Strategic Investment Under Uncertainty and Information Spillovers

Thijssen, J.J.J.; Huisman, K.J.M.; Kort, P.M.

Publication date:
2001

Link to publication

Citation for published version (APA):
Thijssen, J. J. J., Huisman, K. J. M., & Kort, P. M. (2001). *Strategic Investment Under Uncertainty and Information Spillovers*. (CentER Discussion Paper; Vol. 2001-91). Tilburg: Microeconomics.

No. 2001-91

STRATEGIC INVESTMENT UNDER UNCERTAINTY AND INFORMATION SPILLOVERS

By Jacco J.J. Thijssen, Kuno J.M. Huisman and Peter M. Kort

November 2001

ISSN 0924-7815

Discussion paper

Strategic Investment under Uncertainty and Information Spillovers *

Jacco J.J. Thijssen[†] Kuno J.M. Huisman[‡] Peter M. Kort[§]

August 14, 2001

Abstract

In this paper a new market model is considered where two firms compete in investing in a risky project. The model incorporates a Stackelberg advantage for the first mover and information spillovers that may constitute a second mover advantage. At certain points in time the firms obtain information about the profitability of the project. The threshold beliefs in a profitable project for which investment is optimal are calculated. It is shown that both a preemption game as well as a war of attrition can arise for specific parametrizations of the model, depending on the levels of the first and second mover advantages. Furthermore, it is shown that more competition does not necessarily lead to higher social welfare.

Keywords: Market uncertainty, Strategic investment, real options, Welfare.

JEL codes: C61, D43, D81.

*We thank Eric van Damme for inspiring this paper and Jan Boone for some useful comments. Mathematical assistance by Miranda van Uitert is gratefully acknowledged. The usual disclaimer applies.

[†]Corresponding author. Department of Econometrics & Operations Research and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: `J.J.J.Thijssen@kub.nl`.

[‡]Centre for Quantitative Methods, Eindhoven, The Netherlands.

[§]Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, The Netherlands.

1 Introduction

Two main forces that influence a firm's investment decision are uncertainty about the profitability of the investment project, and the behaviour of potential competitors, that have an option to invest in the same project. Most of the literature on optimal investment deals with either aspect. The real options theory concerns itself with investment decisions under uncertainty (cf. Dixit and Pindyck (1994)). In this literature nature chooses a state of the world at each point in time, influencing the profitability of the investment project. The problem is then to find an optimal threshold level of an underlying variable (e.g. price or output value of the firm), above which the investment should be undertaken.

In the strategic interaction literature a number of models have been developed, dealing with different situations such as patent races and new technology adoption. In general, a distinction can be made between two types of models. First, there are preemption games in which two firms try to preempt each other in investing (cf. Fudenberg and Tirole (1991)). The equilibrium concept used in such games was developed in Fudenberg and Tirole (1985). Another class is the war of attrition, which is first introduced by Maynard Smith (1974) in the biological literature and later adopted for economic situations (cf. Tirole (1988)). Originally, the war of attrition describes two animals fighting over a prey. In an economic context one can think of two firms considering adopting a new technology. From a certain point onwards both know that for one firm it would be optimal to invest, but however, both do not want to be the first to invest, since waiting for an even newer technology would be better. The equilibrium concept used in this type of game is introduced in Hendricks et al. (1988).

The literature combining both aspects is small indeed. Jensen (1982) was the first to introduce uncertainty in a technology adoption model. Hoppe (2000) extends this paper to a model where second mover advantages can arise in equilibrium due to information spillovers. A first attempt to combine real option theory with timing games was made in Smets (1991). Huisman (2000) provides some extensions to this approach.

In Hoppe (2000) it is assumed that an investment is either profitable or not. As soon as one firm invests, the true profitability of the project becomes known. This creates informational spillovers that yields a second mover advantage. The probability with which the project is profitable is exogenously given, fixed and common knowledge. we present a framework where the succes probability is updated over time due to information that becomes available via signals that arrive according to a Poisson process. The signal can either be good or bad: in the first case it indicates that the project is profitable, whereas in the latter case it is signalled that the project is a bad one, in which case investment yields a loss. However, the signals need not provide perfect information in the sense that with a certain probability $\lambda \in (1/2, 1]$ the signal gives the correct information. For simplicity, it is assumed that the signals can costlessly be observed. They can be thought of for example as arising from media or publicly available marketing research.

As an example of the duopoly model with signals, consider two soccer scouts who are considering to contract a player. In order to obtain information on the player's quality both scouts go to matches in which the desired player plays. If he performs well, this can be seen as a signal indicating high revenues, but if he performs poorly, this is a signal that the investment is not profitable. This induces an option value of waiting for more signals to arrive and hence getting a better approximation of the actual profitability of the project.

In this paper it is shown that, depending on the prior beliefs on the profitability of the project and the first and second mover advantages, either a preemption game or a war of attrition arises. If the information spillover exceeds the first mover Stackelberg effect, then a war of attrition may result. In the reverse case a preemption game arises. Even both types of games may occur in the same scenario. Suppose that the information spillover prevails and that the prior beliefs are such that a war of attrition arises. With positive probability it would be the case that no firm makes the investment. Over time new signals arrive that influence the belief in the profitability of the project. Then it may happen – if enough good signals arrive – that at a certain point in time the first mover advantage outweighs the information spillover, hence inducing a preemption game. As soon as the game becomes a preemption game, in equilibrium one of the firms or both firms invest.

From the standard theory of industrial organization it is well-known that social welfare is not always higher in the case of competition than in the case of a monopoly. For example, Mankiw and Whinston (1986) develop a model in which competition can yield a number of firms operating in the industry that is either too high or too low from a social welfare point of view. In this paper we show that the presence of uncertainty concerning the profitability of a new market can strengthen the effect of having social welfare under competition. However, it can go either way. Three effects are at work here, the first two of which are standard arguments. Firstly, the dead-weight loss in monopoly is higher than in the case of duopoly. Secondly, the total sunk costs in duopoly are higher than in monopoly. Finally, in the duopoly case there can be a preemption effect, which induces firms to invest too soon from a social welfare perspective, since at that specific time the economic prospects are too uncertain for an investment to be undertaken optimally. In order to facilitate welfare analysis a measure for ex ante expected total surplus is introduced that incorporates the distribution of first passage times through the various critical levels for monopoly and duopoly.

The present paper is related to Dcamps and Mariotti (2000) in which also a duopoly model is considered where signals arrive over time. Differences are that in Dcamps and Mariotti (2000) only bad signals exist and that signals are perfectly informative. This means that after receiving one signal the game is over since the firms are sure that the project is not profitable (a soccer player who plays one bad game is definitely a bad player who should not be contracted), while in our framework it could still be possible that the project is good. In Dcamps and Mariotti it holds that, as long as no signal arrives, the probability that the project is good continuously increases over time. Furthermore the firms are assumed to be asymmetric, which also induces uncertainty regarding the players' types. This implies that Dcamps and Mariotti need to apply the Bayesian equilibrium concept, whereas in our model this is not the case. Another implication is that the coordination problem between the two firms that is analysed in our framework is not present in Dcamps and Mariotti (2000). This coordination problem concerns the issue of which firm will be the first to invest in the preemption equilibrium. Another duopoly paper where information arrives over time is Lambrecht and Perraudin (1999), but there the information relates to the behaviour of the competitor: one firm has a certain belief about when the other firm would invest and this belief is updated by observing the other firm's behaviour.

The paper is organized as follows. In Section 2 the model is described. Then, in Section 3 we analyse the model for the scenario that the firm roles, i.e. leader and follower, are exogenously determined. In Section 4 the exogenous firm roles are dropped and the model is analysed for the case where the firms are completely symmetrical. Section 5 reconsiders two extreme cases, namely situations where either the information spillover or the Stackelberg effect is absent. In Section 6 a welfare measure is introduced and welfare effects are discussed. Finally, in Section 7 some conclusions will be drawn.

2 The Model

The model presented in this section describes a new market situation where two symmetric firms have the opportunity to invest in a project with uncertain revenues. The first firm to invest becomes the Stackelberg leader. Its revenues can be either high, U_L^H, or low, U_L^L, where the latter is normalized to zero. A bad project is also not profitable for the follower and as soon as one firm has invested, the true state of the project is revealed. If the project turns out to be good, the other firm, the follower, can decide to invest and get revenues U_F^H. It is assumed that $U_L^H > U_F^H$. Hence, there is a first mover Stackelberg advantage if the project turns out to yield high revenues. If both firms invest simultaneously and the project turns out to be good, both receive U_M^H, where $U_F^H < U_M^H < U_L^H$. The revenues can be seen as an infinite stream of payoffs π_j^i discounted at rate r, i.e. $U_j^i = \int_0^\infty e^{-rt}\pi_j^i dt = \frac{1}{r}\pi_j^i$, $i = H, L$, $j = L, M, F$. Example 2.1 illustrates this framework.

