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Cooperative games with a simplicial core

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Abstract

In this paper n-person cooperative games having the property that the core
is a subsimplex of the imputation set are characterized. Also a characterization
of games where the core is a subsimplex of the dual imputation set is given by
using some duality relations for games. We also give a geometric characteriza-
tion of games with a non-empty core, which follows easily from the well-known
Bondareva–Shapley theorem.

MSC 2000: 91A12

Key words: cooperative games, imputation set, core

1 Introduction

A cooperative n-person game is a pair < N, v >, where N = {1, 2, ..., n} is the set
of players and v : 2^N → R is the characteristic function with domain the family of
subsets of N. Such subsets are called coalitions and v(S) is called the value of coalition
S ∈ 2^N. Such a game models a situation where a group N of persons can cooperate
and also subgroups. For each subgroup S the value v(S) indicates the amount of
money which they can obtain when cooperating. There is only one restriction on
the characteristic function, namely v(ϕ) = 0, the value of the empty coalition is 0.
This implies that the set of characteristic functions of n-person games forms with the
obvious operations a (2^n − 1)-dimensional linear space G^N. Often v and < N, v >
will be identified. The question: "How to divide v(N), if all the players in N are
cooperating?" has given rise to many proposals called solution concepts. Of the one-
point solution concepts we mention only the Shapley value [6], the τ-value [9] and
the nucleolus [5]. Sometimes subsets of payoff distributions of v(N) are assigned to
games as solutions; such a subset consists of points which are from a certain point of
view better than the points outside. Three of such subsets, namely the imputation
set, the dual imputation set and the core [4] will play a role in this paper and we
describe them now.

The imputation set I(v) of a game < N, v > is defined by

I(v) = \{ x ∈ R^n | \sum_{i=1}^{n} x_i = v(N), x_i ≥ v(\{i\}) for each i ∈ N \}

and consists of those payoff distributions of v(N), which are individual rational i.e.
player i obtains an amount x_i which is at least as large as his individual value
v(\{i\}), which he can obtain by staying alone. From the geometric point of view,
the imputation set I(v) is equal to the intersection of the efficiency hyperplane
\[
\left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \right\} \text{ and the orthant } \{ x \in \mathbb{R}^n \mid x \geq i(v) \} \text{ of individual rational payoff vectors.}
\]

The imputation set is non-empty iff \( v(N) \geq \sum_{i=1}^{n} v(\{i\}) \). If \( v(N) > \sum_{i=1}^{n} v(\{i\}) \), then \( I(v) \) is an \((n - 1)\)-dimensional simplex with extreme points \( f^1(v), f^2(v), \ldots, f^n(v) \), where the \( k \)-th coordinate \((f^i(v))_k\) of \( f^i(v) \) equals \( v(\{k\}) \) if \( k \neq i \) and \((f^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v(\{k\})\).

For an \( n \)-person game \(<N,v>\) and \( S \subseteq 2^N \) we define the dual value \( v^*(S) \) of \( S \) as \( v^*(S) = v(N) - v(N \setminus S) \).

The amount \( v^*(S) \) can be seen as the marginal contribution of \( S \) to the grand coalition or also as a sort of blocking power of \( S \). The dual imputation set \( I^*(v) \) of the game \( v \) is given by

\[
I^*(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N), \ x_i \leq v^*(\{i\}) \text{ for each } i \in N \right\}.
\]

It consists of distributions of \( v(N) \), where no player gets more than his marginal contribution to the grand coalition. From the geometric point of view \( I^*(v) \) is equal to the intersection of the efficiency hyperplane and the orthant \( \{ x \in \mathbb{R}^n \mid x \leq u(v) \} \) of subtopic vectors.

