games is related with the increasing marginal contribution property (2) for players in crisp games. It states that a level increase of a player in a fuzzy coalition has more beneficial effect in a larger coalition than in a smaller coalition.

Proposition 3 *Let $v \in FG^N$ be a convex game. Let $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$ and let $\varepsilon \in \Re_+$ with $0 \leq \varepsilon \leq 1 - s_i^2$ for all $i \in N$. Then*

$$v\left(s^1 + \varepsilon e^i\right) - v\left(s^1\right) \leq v\left(s^2 + \varepsilon e^i\right) - v\left(s^2\right). \tag{6}$$

Proof. Suppose $N = \{1, 2, \ldots, n\}$. Define the fuzzy coalitions $c^0, c^1, c^2, \ldots, c^n$ by $c^0 = s^1$, and $c^k = c^{k-1} + (s_k^2 - s_k^1)\, e^k$ for $k \in \{1, 2, \ldots, n\}$. Then $c^n = s^2$. To prove (6) it is sufficient to show that for each $k \in \{1, 2, \ldots, n\}$ the inequality $\left(I^k\right)$ holds

$$v\left(c^k + \varepsilon e^i\right) - v\left(c^k\right) \geq v\left(c^{k-1} + \varepsilon e^i\right) - v\left(c^{k-1}\right). \tag{I^k}$$

Note that (I^i) follows from the coordinate−wise convexity of v and $\left(I^k\right)$ for $k \neq i$, from the supermodularity property (5) with $c^{k-1} + \varepsilon e^i$ in the role of s and c^k in the role of t. Then $s \vee t = c^k + \varepsilon e^i$, $s \wedge t = c^{k-1}$. ■

Also an analogue of the increasing marginal contribution property for coalitions (3) holds as we see in

Proposition 4 *Let $v \in FG^N$ be a convex game. Let $s, t \in \mathcal{F}^N$ and $z \in \Re_+^N$ such that $s \leq t \leq t + z \leq e^N$. Then*

$$v(s+z) - v(s) \leq v(t+z) - v(t). \tag{7}$$

Proof. For each $k \in \{1, 2, \ldots, n\}$ it follows from Proposition 3 (with $s + \sum_{r=1}^{k-1} z_r e^r$ in the role of s^1, $t + \sum_{r=1}^{k-1} z_r e^r$ in the role of s^2, k in the role of i, and z_k in the role of ε) that

$$v\left(s + \sum_{r=1}^{k} z_r e^r\right) - v\left(s + \sum_{r=1}^{k-1} z_r e^r\right) \leq v\left(t + \sum_{r=1}^{k} z_r e^r\right) - v\left(t + \sum_{r=1}^{k-1} z_r e^r\right).$$

Adding these n inequalities yields the inequality (7). ■

Important is the following proposition.

Proposition 5 *Let $v \in FG^N$ be a convex game. Let $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$ and let $\varepsilon_1, \varepsilon_2 \in \Re_{++}$ with $s_i^1 + \varepsilon_1 \leq s_i^2 + \varepsilon_2 \leq 1$ for each $i \in N$. Then*

$$\varepsilon_1^{-1}\left(v\left(s^1 + \varepsilon_1 e^i\right) - v\left(s^1\right)\right) \leq \varepsilon_2^{-1}\left(v\left(s^2 + \varepsilon_2 e^i\right) - v\left(s^2\right)\right). \tag{8}$$

Proof. From Proposition 3 (with s^1, $(s^2 + (s_i^1 - s_i^2)\, e^i)$ and ε_1 in the roles of s^1, s^2 and ε respectively) it follows that

$$\varepsilon_1^{-1}\left(v\left(s^2 + \left(s_i^1 - s_i^2 + \varepsilon_1\right) e^i\right) - v\left(s^2 + \left(s_i^1 - s_i^2\right) e^i\right)\right) \geq$$

$$\varepsilon_1^{-1}\left(v\left(s^1 + \varepsilon_1 e^i\right) - v\left(s^1\right)\right).$$