Example 2.1 *Consider a new market for a homogeneous good. Two firms have the opportunity to enter the market, which can be either good or bad. Let market demand be given by $P(Q) = Y - Q$ for some $Y > 0$ if the market is good (H) and by $P(Q) = 0$ if the market is bad (L). The cost function is given by $C(q) = cq$, for some $c \geq 0$. It is assumed that if the firms invest they engage in Cournot competition. If the market turns out to be bad, then the action to take is not to produce in any case, i.e. $U_L^L = U_F^L = U_M^L = 0$. Suppose that there is one firm that invests in the* *market first. This firm then is the Stackelberg leader.*[1] *The follower solves the following profit maximization problem*

$$\max_{q_F} \frac{1}{r} q_F [P(q_L + q_F) - c],$$

where r *is the discount rate. This yields* $q_F = \frac{Y-c-q_L}{2}$. *Using this reaction, the leader maximizes its stream of profits. Solving the corresponding maximization problem yields* $q_L = \frac{Y-c}{2}$, *which results in* $q_F = \frac{Y-c}{4}$, *and the payoffs* $U_L^H = \frac{(Y-c)^2}{8r}$ *and* $U_F^H = \frac{(Y-c)^2}{16r}$, *respectively. In case both firms invest simultaneously, the Cournot-Nash outcome prevails. Straightforward computations yield* $U_M^H = \frac{(Y-c)^2}{9r}$. *Note that* $U_L^H > U_M^H > U_F^H$.

Investing in the project implies incurring a sunk cost I. It is assumed that $0 < I < U_F^H$, so that it is always worthwhile for the follower to enter when the market is good. But if the project is bad the follower observes this and thus refrains from investment. This implies that in case of a bad project only the leader incurs a loss that is equal to the sunk costs of investment. Hence, the presence of an information spillover leads to a second mover advantage. To see who is in the best position, the leader or the follower, this second mover advantage has to be compared with the first mover advantage of being a Stackelberg leader.

Given the belief p that the project is good, the ex ante expected payoff for the leader is given by

$$L(p) = p(U_L^H - I) + (1-p)(-I) = pU_L^H - I. \tag{1}$$

Since the follower only invests in case of a good project, the payoff for the follower is given by

$$F(p) = p(U_F^H - I). \tag{2}$$

In case of mutual investment each firm has an ex ante payoff that equals

$$M(p) = pU_M^H - I. \tag{3}$$

Define by p_M the belief such that the ex-ante expected profit for the follower equals the ex-ante expected profit of mutual investment, i.e. p_M is such that

[1]It is assumed that firms can only set capacity once, thereby fixing the production level forever. This resolves the problem mentioned in Dixit (1980).

$F(p_M) = M(p_M)$. Note that, as soon as $p \geq p_M$, the follower will always invest simultaneously with the leader, yielding payoffs

$$l(p) = \begin{cases} L(p) & \text{if } p < p_M, \\ M(p) & \text{if } p \geq p_M, \end{cases} \tag{4}$$

and

$$f(p) = \begin{cases} F(p) & \text{if } p < p_M, \\ M(p) & \text{if } p \geq p_M. \end{cases} \tag{5}$$

A graphical representation of these payoffs is given in Figure 1.

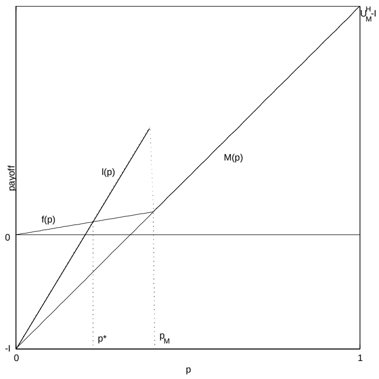

Figure 1: Payoff functions.

At the moment that the investment opportunity becomes available, both firms have an identical prior belief about the project yielding high revenues, say p_0, which is common knowledge. Occasionally, the firms obtain signals about the profitability of the project. These signals are observed by both firms. A signal can either indicate high revenues (an h-signal) or low revenues (an l-signal). A signal revealing the true state of the project occurs with probability $\lambda > \frac{1}{2}$, see Table 1. The signals arrive stochastically over time following a Poisson process with parameter $\mu > 0$.

Let n denote the number of signals and let $g \leq n$ be the number of h-signals. Given that at a certain point in time n signals have arrived, g of which were h-signals, the firms then calculate their beliefs in a good project in a Bayesian way. Note that only the relative difference between n and g is interesting for the firm. By defining $k := 2g - n$ and $\zeta := \frac{1-p_0}{p_0}$, it can be

	h	l
H	λ	$1-\lambda$
L	$1-\lambda$	λ

Table 1: Conditional probabilities of h and l-signals.

shown that the belief in a good project is a function of k and is given by[2]

$$p(k) = \frac{\lambda^k}{\lambda^k + \zeta(1-\lambda)^k}. \tag{6}$$

Note that the inverse of this function gives the number of h-signals in excess of l-signals that is needed to obtain a belief equal to p. The inverse is given by

$$k(p) = \frac{\log(\frac{p}{1-p}) + \log(\zeta)}{\log(\frac{\lambda}{1-\lambda})}. \tag{7}$$

3 Exogenous Firm Roles

Before we turn to the more interesting case where it is endogenously determined which firm invests first, we now look at the simpler case of exogenous firm roles. Suppose w.l.o.g. that only firm 1 is allowed to be the first investor. Then, firm 1 does not need to take into account the possibility that firm 2 preempts. Firm 2 can choose between the follower role and investing at the same time as the leader. In the first case firm 1 is the Stackelberg leader and in the other case a Nash equilibrium results. Firm 1 should invest at the moment that its belief in a good project exceeds a certain threshold. From Thijssen et al. (2001) it follows that this threshold belief p_L is given by

$$p_L = \frac{1}{\Psi(U_L^H/I - 1) + 1}, \tag{8}$$

where

$$\Psi = \frac{\beta(r+\mu)(r+\mu(1-\lambda)) - \mu\lambda(1-\lambda)(r+\mu(1+\beta-\lambda))}{\beta(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta+\lambda))},$$

[2]Note that k increases with one if an h-signal arrives and decreases with one if an l-signal arrives. See Thijssen et al. (2001) for a derivation.

and

$$\beta = \frac{r+\mu}{2\mu} + \tfrac{1}{2}\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}.$$

Hence, as soon as p exceeds p_L, the leader invests. Then, the follower decides whether or not to invest, based on the true state of the project that is immediately revealed after the investment by the leader. Note that p_L will not be reached exactly, since the belief $p(k)$ jumps along with the discrete variable k. Therefore, it is worthwhile to define by $\tilde{p}_L$ the belief that is reached first after exceeding p_L, i.e. $\tilde{p}_L = p(\lceil k_L \rceil)$, where $k_L = k(p_L)$.

As soon as p enters the region $(p_M, 1]$, both firms will imediately invest, yielding for both a discounted payoff stream U_M^H if the project is good, and 0 if the project is bad. Here the belief is that high that the follower prefers to receive the Nash equilibrium payoff rather than being a Stackelberg follower, implying that it takes the risk of making a loss that equals the sunk costs of investment when the project value is low.

4 Endogenous Firm Roles

Let the firm roles now be endogenous. This implies that both firms are allowed to be the first investor. Define the preemption belief, p^* to be the belief at which $L(p^*) = F(p^*)$ (cf. Figure 1). Hence, as soon as p raises beyond p^* (if ever), both firms want to be the leader and try to preempt each other. Define $\tilde{p}^* = p(\lceil k^* \rceil)$, where $k^* = k(p^*)$. In a similar way one can define $\tilde{p}_M$ to be the first level of belief that is reached after the value of mutual investment equals the follower value. Attached to $\tilde{p}_L$, $\tilde{p}_M$ and $\tilde{p}^*$ one can define the accompanying levels of k, denoted by $\tilde{k}_L$, $\tilde{k}_M$ and $\tilde{k}^*$, respectively. For the analysis an important part is played by the positioning of k_L, which can be smaller or larger than k^*. It can be shown that[3]

$$k_L > k^* \Leftrightarrow \Psi < \frac{U_L^H - U_F^H}{U_L^H - I}. \tag{9}$$

[3]Equalizing $F(p)$ and $L(p)$, yields p^*, i.e.

$$p^* = \frac{I}{U_L^H - U_F^H + I}.$$

From eq. (8) and the monotonicity of the mapping $p \mapsto \frac{\log(\frac{p}{1-p}) + \log(\zeta)}{\log(\frac{\lambda}{1-\lambda})}$, the result follows.