Note that \( I^*(v) \neq \emptyset \) iff \( \sum_{i=1}^{n} v^*(\{i\}) \geq v(N) \). In the case of strict inequality, \( I^*(v) \) is an \((n - 1)\)-dimensional simplex with extreme points \( g^1(v), g^2(v), \ldots, g^n(v) \), where

\[
(g^i(v))_k = v^*(\{k\}) \text{ if } k \neq i, \text{ and } (g^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v^*(\{k\}).
\]

The core \( C(v) \) of a game \(<N,v>\) is a subset of \( I(v) \cap I^*(v) \) defined by

\[
C(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N), \ \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq 2^N \right\}.
\]

Note that \( v(\{i\}) \leq x_i \leq v^*(\{i\}) \) for all \( i \in N \) and \( x \in C(v) \). So, the core is the bounded solution set of a set of linear inequalities, which means that the core is a polytope i.e. the convex hull of a finite set of vectors in \( \mathbb{R}^n \). When proposing a core element for the division of \( v(N) \) among the players, no subgroup \( S \subset N \) will have an incentive to split off. However, the core may be empty. Independently, Bondareva in [2] and Shapley in [8] gave necessary and sufficient conditions for the non-emptiness of the core: let \(<N,v> be an n-person game; then \( C(v) \neq \emptyset \) iff \( v(N) \geq \sum_{S \subseteq 2^N \setminus \emptyset} \lambda_S v(S) \) in case \( \lambda_S \geq 0 \) for all \( S \subseteq 2^N \setminus \emptyset \) and \( \sum_{S \subseteq 2^N \setminus \emptyset} \lambda_S e_S = e^N \).

Here \( e^S \in \mathbb{R}^N \) is the characteristic vector of the coalition \( S \), with \((e^S)_i = 0 \) if \( i \notin S \), and \((e^S)_i = 1 \) otherwise.
In Section 2 we like to reformulate this Bondareva–Shapley result in geometric terms. In Section 3 we characterize $n$-person simplex games where the core is a subsimplex of the imputation set, and in Section 4 duality results for games lead to a characterization of dual simplex games, where the core is a subsimplex of the dual imputation set. Section 5 concludes.

2 Geometric characterization of games with a non-empty core

We define the per capita value $\bar{v}(S)$ of coalition $S \neq \emptyset$ by $\frac{1}{|S|} v(S)$ where $|S|$ is the cardinality of $S$. Further, let for a subsimplex $\Delta(S, v) = \text{conv}\{f_i(v) \mid i \in S\}$ of $I(v) = \Delta(N, v)$, the barycenter $\frac{1}{|S|} \sum_{i \in S} f_i(v)$ be denoted by $b(S, v)$. Then we obtain the following characterization of games with a non-empty core.

**Theorem 2.1.** The game $< N, v >$ has a non-empty core iff $\sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S b(S, v) = b(N, v)$ with $\mu_S \geq 0$, $\sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S = 1$, implies that $\sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S \bar{v}(S) \leq \bar{v}(N)$.

The theorem tells that $< N, v >$ has a non-empty core iff for each way of writing the barycenter of the imputation set as a convex combination of barycenters of subcoalitions, the per capita value of $N$ is at least as large as the corresponding convex combination of per capita values of the subcoalitions.

**Proof of Theorem 2.1.** For $\lambda = (\lambda_S)_{S \in 2^N \setminus \{\emptyset\}}$, let $\mu = (\mu_S)_{S \in 2^N \setminus \{\emptyset\}}$ be defined by $\mu_S = n^{-1} |S| \lambda_S$. Then

$$\lambda_S \geq 0, \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S e^S = e^N \text{ iff } \sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S \frac{e^S}{|S|} = \frac{e^N}{|N|}, \mu_S \geq 0, \sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S = 1.$$ 

This implies

(i) $\lambda_S \geq 0, \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S e^S = e^N \text{ iff } \sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S b(S, v) = b(N, v)$

since $b(S, v) = (v(\{1\}), v(\{2\}), \ldots, v(\{n\})) + \alpha |S| e^S$ for each $S \in 2^N \setminus \{\emptyset\}$

where $\alpha = v(N) - \sum_{i \in N} v(\{i\})$.