Further, from the coordinate$-$wise convexity (by noting that $s_i^2 + \varepsilon_2 \geq s_i^2 + (s_i^1 - s_i^2 + \varepsilon_1)$, $s_i^2 \geq s_i^2 + (s_i^1 - s_i^2)$) it follows that

$$\varepsilon_2^{-1}\left(v\left(s^2 + \varepsilon_2 e^i\right) - v\left(s^2\right)\right) \geq$$

$$\varepsilon_1^{-1}\left(v\left(s^2 + \left(s_i^1 - s_i^2 + \varepsilon_1\right) e^i\right) - v\left(s^2 + \left(s_i^1 - s_i^2\right) e^i\right)\right),$$

resulting in the desired inequality. ■

We will call inequality (8) in Proposition 5 the *increasing average marginal return property* (IAMR−property). It expresses the fact that for a convex game an increase in participation level of any player in a smaller coalition yields per unit of level less than an increase in a bigger coalition under the condition that the reached level of participation in the first case is still not bigger than the reached level in the second case. The IAMR−property turns out to be crucial for convex fuzzy games as we see in Theorem 6.

Theorem 6 *Let $v \in FG^N$. Then the following assertions are equivalent:*
(i) v is a convex game;
(ii) v satisfies the increasing average marginal return property (IAMR−property).

Proof. We know from Proposition 5 that a convex game satisfies the IAMR−property. On the other hand it is clear that the IAMR−property implies the coordinate−wise convexity property. Hence, we only have to prove that the IAMR−property implies the supermodularity property. So, given $s, t \in \mathcal{F}^N$ we have to prove that the supermodularity inequality (5) holds.

Let $P = \{i \in N \mid t_i < s_i\}$. If $P = \emptyset$, then (5) follows from the fact that $s \vee t = t, s \wedge t = s$. For $P \neq \emptyset$, arrange the elements of P in a sequence $\sigma(1), \sigma(2), \ldots, \sigma(p)$, where $p = |P|$, and put $\varepsilon_{\sigma(k)} = s_{\sigma(k)} - t_{\sigma(k)} > 0$ for $k \in \{1, 2, \ldots, p\}$. Note that in this case

$$s = s \wedge t + \sum_{k=1}^{p} \varepsilon_{\sigma(k)} e^{\sigma(k)}, \quad s \vee t = t + \sum_{k=1}^{p} \varepsilon_{\sigma(k)} e^{\sigma(k)}.$$

Hence,

$$v(s) - v(s \wedge t) = \sum_{r=1}^{p} \left(v\left(s \wedge t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) - v\left(s \wedge t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) \right),$$

$$v(s \vee t) - v(t) = \sum_{r=1}^{p} \left(v\left(t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right) - v\left(t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right)\right).$$

From these equalities the supermodularity inequality (5) follows because the IAMR$-$property implies for each $r \in \{1, 2, \ldots, p\}$:

$$v\left(s \wedge t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right) - v\left(s \wedge t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right) \leq$$

$$v\left(t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right) - v\left(t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right).$$

■

As we see in the following theorem the stable marginal vector property (4) also holds for convex fuzzy games and the Weber set coincides with the core. So the core is large; moreover it coincides with the core of the corresponding crisp game.

Theorem 7 *Let $v \in FG^N$ be a convex game. Then*
(i) $m^\sigma(v) \in C(v)$ for each $\sigma \in \Pi(N)$;
(ii) $C(v) = W(v)$;
(iii) $C(v) = C(cr(v))$.

Proof. (i) For each $\sigma \in \Pi(N)$ we have $\sum_{i \in N} m_i^\sigma(v) = v\left(e^N\right)$. Further for each $\sigma \in \Pi(N)$ and $s \in \mathcal{F}^N$

$$\sum_{i \in N} s_i m_i^\sigma(v) = \sum_{k=1}^{n} s_{\sigma(k)} m_{\sigma(k)}^\sigma(v) =$$

$$\sum_{k=1}^{n} s_{\sigma(k)} \left(v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right)\right) \geq$$

$$\sum_{k=1}^{n} \left(v\left(\sum_{r=1}^{k} s_{\sigma(r)} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} s_{\sigma(r)} e^{\sigma(r)}\right)\right) =$$

$$v\left(\sum_{r=1}^{n} s_{\sigma(r)} e^{\sigma(r)}\right) = v(s),$$

where the inequality follows by applying n times Proposition 5. Hence $m^\sigma(v) \in C(v)$ for each $\sigma \in \Pi(N)$.

(ii) From assertion (i) and the convexity of the core we obtain $W(v) = conv\,\{m^\sigma(v) \mid \sigma \in \Pi(N)\} \subset C(v)$. The reverse inclusion follows from Proposition 1.