Note that if $k_L > k^*$ then $\tilde{k}_L \geq \tilde{k}^*$. The right-hand side of the second inequality in (9) can be seen as the relative price that the follower pays for waiting to obtain the information spillover if the market is good. Since Ψ decreases with μ and λ, Ψ increases with the value of the information spillover. For if Ψ is low, then this implies that the quality and the quantity of the signals is relatively high. Therefore, by becoming the leader for low values of Ψ, a firm provides relatively less information to its competitor than when Ψ is high. So, expression (9) implies a comparison between the value that the leader transfers to the follower, i.e. the information spillover, and the price that the follower pays to obtain this information, i.e. the Stackelberg advantage. In what follows we consider the two cases: $\tilde{k}_L \geq \tilde{k}^*$ and $\tilde{k}_L < \tilde{k}^*$.

4.1 The Case Where the Stackelberg Effect Outweighs the Information Spillover

In this case it holds that $\tilde{k}_L \geq \tilde{k}^*$. This implies that firms try to preempt each other in investing in the project. We will use the equilibrium concept introduced in Fudenberg and Tirole (1985) which is extended for the present setting involving uncertainty in Appendix A, to solve the game. The application of this equilibrium concept requires using several stopping times. Define $T^* = \inf\{t \geq 0 | p_t \geq p^*\}$ and $T_M = \inf\{t \geq 0 | p_t \geq p_M\}$, where $p_t \equiv p(k_t)$. Note that $T_M \geq T^*$ a.s. As soon as $t \geq T_M$ the value of mutual investment is higher than the value of being the second investor. This implies that no firm wants to be follower and hence that both firms will invest immediately. Note that whether or not $p_M > p_L$ is irrelevant, since if it were not the case, then no firm would be willing to wait until p_L is reached, because of the sheer fear of being preempted by the other firm.

Consider now the region (T^*, T_M). Again both firms try to preempt as soon as this region is reached. This means that in a symmetric equilibrium[4] each firm invests with a positive probability. Here both firms want to be the first investor, since the expected Stackelberg leader payoff is sufficiently large that it is optimal to take the risk that the project has a low payoff. On the other hand, if both firms invest with positive probability, the probability

[4]Since the firms are identical, a symmetric equilibrium seems to be the most plausible candidate.

that both firms simultaneously invest is also positive. This would lead to the Nash equilibrium payoff. However, since $t < T_M$ this payoff is not large enough for the investment as such to be optimal. We conclude that there is a trade-off here between the probability of being the leader on the one hand and the probability of simultaneous investment on the other hand. As is proved in Proposition 4.1 below the probability that the firms invest equals $\frac{L(p)-F(p)}{L(p)-M(p)}$. Hence, this probability increases with the Stackelberg advantage and decreases with the difference between the leader and the mutual payoff. The latter makes sense because if this difference is large the firms will try to avoid simultaneous investment by lowering their investment probability.

From Fudenberg and Tirole (1985) one knows that the game must end as soon as the preemption region is reached. This means that if no investment takes place (which happens with positive probability) the game is replayed instantly. It is assumed that this replay occurs at exactly the same time instance, so it does not take any time. The replay goes on until firm 1 or firm 2 invests.[5] Hence, exactly at the point in time where the preemption region is reached, the game ends. Again, the position of p_L is of no importance, since the leader curve lies above the follower curve, implying that both firms will try to become the leader.

The last region is the region where $t < T^*$. In this region the leader curve lies under the follower curve, and since in this case $k_L \geq k^*$, p_L has not been reached yet. Hence, no firm wants to be the leader and both firms abstain from investment until enough h-signals have arrived to make investment more attractive than waiting.

Formally, the above discussion can be summarized in the following subgame perfect equilibrium.[6]

Proposition 4.1 *If* $\Psi \leq \frac{U_L^H - U_F^H}{U_L^H - I}$*, then a subgame perfect equilibrium is given by the tuple of closed-loop strategies* $\left((G_1^t, \alpha_1^t), (G_2^t, \alpha_2^t)\right)_{t \in [0,\infty)}$*, where*

[5]See Huisman (2000) for a concise treatment of the arguments.

[6]See Appendix A for a formal definition of subgame perfect equilibrium.

for $i = 1, 2$

$$G_i^t(s) = \begin{cases} 0 & \textit{if } s < T^*, \\ 1 & \textit{if } s \geq T^*, \end{cases} \tag{10}$$

$$\alpha_i^t(s) = \begin{cases} 0 & \textit{if } s < T^*, \\ \frac{L(p_s)-F(p_s)}{L(p_s)-M(p_s)} & \textit{if } T^* \leq s < T_M, \\ 1 & \textit{if } s \geq T_M. \end{cases} \tag{11}$$

For a proof of this proposition, see Appendix B.

4.2 The Case Where the Information Spillover Outweighs the Stackelberg Effect

In this case it holds that $p_L < p^*$. Now the problem becomes somewhat different. We know that the game ends as soon as T^* is reached. Note however that before this happens p_L can be reached several times, depending on the occurrence of h and l-signals. Note that p only changes if a new signal arrives. Therefore, define the following increasing sequence of stopping times: $T_1 = \inf\{t \geq 0 | p_t > p_0\}$ and $T_{n+1} = \inf\{t > T_n | p_t > p_{T_n}\}$, $n = 1, 2, 3, \ldots$. Note that n is the number of signals that have arrived up untill and including time T_n. For $t > T^*$, the game is exactly the same as in the former case. The difference arises if t is such that $p_t \in [p_L, p^*)$. In this region it is optimal to invest for the leader had the leader role been determined exogenously. However, since the leader role is endogenous and the leader curve lies below the follower curve, both firms prefer to be the follower. In other words, a war of attrition (cf. Hendricks et al. (1988)) arises. Two asymmetric equilibria of the war of attrition arise trivially: one firm invests always with probability one and the other always with probability zero, and vice versa. However, since the firms are assumed to be identical there is no reason to expect that one of these asymmetric equilibria will actually be played. Furthermore, very strong assumptions would have to be made on the level of coordination between the two firms to reach these equilibria.

To find a symmetric equilibrium we argue in line with Fudenberg and Tirole (1991) that for a symmetric equilibrium it should hold that for each point in time, the expected revenue of investing directly exactly equals the value of waiting a small period of time dt and investing when a new signal arrives.[7] The expected value of investing at each point in time depends on the value of k at that point in time. Denoting the probability that the other firm invests at belief k by $\gamma(k)$, the expected value of investing equals

$$V_1(p_t) = \gamma(k_t)M(p_t) + (1 - \gamma(k_t))L(p_t). \tag{12}$$

The value of waiting for a small period of time equals the weighted value of becoming the follower and of both firms waiting, i.e.

$$V_2(p_t) = \gamma(k_t)F(p_t) + (1 - \gamma(k_t))\tilde{V}(p_t), \tag{13}$$

where $\tilde{V}(p)$ is the value of waiting when both firms do so. Let $\gamma(\cdot)$ be such that $V_1(\cdot) = V_2(\cdot)$. Furthermore, define

$$g(t) = \begin{cases} \gamma(k_t) & \text{if } p_L \leq p_t < p^*, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Finally, define $n_t = \sup\{n|T_n \leq t\}$ to be the number of signals that have arrived up untill time t. In the following proposition a symmetric subgame perfect equilibrium is given.

Proposition 4.2 *If $\Psi > \frac{U_L^H - U_F^H}{U_L^H - I}$, then a subgame perfect equilibrium is given by the tuple of closed-loop strategies $\big((G_1^t, \alpha_1^t), (G_2^t, \alpha_2^t)\big)_{t\in[0,\infty)}$, where for $i = 1, 2$*

$$G_i^t(s) = \begin{cases} \sum_{n=n_t}^{n_s} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_t}^{n} \big(1 - g(T_{n'})\big) & \text{if } t < T^*, \\ 1 & \text{if } s \geq T^*, \end{cases} \tag{15}$$

$$\alpha_i^t(s) = \begin{cases} 0 & \text{if } s < T^*, \\ \frac{L(p_s)-F(p_s)}{L(p_s)-M(p_s)} & \text{if } T^* \leq s < T_M, \\ 1 & \text{if } s \geq T_M. \end{cases} \tag{16}$$

The proof of Proposition 4.2 can be found in Appendix C.