(ii) $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S v(S) \leq v(N) \text{ iff } \sum_{S \in 2^N \setminus \{\emptyset\}} \mu_S \bar{v}(S) \leq \bar{v}(N)$.

From these observations the proof of Theorem 2.1 follows easily. □
3 Characterization of simplex games

Let us call a game \( <N, v> \) a \( T \)-simplex game, where \( \phi \neq T \subset N \), if \( v(N) > \sum_{i=1}^n v(\{i\}) \). Then \( C(v) = \text{conv} \{ f^i(v) \mid i \in T \} = \Delta(T, v) \). Note that for a \( T \)-simplex game the imputation set is an \( (n - 1) \)-dimensional simplex with \( f^1(v), f^2(v), \ldots, f^n(v) \) as extreme points and the core is a \( |T| - 1 \)-dimensional sub-simplex. In [11] \( N \)-simplex games (and also dual \( N \)-simplex games) were introduced and the family of these games was denoted by \( \text{SI}_N \) (and \( \text{SI}_N^T \)). The main results obtained were:

(i) \( \text{SI}_N = \left\{ v \in G^N \mid v(S) \leq \sum_{i \in S} v(\{i\}) \text{ for all } S \neq N, \sum_{i \in N} v(\{i\}) < v(N) \right\} \) is a cone and the core correspondence is additive of \( \text{SI}_N^T \);

(ii) For \( v \in \text{SI}_N : \text{CIS}(v) = \text{ENS}(v) = \tau(v), \text{CC}(v) = C(v) \), where \( \text{CIS}(v) \) is the center of the imputation set and \( \text{ENS}(v) \) is the center of the dual imputation set. For the definition of the \( \tau \)-value we refer to [9], [1] or to [8].

In the next Theorem 3.1 we give some properties for games \( v \in \text{SI}_T \), the set of \( T \)-simplex games. In Theorem 3.2 it turns out that these properties are characterizing properties. Then we show in Example 3.4 that \( \text{SI}_T \) is not necessarily a cone of games if \( T \neq N \).

For a game \( <N, v> \) the zero-normalization is the game \( <N, 0> \) with

\[
v_0(S) = v(S) - \sum_{i \in S} v(\{i\}) \text{ for each } S \in 2^N.
\]

Theorem 3.1. Let \( v \in \text{SI}_T \) for \( \emptyset \neq T \subset N \) and let \( v_0 \) be the corresponding zero-normalization. Then

(i) (Losing property) \( v_0(S) \leq 0 \) for each \( S \in 2^N \) with \( T \setminus S \neq \emptyset \);

(ii) (Veto player property) \( T = \cap \{S \in 2^N \mid v_0(S) = v_0(N)\} \);

(iii) \( (N, 0) \)-monotonicity property \( v_0(S) \leq v_0(N) \) for all \( S \in 2^N \).

Remarks. In the spirit of [7] we call \( S \) with \( v_0(S) \leq 0 \) losing-coalitions and those with \( v_0(S) = v_0(N) \) winning coalitions. Players who are in each winning coalition are called veto players. Property (ii) says then that the set of veto players is equal to \( T \).

Proof of Theorem 3.1.

(i) Take \( S \in 2^N \) such that there is a \( k \in T \setminus S \). Then \( f^k(v) \in C(v) \) which implies that \( v(S) \leq \sum_{i \in S} (f^k(v))_i = \sum_{i \in S} v(\{i\}) \), \( v_0(S) \leq 0 \).

(ii) By (i), for \( S \in 2^N \) with \( T \setminus S \neq \emptyset : v_0(S) \leq 0 \leq v_0(N) \). For \( S \in 2^N \) with \( T \subset S \) and each \( x \in C(v) = \text{conv} \{ f^i(v) \mid i \in T \} \) we have \( v(S) \leq \sum_{i \in S \setminus T} x_i + \sum_{i \in T} x_i = \sum_{i \in S \setminus T} v(\{i\}) + \left( v(N) - \sum_{i \in N \setminus T} v(\{i\}) \right) \).
which is equivalent with $v_0(S) \leq v_0(N)$.