(iii) Since $cr(v)$ is a convex crisp game by Proposition 2, we have $C(cr(v)) = W(cr(v))$, and $W(cr(v)) = W(v) = C(v)$ by (ii). ■

It follows from Theorem 7 that $\phi(v)$ has a central position in the core if v is a convex fuzzy game. For crisp games it holds that a game v is convex if and only if $C(v) = W(v)$ (Ichiishi, 1981). For fuzzy games the implication is only in one direction. Example 5 gives a game which is not convex and where the core and the Weber set coincide.

Example 5 *Let v be the two–person fuzzy game with $v(s_1, s_2) = s_1 s_2$ if $(s_1, s_2) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. Then v is not a convex game, but $C(v) = W(v) = conv\,\{(0,1), (1,0)\}$.*

Example 6 *Let us consider one–person fuzzy games $v : [0,1] \to \Re$. Then v is a convex game iff v is a convex function. Further v has a non–empty core $C(v) = \{v(1)\}$ iff $v(s) \leq sv(1)$ for each $s \in [0,1]$, and always $C(v) = W(v)$ if $C(v) \neq \emptyset$.*

Example 7 *(A public good game) Suppose n players want to create a facility for joint use. The cost of the facility depends on the sum of the participation levels of the players and it is described by $k\left(\sum_{i=1}^{n} s_i\right)$, where k is a continuous monotonic increasing function on $[0,n]$, with $k(0) = 0$, and where $s_1, s_2, \ldots, s_n \in [0,1]$ are the participation levels of the players. The gain of* *a player i with participation level s_i is given by $g_i(s_i)$, where $g_i : [0,1] \to \Re$ is a continuous monotone increasing function with $g_i(0) = 0$. This situation leads to a fuzzy game $v \in FG^N$ where $v(s) = \sum_{i=1}^{n} g_i(s_i) - k\left(\sum_{i=1}^{n} s_i\right)$ for each $s \in \mathcal{F}^N$. In case the functions $g_1, g_2, \ldots, g_n$ and $-k$ are convex the fuzzy game v is a convex game.*

For fuzzy games the core is a superadditive solution, i.e.

$$C(v+w) \supset C(v) + C(w) \text{ for all } v, w \in FG^N$$

and the fuzzy games with a non−empty core form a cone.

For convex fuzzy games the core turns out to be an additive correspondence as we see in

Proposition 8 *The core of a convex fuzzy game and the fuzzy Shapley value are additive solutions.*

Proof. Let v, w be convex fuzzy games. Then

$$C(v+w) = C(cr(v+w)) =$$

$$C(cr(v) + cr(w)) = C(cr(v)) + C(cr(w)) = C(v) + C(w),$$

where the first equality follows from Theorem 7 (*iii*) and the third equality follows from the additivity of the core for convex crisp games (cf. Brânzei and Tijs, 2001). Further from $\phi(v) = \phi(cr(v))$ and the additivity of the Shapley value for convex crisp games it follows that $\phi(v+w) = \phi(v) + \phi(w)$. ■

Now we define fuzzy unanimity games and study some properties of these games. In the theory of cooperative crisp games unanimity games play an important role. Crisp unanimity games are all convex and have therefore a non−empty core.

For $t \in \mathcal{F}^N$, we denote by u_t the simple fuzzy game defined by $u_t(s) = 1$ if $s \geq t$ and $u_t(s) = 0$ otherwise. We call this game the *unanimity game based on* t : a fuzzy coalition s is winning if the participation levels of s exceed weakly the corresponding participation levels of t; otherwise the coalition is losing, i.e. has value zero.

Note that for the unanimity game u_t, the corresponding crisp game $cr(u_t)$ is equal to u_T, where u_T is the crisp unanimity game based on $T = supp(t) = \{i \in N \mid t_i > 0\}$.

Conversely, $ml(u_T)$ is for no $T \in 2^N \diagdown \{\emptyset\}$ a fuzzy unanimity game because $ml(u_T)$ has a continuum of values: $ml(u_T)(s) = \prod_{i \in T} s_i$ for each $s \in \mathcal{F}^N$.

The next proposition shows that for an unanimity game u_t the gain $1 = u_t(e^N)$ in a core element is divided among the members of t with full participation levels. Further only unanimity games u_t, where all the participation levels of the players in t are 0 or 1 are convex.