To actually calculate the symmetric equilibrium, we use the fact that only for certain values of p the probability of investment needs to be calculated. These probabilities are the beliefs that result from the signals, i.e. for the beliefs p such that $p = p(k)$, $k \in \mathbb{Z}$. Therefore, from now on we will

[7] It might seem strange that a firm then also invests when a bad signal arrives. Note however that it is always optimal for one firm to invest in the war of attrition region.

analyse the model in terms of k. For notational convenience we will take k as dependent variable instead of p. For example, we write $V(k)$ instead of $V(p(k))$. To calculate the isolated atoms – the probabilities of investment – in the war of attrition, $\gamma(\cdot)$, the value of waiting $\tilde{V}(\cdot)$ needs to be determined. It is governed by the following equation:

$$\begin{aligned}\tilde{V}(k) =& e^{-rdt}\{(1-\mu dt)\tilde{V}(k) + \mu dt[p(k)(\lambda V_1(k+1) + (1-\lambda)V_1(k-1)) + \\ &+ (1-p(k))(\lambda V_1(k-1) + (1-\lambda)V_1(k+1))]\}.\end{aligned} \tag{17}$$

Eq. (17) arises from equalizing the value of $\tilde{V}(k)$ to the value a small amount of time later. In this small time interval, nothing happens with probability $1-\mu dt$. With probability μdt a signal arrives. The belief a firm has in a good project is given by $p(k)$. If the project is indeed good, an h-signal arrives with probability λ, and an l-signal arrives with probability $1-\lambda$. Vice versa if the project is bad. If a signal arrives then investing yields either $V_1(k+1)$ or $V_1(k-1)$. After letting $dt \downarrow 0$ and substituting eqs. (6) and (12) into eq. (17) it is obtained that

$$\begin{aligned}\tilde{V}(k) =& \frac{\mu}{r+\mu}\Big[\frac{\lambda^{k+1}+\zeta(1-\lambda)^{k+1}}{\lambda^k+\zeta(1-\lambda)^k}\big(\gamma(k+1)M(k+1) + (1-\gamma(k+1)) \\ & L(k+1)\big) + \lambda(1-\lambda)\frac{\lambda^{k-1}+\zeta(1-\lambda)^{k-1}}{\lambda^k+\zeta(1-\lambda)^k}\big(\gamma(k-1)M(k-1) \\ & + (1-\gamma(k-1))L(k-1)\big)\Big].\end{aligned} \tag{18}$$

Substituting eq. (18) into eq. (13) yields, after equating eqs. (13) and (12) and rearranging

$$a_k\gamma(k) + b_k = (1-\gamma(k))(c_k\gamma(k+1) + d_k\gamma(k-1) + e_k), \tag{19}$$

where

$$
\begin{aligned}
a_k =& M(k) - L(k) - F(k),\\
b_k =& L(k),\\
c_k =& \frac{\mu}{r+\mu}\frac{\lambda^{k+1}+\zeta(1-\lambda)^{k+1}}{\lambda^k+\zeta(1-\lambda)^k}\big(M(k+1)-L(k+1)\big),\\
d_k =& \frac{\mu}{r+\mu}\lambda(1-\lambda)\frac{\lambda^{k-1}+\zeta(1-\lambda)^{k-1}}{\lambda^k+\zeta(1-\lambda)^k}\big(M(k-1)-L(k-1)\big),\\
e_k =& \frac{\mu}{r+\mu}\Big(\frac{\lambda^{k+1}+\zeta(1-\lambda)^{k+1}}{\lambda^k+\zeta(1-\lambda)^k}L(k+1)+\\
& \lambda(1-\lambda)\frac{\lambda^{k-1}+\zeta(1-\lambda)^{k-1}}{\lambda^k+\zeta(1-\lambda)^k}L(k-1)\Big).
\end{aligned}
$$

To solve for $\gamma(\cdot)$ note that if $k < \tilde{k}_L$, no firm will invest, since the option value of waiting is higher than the expected revenues of investing. Therefore $\gamma(\tilde{k}_L - 1) = 0$. On the other hand, if $k \geq \tilde{k}^*$ the firms know that they enter a preemption game. Therefore, they know that the game ends at that point and a new game starts with an expected payoff equal to the follower value (cf. Huisman (2000)). Note that it is possible that $\tilde{k}^* = \tilde{k}_M$. Then the game proceeds from the war of attrition directly into the region where mutual investment is optimal. In this case the expected payoff is governed by $M(\cdot)$. Since a new subgame starts we set $\gamma(\tilde{k}^*) = 1$.[8] For other values of k, we have to solve a system of equations, where the k-th entry is given by eq. (19). The complete system is given by

$$
\mathrm{diag}(\gamma)\mathbf{A}\gamma + \mathbf{B}\gamma = \mathbf{b}, \tag{20}
$$

where $\mathrm{diag}(\cdot)$ is the diagonal operator, $\gamma = (\gamma(\tilde{k}_L - 1), \ldots, \gamma(\tilde{k}^*))$, $\mathbf{b} =$

[8]This is purely for computational convenience. According to Proposition 4.2 we have $\gamma(\tilde{k}^*)$=0.

$(0, e_{\tilde{k}_L} - b_{\tilde{k}_L}, \ldots, e_k - b_k, \ldots, e_{\tilde{k}^*-1} - b_{\tilde{k}^*-1}, 1)$,

$$\mathbf{A} = \begin{bmatrix} 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & 0 \\ d_{\tilde{k}_L} & 0 & c_{\tilde{k}_L} & 0 & \ldots\ldots\ldots & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ldots & d_k & 0 & c_k & \ldots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots\ldots & & 0 & d_{\tilde{k}^*-1} & 0 & c_{\tilde{k}^*-1} \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & 0 \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & 0 \\ -d_{\tilde{k}_L} & a_{\tilde{k}_L} + e_{\tilde{k}_L} & -c_{\tilde{k}_L} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \ldots & -d_k & a_k + e_k & -c_k & \ldots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & -d_{\tilde{k}^*-1} & a_{\tilde{k}^*-1} + e_{\tilde{k}^*-1} & -c_{\tilde{k}^*-1} \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & 0 & 1 \end{bmatrix}.$$

The system of equations (20) cannot be solved analytically. However, for any specific set of parameter values, it can be solved numerically.

Example 4.1 *As an example consider a situation whose characteristics are given in Table 2. For this example the preemption moment is given by* $\tilde{p}^* =$

$U_L^H = 13.3$	$r = 0.1$
$U_F^H = 13$	$\mu = 2$
$U_M^H = 13.2$	$\lambda = 0.7$
$I = 2$	$p_0 = 0.5$

Table 2: Parameter values.

0.87. *The minimal belief that an exogenous leader needs to invest optimally is given by* $p_L = 0.51$. *Using eq. (7) this implies that a war of attrition arises for* $k \in \{1, 2\}$. *Solving the system of equations given in (20) yields the vector of probabilities with which each firm invests in the project. It yields* $\gamma(1) = 0.4547$, *and* $\gamma(2) = 0.7613$.

From this example one can see that the probability of investment increases rapidly and is substantial. The reason is that in this particular case it holds that $\tilde{k}^* = \tilde{k}_M$. Therefore, the belief process jumps immediately to the mutual investment region if sufficient good signals arrive. Both firms know that it is better to become the leader, so as the mutual investment region comes closer, both invest with higher probability.

5 Two Extreme Cases

Now that the equilibria of the game are known, one can look at two extreme cases. First, we analyse the game when the information spillover is absent. Afterwards, we investigate the game without Stackelberg advantage.

5.1 The Case Without Information Spillover

If there are no information spillovers, investment by the leader does not generate information for the follower. That is to say, as soon as one of the firms invests, the other firm still does not know whether the project is good or bad. This implies that the revenue function for the follower is now given by

$$f(p) = \max\{0, M(p)\}. \tag{5'}$$

The equivalent of Figure 1 then becomes as depicted in Figure 2. In this

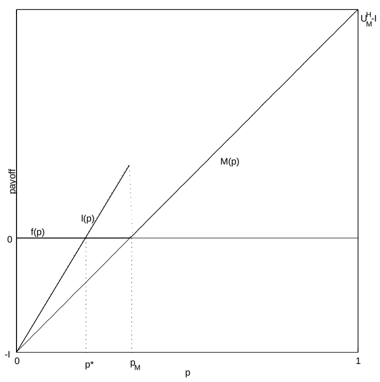

Figure 2: Payoff functions without information spillover.

case, the expected revenue for the leader is higher than the expected payoff of the follower if $p > \min\{p|L(p) = 0\}$. Hence, both firms want to be the leader as soon as the leader's expected profits are non-negative, which is at $\tilde{p}^*$. As soon as one firm has invested, the other firm has an incentive to wait because the follower value is higher than the mutual investment value. Therefore, the follower waits to see whether the project is good or not. He invests if the project is indeed good.

5.2 The Case Without Stackelberg Advantage

In this case, the firm that is first to invest does not gain anything by being the leader. This is the case because in absence of the Stackelberg effect it holds that $U_L^H = U_F^H = U_M^H$. The payoff functions for this case are given in Figure 3. Since the follower curve always lies above the leader curve it is

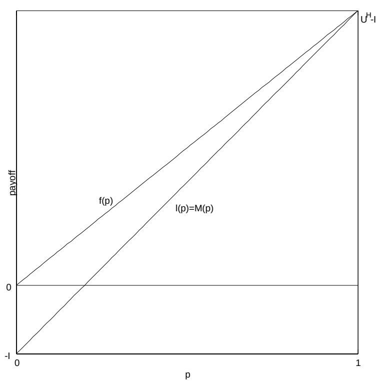

Figure 3: Payoff functions without Stackelberg advantage.

never optimal to become the leader. Therefore, there will never be a point in time where in equilibrium preemption takes place. However, as soon as $\tilde{k}_L$ has been reached each firm realizes that investment by one of the firms is optimal, but neither firm wants to be the first one to invest. Hence, a war of attrition arises in which both firms invest with a small but positive probability. Note that the region for which a war of attrition arises is given by $[\lceil \tilde{k}_L \rceil, \infty)$. These probabilities can be found numerically by equating for each k in this region the value of investing immediately and the value of investing after the next signal has arrived. Of course, the asymmetric strategies prescribing one firm to invest always and the other firm never to invest still constitute an equilibrium.

6 Welfare Analysis

Welfare effects resulting from investment under uncertainty have been reported by e.g. Jensen (1992) and Stenbacka and Tombak (1994). In both cases the timing of investment does not depend on the arrival of signals. In these papers the uncertainty comprises the time needed to succesfully implement the investment, i.e. the time between investment and the succesful implementation of the investment is stochastic. The models in Jensen (1992) and Stenbacka and Tombak (1994) allow for the critical levels to be explicit points in time. In our model, the critical level is not measured in units of time but measured as a probability, i.e. a belief. To perform a welfare analysis it is necessary to incorporate the time element in the model.

Suppose for the sake of convenience that $p_0 < p^* < p_L$, i.e. a symmetric subgame perfect equilibrium of this game is given by Proposition 4.1. This equilibrium implies that as soon as $\tilde{k}^*$ is reached, at least one firm invests and the game ends. Given the belief in a good project p, the probability of mutual investment is given by[9]

$$b(p) = \frac{\alpha(p)}{2 - \alpha(p)}. \tag{21}$$

For $p \in [p^*, p_M)$ this implies that

$$b(p) = \frac{L(p) - F(p)}{L(p) - 2M(p) + F(p)}. \tag{22}$$

Let CS^l_M denote the discounted value of consumer surplus if the project is $l \in \{L, H\}$ and simultaneous investment takes place. Furthermore, let CS^H_S and CS^L denote the discounted stream of consumer surplus in the Stackelberg equilibrium if the project is good, and the discounted stream of consumer surplus if the project is bad and one firm invests, respectively.

If the critical number of h-signals in excess of l-signals is given by $k \geq 0$ with first passage time t, the expected discounted total surplus if the project gives high revenues is given by

$$\begin{aligned} ES^H(k,t) =& e^{-rt}\Big[(b \circ p)(k)(2U^H_M + CS^H_M) \\ &+ \big(1 - (b \circ p)(k)\big)(U^H_L + U^H_F + CS^H_S) - 2I\Big], \end{aligned} \tag{23}$$

[9]See Appendix A.

whereas the expected total surplus if the project gives low revenue is given by

$$ES^L(k,t) = e^{-rt}\Big[(b\circ p)(k)(CS^L_M - 2I) + \big(1-(b\circ p)(k)\big)(CS^L - I)\Big]. \tag{24}$$

The expected total surplus with critical level k with first passage time t is then given by

$$W(k,t) = p(k)ES^H(k,t) + (1-p(k))ES^L(k,t). \tag{25}$$

So far, there is no difference with the ideas in Jensen (1992) and Stenbacka and Tombak (1994). To incorporate the uncertainty regarding the first passage time through k, we define the ex ante expected total surplus $W(k)$ to be the expectation of $W(k,t)$ over the first passage time through k. That is,

$$\begin{aligned} W(k) &= \mathbb{E}_\mathrm{k}\big(\mathrm{W(k,t)}\big) \\ &= \int_0^\infty W(k,t)f_k(t)dt, \end{aligned} \tag{26}$$

where $f_k(\cdot)$ is the pdf of the first passage time through k.

Denote the modified Bessel function with parameter ρ by $I_\rho(\cdot)$, i.e.

$$I_\rho(x) = \sum_{l=0}^\infty \frac{1}{l!\Gamma(l+\rho+1)}\Big(\frac{x}{2}\Big)^{2l+\rho},$$

where $\Gamma(\cdot)$ denotes the gamma function. The pdf of the first passage time through $k \geq 0$ can now be established as is done in the following proposition, the proof of which can be found in Appendix D.

Proposition 6.1 *The probability density function $f_k(\cdot)$ of the first passage time through $k \geq 0$ is given by*

$$f_k(t) = \tfrac{\lambda^k+\zeta(1-\lambda)^k}{1+\zeta}\big(\lambda(1-\lambda)\big)^{-k/2}\frac{k}{t}I_k(2\mu\sqrt{\lambda(1-\lambda)}t)e^{-\mu t},$$

for all $t \geq 0$.

It is assumed that a social planner faces the same uncertainty about the project being good or bad as the firms do. A social planner maximizing ex ante expected total surplus therefore has to determine a critical level for k.

Note that, because of the sunk costs $I > 0$, in the case of non-decreasing returns to scale it is always optimal for the social planner to have one active firm. Denoting the maximal sum of discounted consumer and producer surplus if investment takes place at critical level k with first passage time t by $W_{soc}(k,t)$, the social planner maximizes ex ante expected total surplus W_{soc},

$$W_{soc} = \max_{k \geq 0} \Big\{ \mathrm{I\!E}\big(\mathrm{W_{soc}(k,t)}\big) \Big\}. \tag{27}$$

From the standard theory of industrial organization it is well-known that monopoly gives lower social welfare than competition. However, in the following example it is shown that in the presence of uncertainty this need not hold. In the remainder let CS_{mon} and W_{mon} denote the present value of the infinite flow of consumer surplus and the ex ante expected total surplus, respectively, in the case of a monopolist. The critical level of investment for the monopoly case is calculated as in Thijssen et al. (2001).

Example 6.1 (Example 2.1 continued) *Reconsider the case of a new market model with linear demand and linear costs as given in Example 2.1. Consider the parameterization as given in Table 3. From Example 2.1*

$Y = 10$	$r = 0.1$
$c = 5$	$\mu = 4$
$I = 12$	$p_0 = 0.4$

Table 3: Parameter values.

and Thijssen et al. (2001) we can conclude that the monopoly price is given by $P_{mon} = \frac{Y+c}{2}$*, the price in case of mutual investment is given by* $P_M = \frac{Y+2c}{3}$*, and the price in the Stackelberg case is given by* $P_S = \frac{Y+3c}{4}$*. Given that the market is good, the flow of consumer surplus is then given by* $\int_{P^*}^{Y} P^{-1}(p)dp = \frac{1}{2}(Y - P^*)^2$*, where* P^* *is the equilibrium price. Hence,* $CS^H_{mon} = \int_0^\infty e^{-rt}\frac{1}{2}(Y - P_{mon})^2 dt = \frac{(Y-P_{mon})^2}{8r}$*. Similarly,* $CS^H_M = \frac{(Y-P_M)^2}{6r}$*,* $CS^H_S = \frac{(Y-P_S)^2}{32r}$*, and* $CS^L_{mon} = CS^L_M = CS^L = 0$*.*

If $\lambda = 0.8$*, then in the duopoly case a Nash equilibrium occurs. This happens because* $p^* < p_M < \tilde{p}^*$*. The ex ante expected total surplus in case of a duopoly is given by* $W_{duo} = 32.16$*. For the monopoly case we get* $W_{mon} = 29.75$ *and for the social planner* $W_{soc} = 41.68$*.*

Now consider the situation where uncertainty increases in the sense that the signals become less informative. As an example we take $\lambda = 0.6$*. In this case too, duopoly gives a Nash equilibrium. Furthermore we get:* $W_{mon} = 25.25$*,* $W_{duo} = 20.20$*, and* $W_{soc} = 36.56$*. Hence in case of a monopoly the ex ante expected total surplus is higher than for a duopolistic market.*

From Example 6.1 it can be concluded that comparing social welfare under monopoly and duopoly leads to ambiguous results caused by opposing effects. First, as is well-known from the industrial organization literature (e.g. Tirole (1988)), the dead-weight loss is highest under monopoly. Second, more firms entering the market reduces the dead-weight loss but, on the other hand, the total amount of sunk cost investments is increased, which has a negative effect on the welfare level. Third, social welfare is influenced by the timing of investment. Tempted by the Stackelberg advantage (if it outweighs the information spillover), the leader in a duopoly might invest too soon in the sense that the payoff in the new market is too uncertain.

In the above analysis only the preemption case is considered. From a mathematical point of view the advantage of considering the preemption case is that one knows that the game stops as soon as the preemption level is reached. This allows for the use of the distribution of the first passage time in the definition of ex ante expected total surplus. In the case where the information spillover outweighs the Stackelberg effect a war of attrition arises. To make a comparable welfare analysis for this case one has to consider all possible paths for the arrival of signals. So, not only the distribution for the first passage time, but also the distribution of all passage times have to be considered, conditional on the fact that the preemption value is not reached. Such an analysis is not analytically tractable. However, one could estimate the ex ante expected total surplus by use of simulations. Also in this case ambiguous results regarding the welfare effects of monopoly and duopoly can be expected, depending on the position of the critical investment level for a monopolist relative to p_L. An additional effect concerning the welfare comparison of monopoly and duopoly in case of a war of attrition is the free rider effect. In a duopoly both firms like the other to invest first so that it does not need to take the risk that the project has low value. Consequently firms invest too late, leading to a low consumer surplus.