Proposition 9 *Let u_t be the unanimity game based on the fuzzy coalition t. Then*

(i) The core $C(u_t)$ is non-empty iff $t_k = 1$ for some $k \in N$. In fact the core $C(u_t)$ equals conv $\left\{e^k \mid k \in N, t_k = 1\right\}$.

(ii) The game u_t is convex iff $t = e^T$ for some $T \in 2^N \setminus \{\emptyset\}$.

Proof. (*i*) If $t_k = 1$ for some $k \in N$, then $e^k \in C(u_t)$. Therefore, $conv\left\{e^k \mid k \in N, t_k = 1\right\} \subset C(u_t)$.

Conversely, $x \in C(u_t)$ implies that $\sum_{i=1}^n x_i = 1 = u_t(e^N)$, $\sum_{i=1}^n t_i x_i \geq 1 = u_t(t)$, $x_i \geq u_t(e^i) \geq 0$ for each $i \in N$. So $x \geq 0$, $\sum_{i=1}^n x_i(1-t_i) \leq 0$, which implies that $x_i(1-t_i) = 0$ for all $i \in N$. Hence $supp(x) \subset \{i \in N \mid t_i = 1\}$, and, consequently $x \in conv\left\{e^k \mid k \in N, t_k = 1\right\}$.
So $C(u_t) \subset conv\left\{e^k \mid k \in N, t_k = 1\right\}$.

(ii) Suppose $t \neq e^T$ for some $T \in 2^N \backslash \{\emptyset\}$. Then there is a $k \in N$ such that $\varepsilon = \min\{t_k, 1 - t_k\} > 0$ and $0 = u_t(t + \varepsilon e^k) - u_t(t) < u_t(t) - u_t(t - \varepsilon e^k) = 1$, implying that u_t is not convex.

Conversely, suppose that $t = e^T$ for some $T \in 2^N \setminus \{\emptyset\}$. Then we show that u_t has the supermodularity property and the coordinate−wise convexity property. Take s and k in $\mathcal{F}^N$. We can distinguish three cases.

(1) $u_t(s \vee k) + u_t(s \wedge k) = 2$. Then $u_t(s \wedge k) = 1$, so $u_t(s) + u_t(k) \geq 2u_t(s \wedge k) = 2$, $u_t(s) + u_t(k) = 2$.

(2) $u_t(s \vee k) + u_t(s \wedge k) = 0$. Then $u_t(s \vee k) = 0$, so $u_t(s) + u_t(k) \leq 2u_t(s \vee k) = 0$, $u_t(s) + u_t(k) = 0$.

(3) $u_t(s \vee k) + u_t(s \wedge k) = 1$. Then $u_t(s \vee k) = 1, u_t(s \wedge k) = 0$ and, consequently, at least one of the numbers $u_t(s)$ and $u_t(k)$ equals 0. So $u_t(s) + u_t(k) \leq 1$.

Hence, the supermodularity property holds for u_{e^T}.

Secondly, to prove the coordinate−wise convexity of u_{e^T}, note that all functions $g_{s^{-i}}$ in the definition of coordinate−wise convexity are convex because they are either constant with value 0 or with value 1, or they have value 0 on $[0, 1)$ and value 1 in 1. So u_{e^T} is a convex game. $\blacksquare$

5 Participation monotonic allocation schemes

Inspired by Sprumont (1990) (see also Hokari, 2000), who considers the interesting notion of population monotonic allocation scheme (pmas) for cooperative crisp games, we introduce here for fuzzy games the notion of participation monotonic allocation scheme (pamas). In a pmas for the crisp game and for each crisp subgame there is given a core element and the core elements are related via a monotonicity condition. To be more precise, a pmas for a crisp game $v : 2^N \to \Re$ is an allocation scheme $[a_{S,i}]_{S\in 2^N\setminus\{\emptyset\},i\in S}$ such that:

(i) $(a_{S,i})_{i\in S} \in C(v_S)$ for each $S \in 2^N \setminus \{\emptyset\}$, where v_S is the subgame corresponding to S, i.e. $v_S : 2^S \to \Re$ is the restriction of $v : 2^N \to \Re$ to 2^S;

(ii) $a_{S,i} \leq a_{T,i}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$.

In our approach the role of subgames of a crisp game will be taken over by restricted games of a fuzzy game.