7 Conclusions

Non-exclusivity is a main feature that distinguishes real options from their financial counterparts (Zingales (2000)). A firm having a real investment opportunity often shares this possibility with one or more competitors and this has a negative effect on profits. The implication is that, to come to a meaningful analysis of the value of a real option, competition must be taken into account.

This paper considers a duopoly where both firms have the same possibility to invest in a new market with uncertain payoffs. As time passes uncertainty is gradually resolved by the arrival of new information regarding the quality of the investment project in the form of signals. Generally speaking, each firm has the choice of being the first or second investor. A firm moving first reaches a higher market share by having a Stackelberg advantage. However, being the second investor implies that the investment can be undertaken knowing the payoff with certainty, since by observing the performance in the market of the first investor it is possible to obtain full information regarding the quality of the investment project.

The outcome mainly depends on the speed at which information arrives over time. If the quality and quantity of the signals is sufficiently high, the information advantage of the second investor is low so that the Stackelberg advantage of the first investor dominates, which always results in a preemption game. In the other scenario, initially a war of attrition prevails where it is preferred to wait for the competitor to undertake the risky investment. During the time where this war of attrition goes on it happens with positive probability that both firms refrain from investment. It can then be the case that so many bad signals arrive that the belief in a good project again becomes so low that the war of attrition is ended and that no firm invests for the time being. On the other hand, it can happen that so many positive signals in excess of bad signals arrive that at some point in time the Stackelberg advantage starts to exceed the value of the information spillover. This then implies that the war of attrition turns into a preemption game.

From the industrial organization literature it is known that a monopoly is bad for social welfare. Indeed, in our framework it is possible to find examples where a duopoly does better than a monopoly in terms of ex ante expected total surplus. However, within a duopoly it is also possible that in the case of a preemption equilibrium the first investor is tempted by the Stackelberg advantage to undertake the investment too soon from a social welfare perspective, i.e. when the environment is too risky. As a result it happens that welfare is lower than in the monopoly case. In this sense, our analysis strengthens the results from Mankiw and Whinston (1986).

Finally, departing from the modelling framework of this paper two interesting topics for future research can be distinguished. Firstly, one could include the possibility for firms to invest in the quantity and quality of the signals. This would then give rise to an optimal R&D model, that also includes the problem of optimal sampling. Secondly, it is interesting to allow for entry and exit in this model. This would then lead to an analysis of the optimal number of firms from a social welfare perspective, thereby including the model of Mankiw and Whinston (1986).

Appendix

A Equilibrium Concepts for Timing Games

This appendix introduces the equilibrium notions and accompanying strategy spaces that are introduced in Fudenberg and Tirole (1985) for timing games, extended with the presence of uncertainty. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space and let $\{X_t\}_{t\geq 0}$ be an adapted process. The idea is that we extend the concepts introduced in Fudenberg and Tirole (1985) path-wise. First we define a simple strategy for the subgame starting at t_0.

Definition A.1 *A* simple strategy *for player $i = 1, 2$ in the subgame starting at $t_0 \in [0, \infty)$ is given by a tuple of real-valued functions $(G_i^{t_0}, \alpha_i^{t_0}) : [t_0, \infty) \times \Omega \to [0, 1] \times [0, 1]$, such that for each $\omega \in \Omega$*

1. *$G_i^{t_0}(\cdot; \omega)$ is non-decreasing and right-continuous;*
2. *$\alpha_i^{t_0}(t; \omega) > 0 \Rightarrow G_i^{t_0}(t) = 1$;*
3. *$\alpha_i^{t_0}(\cdot; \omega)$ is right differentiable;*
4. *if $\alpha_i^{t_0}(t; \omega) = 0$ and $t = \inf\{u \geq t_0 | \alpha_i^{t_0}(u; \omega) > 0\}$, then the right derivative of $\alpha_i^{t_0}(t; \omega)$ is positive.*

Since strategies are defined path-wise, we omit ω for notational convenience. Thus, the strategy set of simple strategies of player i in the subgame starting at t_0 is given by

$$S_i^s(t_0) = \{s^{t_0} | s^{t_0} = (G_i^{t_0}, \alpha_i^{t_0}) \text{ is a simple strategy of player } i\}. \tag{28}$$

Furthermore, define the strategy space by $S^s(t_0) = \prod_{i=1,2} S_i^s(t_0)$ and denote the strategy at $p \in [t_0, 1]$ by $s^{t_0}(t) = (G_i^{t_0}(t), \alpha_i^{t_0}(t))_{i=1,2}$.

Note that $G_i^{t_0}(t)$ is the probability that a firm has invested up to and including time t. The function $\alpha_i^{t_0}(\cdot)$ is an atom. Since continuous time modelling does not yield the same results as taking the limit of a discrete time model, this function is used to replicate discrete time results. It is assumed that if $\alpha_i^{t_0}(t) > 0$, then the game is repeated over and over again instantaneously, not consuming any time[10], where firm i invests with probability $\alpha_i^{t_0}(t) > 0$, until at least one of the firms has invested. This time interval can therefore be seen as an interval of atoms. As soon as at least one firm has invested the game continues. To make the importance of the atom function clear suppose that the game is played in discrete time and in each period firm i invests with probability α_i, given that the realisation of the stochastic process remains constant. Let Δt be the size of a period and let T_Δ be such that for some constant T, $T_\Delta \Delta t = T$. Then if we take $\Delta t = 1$ we get for instance

$$\begin{aligned}
&\mathbb{P}(\text{both firms invest at the same time before time } T) \\
&\quad = \alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2)\alpha_1\alpha_2 + \cdots + (1-\alpha_1)^{T-1}(1-\alpha_2)^{T-1}\alpha_1\alpha_2.
\end{aligned}$$

Letting $\Delta t \downarrow 0$ we get a result that is independent of T,

$$\begin{aligned}
\lim_{\Delta t \downarrow 0} \sum_{t=0}^{T_\Delta/\Delta t} (1-\alpha_1)^t(1-\alpha_2)^t\alpha_1\alpha_2 &= \alpha_1\alpha_2 \sum_{t=0}^{\infty} \big((1-\alpha_1)(1-\alpha_2)\big)^t \\
&= \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2} < 1.
\end{aligned}$$

Were we not to use the atom function then the probability of simultaneous investment would equal one, whereas using the interval of atoms can replicate the limiting case of the discrete time game.

The expected discounted value of the subgame starting at t_0 and at strategy $s^{t_0} \in S^s(t_0)$ for player $i = 1, 2$ is denoted by $V_i(t_0, s^{t_0})$. A Nash equilibrium for the subgame starting at t_0 is then defined as follows.

[10] Hence, X_t remains constant.

Definition A.2 *A tuple of simple strategies* $s^* \in S^s(t_0)$ *is a* Nash equilibrium *for the subgame starting at* t_0 *if*

$$\forall_{i\in\{1,2\}}\forall_{s_i\in S_i^s(t_0)} : V_i(t_0, s^*) \geq V_i(t_0, s_i, s_{-i}^*).$$

To define perfect equilibrium, the notion of closed loop strategy is needed.

Definition A.3 *A* closed loop strategy *for player* $i \in \{1,2\}$ *is a collection of simple strategies* $\{(G_i^t, \alpha_i^t)_{t\in[0,\infty)} | (G_i^t, \alpha_i^t) \in S_i^s(t)\}$*, that satisfies the following intertemporal consistency conditions:*

1. $\forall_{0\leq t\leq u\leq v<\infty} : G_i^t(v) = G_i^t(u) + (1 - G_i^t(u))G_i^u(v)$;
2. $\forall_{0\leq t\leq u\leq v<\infty} : \alpha_i^t(v) = \alpha_i^u(v) \equiv \alpha_i(v)$.

The set of closed loop strategies for player $i \in \{1,2\}$ is denoted by S_i^{cl}. As before, we define the strategy space to be $S^{cl} = \prod_{i\in\{1,2\}} S_i^{cl}$. Now, subgame perfect equilibrium can be defined.

Definition A.4 *A tuple of closed loop strategies* $s^* \in S^{cl}$ *is a* subgame perfect equilibrium *if for every* $t \in [0,\infty)$*, the corresponding tuple of simple strategies* $(G_i^t, \alpha_i^t)_{i=1,2}$ *is a Nash equilibrium.*

B Proof of Proposition 4.1

First notice that for each $\omega \in \Omega$ the strategy $(G_i^t, \alpha_i^t)_{t\in[0,\infty)}$ satisfies the intertemporal consistency conditions of Definition A.3. Hence, the closed loop strategies are well-defined. The proof follows closely the proof of Lemma 1 in Fudenberg and Tirole (1985). However, since it is an instructive proof, we will present it in some detail. Let $t \in [0,\infty)$. It will be shown that (G_i^t, α_i^t) is a Nash equilibrium for the game starting at t. We consider three cases.