Definition 2 *Let $v \in FG^N$ and $t \in \mathcal{F}^N$. Then the $t-$**restricted game** of v is the game $v_t : \mathcal{F}^N \to \Re$ given by $v_t(s) = (t * s)$ for all $s \in \mathcal{F}^N$. Here $t * s$ is the coordinate-wise product of t and s, so $(t * s)_i = t_i s_i$.*

Remark 4 *When $t = e^T$ then $v_t(s) = v(e^T * s) = v(\sum_{i\in T} s_i e^i)$ for each $s \in \mathcal{F}^N$, and for $s = e^S$ we obtain $v_t(e^S) = v(e^{S\cap T})$. This implies that the restriction of $cr(v_{e^T}) : 2^N \to \Re$ to 2^T is the subgame of $cr(v)$ on the player set T. Moreover, in v_{e^T} each player $i \in N \setminus T$ is a zero player, i.e. $v(s + \varepsilon e^i) = v(s)$ for all $s \in \mathcal{F}^N$ and for all $\varepsilon \in [0, 1 - s_i]$.*

Remark 5 *Note that for each core element $x \in C(v_t)$ we have $x_i = 0$ for each $i \notin supp(t)$. This follows from*

$$0 = v(0) = v_t(e^i) \leq x_i = \sum_{k\in N} x_k - \sum_{k\in N\setminus\{i\}} x_k \leq v_t(e^N) - v_t(e^{N\setminus\{i\}}) = 0,$$

where we use that $i \notin supp(t)$ in the second and last equality, and that $x \in C(v_t)$ in the two inequalities.

Remark 6 *If $v \in FG^N$ is a convex game, then also v_t is a convex game for each $t \in \mathcal{F}^N$. This is the fuzzy analogue of the fact that subgames of crisp convex games are convex.*

Definition 3 *Let $v \in FG^N$. A scheme $[a_{t,i}]_{t \in \mathcal{F}^N, i \in N}$ is called a* ***participation monotonic allocation scheme (pamas)*** *if*
(i) $(a_{t,i})_{i \in N} \in C(v_t)$ for each $t \in \mathcal{F}^N$ (stability condition);
(ii) $t_i^{-1} a_{t,i} \geq s_i^{-1} a_{s,i}$ for each $s, t \in \mathcal{F}^N$ with $s \leq t$ and each $i \in supp(s)$ (participation monotonicity condition).

Remark 7 *Note that such a pamas is an $\infty \times n-$matrix, where the columns correspond to the players and the rows to the fuzzy coalitions. In each row t there is a core element of the game v_t. The participation monotonicity condition implies that, if the scheme is used as regulator for the payoff distributions in the restricted fuzzy games, players are paid per unit of participation more in larger coalitions than in smaller coalitions.*

Remark 8 *Note that the collection of participation monotonic allocation schemes of a fuzzy game v is a convex set of $\infty \times n-$matrices.*

Remark 9 *In Tsurumi et al. (2001) inspired by Sprumont (1990), the notion of fuzzy population monotonic allocation scheme (FPMAS) is introduced. The relation between such a scheme and core elements is not studied there.*

Remark 10 *A necessary condition for the existence of a pamas for v is the existence of core elements for v_t for each $t \in \mathcal{F}^N$. But this is not sufficient as Example 8 shows. A sufficient condition is the convexity of a game as we see in Theorem 10.*

Example 8 *Consider the game $v \in FG^N$, where $N = \{1, 2, 3, 4\}$ and $v(s) = \min\{s_1 + s_2, s_3 + s_4\}$. Suppose for a moment that $[a_{t,i}]_{t \in \mathcal{F}^N, i \in N}$ is a pamas. Then for $t^1 = e^{N \setminus \{2\}}$, $t^2 = e^{N \setminus \{1\}}$, $t^3 = e^{N \setminus \{4\}}$, and $t^4 = e^{N \setminus \{3\}}$ we have $C(v_{t^k}) = \{e^k\}$ (see Example 2), and so $(a_{t^k,i})_{i \in N} = e^k$ for $k \in N$. But* *then* $\sum_{k\in N} a_{e^N,k} \geq \sum_{k\in N} a_{t^k,k} = 4 > 2 = v\left(e^N\right)$*, and this implies that there does not exist a pamas. Note that* $C(v_t) \neq \emptyset$ *for each* $t \in \mathcal{F}^N$*, because* $(t_1, t_2, 0, 0) \in C(v_t)$ *if* $t_1 + t_2 \leq t_3 + t_4$*; and* $(0, 0, t_3, t_4) \in C(v_t)$ *otherwise.*

Definition 4 *Let* $v \in FG^N$ *and* $x \in C(v)$*. Then we call* x ***pamas−extendable*** *if there exists a pamas* $[a_{t,i}]_{t\in\mathcal{F}^N, i\in N}$ *such that* $a_{e^N,i} = x_i$ *for each* $i \in N$*.*

In the next theorem we see that convex games have a pamas. Moreover, each core element is pamas−extendable.