1. $t \geq T_M$

   Given that firm j plays its closed loop strategy, firm i has three possible strategies. First, firm i can play $G_i^t(t) = 0$, i.e. it does not invest. Then firm i's expected payoff equals $F(p_t)$. If firm i invests with an isolated atom equal to $\lambda > 0$, then the expected payoff equals $F(p_t) + \lambda(M(p_t) - F(p_t)) \geq F(p_t)$. Finally, suppose that $G_i^t(t) = 1$

and $\alpha_i^t(t) = a > 0$. Using the theory from Appendix A one can calculate the following probabilities.

$$\mathbb{P}(\text{firm } i \text{ invests first}) = \frac{a(1-\alpha_j^t(t))}{a+\alpha_j^t(t)-a\alpha_j^t(t)},$$
$$\mathbb{P}(\text{firm } j \text{ invests first}) = \frac{(1-a)\alpha_j^t(t)}{a+\alpha_j^t(t)-a\alpha_j^t(t)},$$
$$\mathbb{P}(\text{firms invest simultaneously}) = \frac{a\alpha_j^t(t)}{a+\alpha_j^t(t)-a\alpha_j^t(t)}.$$

Since $\alpha_j^t(t) = 1$, the expected payoff for firm i is given by

$$\begin{aligned}&\tfrac{1}{a+\alpha_j^t(t)-a\alpha_j^t(t)}\Big(a(1-\alpha_j^t(t))L(p_t) + (1-a)\alpha_j^t(t)F(p_t) + a\alpha_j^t(t)M(p_t)\Big)\\ &= F(p_t) + a(M(p_t) - F(p_t)) \geq F(p_t).\end{aligned} \tag{29}$$

So, maximizing the expected payoff gives $a = 1$.

2. $t < T^*$

   Given the strategy of firm j, if firm i does not invest, its value is $W(p_t)$. Since $T_L \geq T^*$, we know it is not optimal to invest yet. Hence, $W(p_t) > L(p_t)$. If firm i invests with an isolated atom equal to $\lambda > 0$, then its expected payoff equals $W(p_t) + \lambda(L(p_t) - W(p_t)) \leq W(p_t)$. Investing with an interval of atoms, i.e. $G_i^t(t) = 1$ and $\alpha_i^t(t) = a > 0$ gives an expected payoff equal to $L(p_t)$. Hence it is optimal to set $G_i^t(t) = 0$.

3. $T^* \leq t < T_M$

   Investing with probability zero, i.e. $G_i^t(t)$ yields an expected payoff equal to $F(p_t)$, given that firm j plays its strategy, i.e. $G_i^t(t) = 1$. If firm i invests with an isolated jump equal to $\lambda > 0$, then $\mathbb{P}(\text{both firms invest simultaneously}) = \lambda\alpha_{\mathrm{j}}^{\mathrm{t}}(\mathrm{t})$, $\mathbb{P}(\text{firm } i \text{ invests first}) = \lambda(1 - \alpha_{\mathrm{j}}^{\mathrm{t}}(\mathrm{t}))$ and $\mathbb{P}(\text{firm } j \text{ invests first}) = 1 - \lambda$. Given $\alpha_j^t(t) = \frac{L(t)-F(t)}{L(t)-M(t)}$ the expected payoff for firm i is given by

$$\lambda\alpha_j^t(t)M(p_t) + \lambda(1-\alpha_j^t(t))L(p_t) + (1-\lambda)F(p_t) = F(p_t). \tag{30}$$

   Finally, if firm i plays $G_i^t(t) = 1$ and $\alpha_i^t(t) = a > 0$, then the expected payoff is given by

$$\frac{1}{a+\alpha_j^t(t)-a\alpha_j^t(t)}\Big(a\alpha_j^t(t)M(p_t)+a(1-\alpha_j^t(t))L(p_t)+(1-a)\alpha_j^t(t)F(p_t)\Big)$$
$$=F(p_t). \tag{31}$$

□

C Proof of Proposition 4.2

It is trivial to see that (G_i^t, α_i^t) satisfies the intertemporal consistency conditions for each $t \in [0, \infty)$. We show it for the most difficult case. Let $T_L \leq u \leq v \leq T^*$ and $t \leq u$. Note that

$$1 - G_i^t(u) = \mathbb{P}(\text{firm } i \text{ has not invested before } u) = \prod_{\mathrm{n=n_t}}^{\mathrm{n_u}} (1 - \mathrm{g(T_n)}).$$

Hence,

$$\begin{aligned}
&G_i^t(u) + \big(1 - G_i^t(u)\big)G_i^u(v) = \\
&\sum_{n=n_t}^{n_u} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_t}^{n} \big(1 - g(T_{n'})\big) + \\
&+ \prod_{n=n_t}^{n_u} (1 - g(T_n)) \sum_{n=n_u}^{n_v} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_u}^{n} \big(1 - g(T_{n'})\big) \\
&= \sum_{n=n_t}^{n_u} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_t}^{n} \big(1 - g(T_{n'})\big) + \sum_{n=n_u}^{n_v} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_t}^{n} \big(1 - g(T_{n'})\big) \\
&= \sum_{n=n_t}^{n_v} \frac{g(T_n)}{1-g(T_n)} \prod_{n'=n_t}^{n} \big(1 - g(T_{n'})\big) = G_i^t(v).
\end{aligned} \tag{32}$$

We now prove that for each subgame starting at t, the simple strategy (G_i^t, α_i^t) is a Nash equilibrium. The case where t is such that $p_t < p_L$ is exactly the same as the case where $t < T^*$ in the proof of Proposition 4.1. The same holds true for the case where $t \geq T_M$. Consider a region for the war of attrition, i.e. t is such that $p_t \in [p_L, p^*)$. Suppose that firm i invests with an interval of atoms and suppose $\alpha_i^t(t) = a$. Then given that firm j

invests with an isolated jump equal to $\gamma(k_t)$. Using the by now familiar reasoning we get

$$\begin{aligned}\mathbb{P}(\text{firm } i \text{ invests first}) &= 1-\gamma(k_t),\\ \mathbb{P}(\text{firm } j \text{ invests first}) &= \gamma(k_t)(1-a),\\ \mathbb{P}(\text{firms invest simultaneously}) &= a\gamma(k_t).\end{aligned}$$

Hence, the expected payoff for firm i is given by

$$a\gamma_j(k_t)M(p_t)+(1-\gamma_j(k_t))L(p_t)+\gamma_j(k_t)(1-a)F(p_t). \tag{33}$$

This expected payoff is maximized for $a=0$. Hence, firm i will not play an interval of atoms. Suppose firm i plays an isolated jump equal to γ. Then his expected payoff equals

$$\gamma V_1(p_t)+(1-\gamma)V_2(p_t), \tag{34}$$

and is hence independent of γ since, by definition, $\gamma_j(k_t)$ is such that $V_1(p_t)=V_2(p_t)$. Therefore, any $\gamma\in[0,1]$ maximizes the expected payoff.

□

D Proof of Proposition 6.1

The proof follows Feller (1971), Section 14.6 and is probabilistic. Note that the process starts at $k=0$. Arriving at $k\neq 0$ at time t can only be possible if a jump has occurred before t. Assume that the first jump occurred at time $t-x$. The conditional probability of the position $k\neq 0$ at time t is denoted by $P_k(t)$. It is the convolution of the probability that the process was at $k+1$ at time x or at $k-1$ at time x and the probability of an arrival of an l-signal or an h-signal, respectively. Since the arrival of signals follows a Poisson process with parameter μ and hence the interarrival times are exponentially distributed with parameter μ, $P_k(t)$ is given by

$$P_k(t)=\int_0^t \mu e^{-\mu(t-x)}\Big[q_1(k-1)P_{k-1}(x)+q_2(k+1)P_{k+1}(x)\Big]dx, \tag{35}$$

where

$$q_1(k-1)=\frac{\lambda^k+\zeta(1-\lambda)^k}{\lambda^{k-1}+\zeta(1-\lambda)^{k-1}}, \tag{36}$$

is the probability of reaching state k from state $k-1$ and

$$q_2(k+1) = \lambda(1-\lambda)\frac{\lambda^k + \zeta(1-\lambda)^k}{\lambda^{k+1} + \zeta(1-\lambda)^{k+1}}, \tag{37}$$

is the probability of reaching state k from state $k+1$. That is, $P_k(t)$ is the convolution of the distribution of reaching $k+1$ or $k-1$ at time $t-x$ and the distribution of the arrival of one signal in the interval $(t-x, t]$. For $k=0$, the probability of no jump up to t, $1 - \int_0^t \mu e^{-\mu t} dt = e^{-\mu t}$ must be added, i.e.