Theorem 10 *Let* $v \in FG^N$ *be a convex game and let* $x \in C(v)$*. Then* x *is pamas− extendable.*

Proof. We know from Theorem 7 that x is in the convex hull of the marginal vectors $m^\sigma(v)$ with $\sigma \in \Pi^N$. In view of Remark 7 we only need to prove that each marginal vector $m^\sigma(v)$ is pamas-extendable, because then the right convex combination of these pamas extensions gives a pamas extension of x.

So take $\sigma \in \Pi^N$ and define $[a_{t,i}]_{t\in\mathcal{F}^N, i\in N}$ by $a_{t,i} = m_i^\sigma(v_t)$ for each $t \in \mathcal{F}^N$, $i \in N$. We claim that this scheme is a pamas extension of $m^\sigma(v)$.

Clearly, $a_{e^N,i} = m_i^\sigma(v)$ for each $i \in N$ since $v\left(e^N\right) = v$. Further, by Remark 5, each $t-$restricted game v_t is a convex game, and from Theorem 7 it follows that $(a_{t,i})_{i\in N} \in C(v_t)$. Hence the scheme satisfies the stability condition.

To prove the participation monotonicity condition, take $s, t \in \mathcal{F}^N$ with $s \leq t$ and $i \in supp(s)$ and let k be the element in N such that $i = \sigma(k)$. We have to prove that $t_i^{-1} a_{t,i} \geq s_i^{-1} a_{s,i}$. Now

$$t_i^{-1} a_{t,i} = t_{\sigma(k)}^{-1} m_{\sigma(k)}^\sigma(v_t) =$$

$$t_{\sigma(k)}^{-1}\left(v\left(\sum_{r=1}^{k} t_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} t_{\sigma(r)}e^{\sigma(r)}\right)\right) \geq$$
$$s_{\sigma(k)}^{-1}\left(v\left(\sum_{r=1}^{k} s_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right) =$$
$$s_{\sigma(k)}^{-1} m_{\sigma(k)}^{\sigma}(v_s) = s_i^{-1} a_{s,i},$$

where the inequality follows from the convexity of v (Proposition 5). So $[a_{t,i}]_{t\in\mathcal{F}^N, i\in N}$ is a pamas extension of $m^{\sigma}(v)$. ■

Further, the *total fuzzy Shapley value* of a convex game $v \in FG^N$, which is the scheme $\left[\phi_{t,i}\right]_{t\in\mathcal{F}^N, i\in N}$ with the fuzzy Shapley value of the restricted game v_t in each row t, is a pamas. For a study of a Shapley function in relation with FPMAS we refer to Tsurumi et al. (2001).

Example 9 *Let $v \in FG^{\{1,2\}}$ be given by $v(s_1, s_2) = 4s_1(s_1 - 2) + 10(s_2)^2$. Then v is convex and $m^{(1,2)}(v) = m^{(2,1)}(v) = \phi(v) = (-4, 10)$ because in fact v is additive: $v(s_1, s_2) = v(s_1, 0)+v(0, s_2)$. For each $t \in \mathcal{F}^N$ the fuzzy Shapley value $\phi(v_t)$ equals $(4t_1(t_1 - 2), 10(t_2)^2)$, and the scheme $[a_{t,i}]_{t\in\mathcal{F}^N, i\in\{1,2\}}$ with $a_{t,1} = 4t_1(t_1 - 2)$, $a_{t,2} = 10(t_2)^2$ is a pamas extension of $\phi(v)$, with the fuzzy Shapley value of v_t in each row t of the scheme, so $[a_{t,i}]_{t\in\mathcal{F}^N, i\in\{1,2\}}$ is the total fuzzy Shapley value of v.*