$$P_0(t) = e^{-\mu t} + \int_0^t \mu e^{-\mu(t-x)} \Big[q_1(-1)P_{-1}(x) + q_2(1)P_1(x)\Big] dx, \tag{38}$$

Denoting the Laplace transform of $P_k(\cdot)$ by $\pi_k(\cdot)$ we get from eqs. (35) and (38)

$$\pi_k(\gamma) = \frac{\mu}{\mu+\gamma}[q_1(k-1)\pi_{k-1}(\gamma) + q_2(k+1)\pi_{k+1}(\gamma)] \quad \text{for } k \neq 0, \tag{39}$$

$$\pi_0(\gamma) = \frac{1}{\mu+\gamma} + \frac{\mu}{\mu+\gamma}[q_1(-1)\pi_{-1}(\gamma) + q_2(1)\pi_1(\gamma)]. \tag{40}$$

By substituting eqs. (36) and (37) into eq. (39) one obtains the following second order linear difference equation

$$\mu\lambda(1-\lambda)F_{k+1}(\gamma) - (\mu+\gamma)F_k(\gamma) + \mu F_{k-1}(\gamma) = 0, \tag{41}$$

where

$$F_k(\gamma) = \frac{\pi_k(\gamma)}{\lambda^k + \zeta(1-\lambda)^k}.$$

The roots of the characteristic equation of eq. (41) are

$$\beta_\gamma = \frac{\mu+\gamma-\sqrt{(\mu+\gamma)^2 - 4\mu^2\lambda(1-\lambda)}}{2\mu\lambda(1-\lambda)},$$

and

$$\begin{aligned}\sigma_\gamma &= \frac{\mu+\gamma+\sqrt{(\mu+\gamma)^2 - 4\mu^2\lambda(1-\lambda)}}{2\mu\lambda(1-\lambda)} \\ &= \frac{4\mu^2\lambda(1-\lambda)}{2\mu\lambda(1-\lambda)\big(\mu+\gamma-\sqrt{(\mu+\gamma)^2 - 4\mu^2\lambda(1-\lambda)}\big)} \\ &= \frac{1}{\lambda(1-\lambda)}\beta_\gamma^{-1}.\end{aligned}$$

The general solution for $k \neq 0$ is therefore given by

$$F_k(\gamma) = A_\gamma \beta_\gamma^k + \frac{1}{\lambda(1-\lambda)} B_\gamma \beta_\gamma^{-k}.$$

Note that for $k \geq 0$ it holds that $\beta_\gamma^k \to 0$ as $\gamma \to \infty$, but that $\sigma_\gamma^k \to \infty$ as $\gamma \to \infty$. Since $\pi_k(\gamma)$ and hence $F_k(\gamma)$ are bounded as $\gamma \to \infty$, we get for $k \geq 0$ that $B_\gamma = 0$. Similarly we get for $k \leq 0$ that $A_\gamma = 0$. So, a solution to eq. (41) is given by

$$F_k(\gamma) = \begin{cases} F_0(\gamma)\beta_\gamma^k & k \geq 0 \\ \frac{1}{\lambda(1-\lambda)} F_0(\gamma)\beta_\gamma^{-k} & k < 0, \end{cases}$$

and hence,

$$\pi_k(\gamma) = \begin{cases} \frac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \beta_\gamma^k \pi_0(\gamma) & k \geq 0 \\ \frac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \frac{1}{\lambda(1-\lambda)} \beta_\gamma^{-k} \pi_0(\gamma) & k < 0, \end{cases} \tag{42}$$

Solving for $\pi_0(\gamma)$ using eq. (40) gives

$$\pi_0(\gamma) = \frac{1}{\beta_\gamma} \frac{\lambda(1-\lambda)}{(\mu+\gamma)\lambda(1-\lambda) - \mu(1+\lambda^2(1-\lambda)^2)}.$$

Hence, eq. (42) is well-defined.

If at time t the process is at $k \geq 0$, the first passage through k must have occurred at time $\tau \leq t$. In this case, the conditional probability of being at k again at time t equals the probability of being at state 0 at time $t - \tau$ times the probability of a first passage through k at time τ, i.e.

$$P_k(t) = \int_0^t F_k(\tau) P_0(t-\tau) d\tau, \tag{43}$$

where $F_k(\cdot)$ is the distribution of the first passage time through k. The Laplace transform of eq. (43) is given by

$$\pi_k(\gamma) = f_k(\gamma)\pi_0(\gamma). \tag{44}$$

From eq. (42) we therefore conclude that the Laplace transform of $F_k(\cdot)$ equals $f_k(\gamma) = \frac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \beta_\gamma^k$. Feller (1971) shows that for $\gamma > 1$, $(\gamma - \sqrt{\gamma^2 - 1})^k$ is the Laplace transform of the density $\frac{k}{t} I_k(t)$. Applying the mapping $\gamma \mapsto \frac{\gamma}{2\mu\sqrt{\lambda(1-\lambda)}}$ is a change of scale and applying the mapping

$\gamma \mapsto \gamma + \mu$ reflects multiplication of the density by $e^{-\mu t}$. Applying both mappings gives

$$(\gamma - \sqrt{\gamma^2 - 1})^k \mapsto \left(\frac{\gamma + \mu - \sqrt{(\gamma+\mu)^2 - 4\mu^2\lambda(1-\lambda)}}{2\mu\sqrt{\lambda(1-\lambda)}} \right)^k$$
$$= \frac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \beta_\gamma^k \left(\frac{1+\zeta}{\lambda^k + \zeta(1-\lambda)^k} \left(\lambda(1-\lambda)\right)^{k/2} \right).$$

Hence, the pdf of the first passage time through k is given by

$$f_k(t) = \tfrac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \left(\lambda(1-\lambda)\right)^{-k/2} \frac{k}{t} I_k(2\mu\sqrt{\lambda(1-\lambda)}t) e^{-\mu t}.$$

□

References

Dcamps, J-P and T. Mariotti (2000). Irreversible Investment and Learning Externalities. Discussion Paper no. 2003534, CREMAQ, Universit des Sciences Sociales, Toulouse.

Dixit, A.K. (1980). The Role of Investment in Entry Deterrence. *The Economic Journal*, **90**, 95–106.

Dixit, A.K. and R.S. Pindyck (1994). *Investment under Uncertainty.* Princeton University Press, Princeton, NJ.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications, volume II* (Second ed.). John Wiley & Sons, New York, NY.

Fudenberg, D. and J. Tirole (1985). Preemption and Rent Equalization in the Adoption of New Technology. *The Review of Economic Studies*, **52**, 383–401.

Fudenberg, D. and J. Tirole (1991). *Game Theory.* MIT-press, Cambridge, Mass.

Hendricks, K., A. Weiss, and C. Wilson (1988). The War of Attrition in Continuous Time with Complete Information. *International Economic Review*, **29**, 663–680.

Hoppe, H.C. (2000). Second-mover Advantages in the Strategic Adoption of New Technology under Uncertainty. *International Journal of Industrial Organization*, **18**, 315–338.

Huisman, K.J.M. (2000). *Technology Investment: A Game Theoretic Real Options Approach*. Ph. D. thesis, CentER for Economic Research, Tilburg University, Tilburg.

Jensen, R. (1982). Adoption and Diffusion of an Innovation of Uncertain Probability. *Journal of Economic Theory*, **27**, 182–193.

Jensen, R. (1992). Innovation Adoptation and Welfare under Uncertainty. *The Journal of Industrial Economics*, **40**, 173–180.

Lambrecht, B. and W. Perraudin (1999). Real Options and Preemption under Incomplete Information. Working Paper, Birbeck College.

Mankiw, N.G. and M.D. Whinston (1986). Free Entry and Social Inefficiency. *Rand Journal of Economics*, **17**, 48–58.

Maynard Smith, J. (1974). The Theory of Games and Evolution in Animal Conflict. *Journal of Theoretical Biology*, **47**, 209–221.

Smets, F. (1991). Exporting versus FDI: The Effect of Uncertainty, Irreversibilities and Strategic Interactions. Working Paper, Yale University, New Haven, Conn.

Stenbacka, R. and M.M. Tombak (1994). Strategic Timing of Adoption of New Technologies under Uncertainty. *International Journal of Industrial Organization*, **12**, 387–411.

Thijssen, J.J.J., E.E.C. van Damme, K.J.M. Huisman, and P.M. Kort (2001). Investment under Vanishing Uncertainty Due to Information Arriving over Time. CentER DP no. 2001-14, Tilburg University, Tilburg.

Tirole, J. (1988). *The Theory of Industrial Organization*. MIT-press, Cambridge, Mass.

Zingales, L. (2000). In Search of New Foundations. *The Journal of Finance*, **55**, 1623–1654.