6 Concluding Remarks

Game theoretic approaches to cooperative situations in fuzzy environments have given rise to several types of cooperative fuzzy games. We mention here games with fuzzy coalitions and games with fuzzy coalition values. For a survey see Nishizaki and Sakawa (2001). Our study concerns cooperative games with fuzzy coalitions. We study in this paper the cone of convex fuzzy games that lies in the cone of cooperative fuzzy games with a participation monotonic allocation scheme. Convex fuzzy games have an interesting large core, where each element is pamas−extendable, and where the fuzzy Shapley values of the game and its restricted games form a pamas. In the theory of cooperative crisp games this pamas corresponds to the population monotonic allocation rule, known as the total Shapley value. For convex crisp games there is another interesting population monotonic allocation rule, namely the pmas-extension of the egalitarian rule of Dutta and Ray (Hokari, 2000). It would be interesting to find such an egalitarian rule for convex fuzzy games, too. Another goal for further research could be to find a subclass of convex fuzzy games, where the diagonal values of the game and its restricted games lead to a pamas.

Also other solutions could be developed and/or studied for convex fuzzy games, e.g. those corresponding to solutions such as: stable set, kernel and bargaining set for cooperative crisp games.

To find variants of convex fuzzy games when other types of fuzziness are considered, seems to be another interesting direction for future research.

References

[1] J.P. Aubin, Coeur et valeur des jeux flous à paiements latéraux, C.R. Acad. Sci. Paris 279 A (1974) 891-894.

[2] J.P. Aubin, Cooperative fuzzy games, Math. Oper. Res. 6 (1981) 1-13.

[3] R. Brânzei, S. Tijs, Additivity regions for solutions in cooperative game theory, Libertas Mathematica XXI (2001) 155-167.

[4] D. Butnariu, Fuzzy games: a description of the concept, Fuzzy Sets and Systems 1 (1978) 181-192.

[5] D. Butnariu, E.P. Klement, Triangular Norm-Based Measures and Games with Fuzzy Coalitions, Kluwer Academic Publishers, Dordrecht, 1993.

[6] I. Curiel, Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems, Kluwer Academic Publishers, Dordrecht, 1997.

[7] I. Curiel, M. Maschler, S. Tijs, Bankruptcy games, Zeitschrift Oper. Res. 31 (1988) 143-159.

[8] I. Curiel, G. Pederzoli, S. Tijs, Sequencing games, Europ. J. Oper. Res. 40 (1989) 344-351.

[9] D.B. Gillies, Some theorems on $n-$person games, Ph.D. Thesis, Princeton University Press, Princeton, New Jersey, 1953.

[10] T. Hokari, Population monotonic solutions for convex games, Int. J. Game Theory 29 (2000) 327-338.

[11] T. Ichiishi, Super-modularity: Applications to convex games and to the greedy algorithm for LP, J. Econ. Theory 25 (1981) 283-286.

[12] S.C. Littlechild, G. Owen, A simple expresion for the Shapley value in a special case, Manag. Sci. 20 (1973) 370-372.

[13] E. Molina, J. Tejada, The equalizer and the lexicographical solutions for cooperative fuzzy games: characterizations and properties, Fuzzy Sets and Systems 125 (2002) 369-387.

[14] J.von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Wiley, New York, 1944.

[15] I. Nishizaki, M. Sakawa, Fuzzy and Multiobjective Games for Conflict Resolution, Physica-Verlag, Heidelberg, 2001.

[16] G. Owen, Multilinear extensions of games, Manag. Sci. 18 (1972) 64-79.

[17] M. Sakawa, I. Nishizaki, A lexicographical concept in an $n-$person cooperative fuzzy game, Fuzzy Sets and Systems 61 (1994) 265-275.

[18] L.S. Shapley, A value for $n-$person games, Ann. Math. Stud. 28 (1953) 307-317.

[19] L.S. Shapley, Cores of convex games, Int. J. Game Theory 1 (1971) 11-26.

[20] Y. Sprumont, Population monotonic allocation schemes for cooperative games with transferable utility, Games Econ. Behavior 2 (1990) 378-394.

[21] M. Tsurumi, T. Tanino, M. Inuiguchi, A Shapley function on a class of cooperative fuzzy games, Europ. J. Oper. Res. 129 (2001) 569-618.

[22] R. Weber, Probabilistic values for games, in: A.E. Roth (Ed.), The Shapley Value: Essays in Honour of LS Shapley, Cambridge University Press, Cambridge, 1988.

[23] C. Weiß, Die Erweiterung nach Choquet und der Shapley-Wert auf Fuzzy-Spielen, Master Thesis, University of Bielefeld, 1998.