(ii) From (i) it follows that $v_0(S) = v_0(N) > 0$ implies that $T \setminus S = \emptyset$, $T \subseteq S$. So $T \cap \{S \in 2^N \mid v_0(S) = v_0(N)\}$. For the converse inclusion we have to prove that

$$(\cap \{S \in 2^N \mid v_0(S) = v_0(N)\}) \setminus T = \emptyset.$$  

Suppose that this set is non-empty and that $r$ is an element of it. We will deduce a contradiction. For each $U \in 2^N$ with $r \notin U$, $U$ is not winning. This implies that $\max\{v_0(U) \mid r \notin U\} < v_0(N)$. Take $\varepsilon \in (0, 1)$ such that $(1 - \varepsilon)v_0(N) > v_0(U)$ for all $U$ with $r \notin U$. Then we claim that for each $t \in T$ the element $z = (1 - \varepsilon)f^*(v) + \varepsilon f^*(v)$ is a core element, which is in contradiction with the fact that $C(v) = \Delta(T, v) = \text{conv}\{f^*(v) \mid t \in T\}$. To prove the claim note that for $S \in 2^N$ with

(a) $t \notin S, r \notin S: \sum_{i \in S} z_i = (1 - \varepsilon)\sum_{i \in S} v(\{i\}) + \varepsilon \sum_{i \in S} v(\{i\}) \geq v(S)$ by (i).

(b) $t \notin S, r \in S: \sum_{i \in S} z_i = (1 - \varepsilon)\sum_{i \in S} v(\{i\}) + \varepsilon \left(\sum_{i \in S} v(\{i\}) + v_0(N)\right) > \sum_{i \in S} v(\{i\}) \geq v(S)$ by (i).

(c) $t \in S, r \notin S: \sum_{i \in S} z_i = (1 - \varepsilon)\sum_{i \in S} v(\{i\}) + (1 - \varepsilon)v_0(N) + \varepsilon \sum_{i \in S} v(\{i\}) = \sum_{i \in S} v(\{i\}) + (1 - \varepsilon)v_0(N) > \sum_{i \in S} v(\{i\}) = v(S)$ in view of the choice of $\varepsilon$.

(d) $t \in S, r \in S: \sum_{i \in S} z_i = \sum_{i \in S} v(\{i\}) + v_0(N) \geq \sum_{i \in S} v(\{i\}) + v_0(S) = v(S)$, where the inequality follows from (iii). \qed

**Example 3.1.** For $T \subseteq N$, the unanimity game $u_T : 2^N \to \mathbb{R}$ is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$, otherwise. The game $u_T$ is a $T$-simplex game with $C(u_T) = \text{conv}\{e^i \mid i \in T\}$ and $I(u_T) = \text{conv}\{e^i \mid i \in N\}$, where $e^i$ is the $i$-th standard basis element in $\mathbb{R}^n$.

**Example 3.2.** A game is called simple [7] if $v(S) \in \{0, 1\}$ for all $S \in 2^N$ and $v(N) = 1$. The United Nations Security Council Game $< N, v >$ with $N = \{1, 2, \ldots, 15\}$ and

$v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subseteq S$ and $|S| \geq 9,$

$v(S) = 0$ otherwise

is a $\{1, 2, 3, 4, 5\}$-simplex game.

It corresponds to the situation when a bill can pass only if at least nine members agree with, among them the five veto players 1, 2, 3, 4 and 5 are. In fact, all simple games with a non-empty core and with $v(\{i\}) = 0$ for each $i \in N$ are $T$-simplex games (see Corollary 3.1), where $T$ is the non-empty set of veto players.

**Example 3.3.** Let $N = \{1, 2, 3, 4\}$, $v(\{1, 2\}) = v(\{1, 2, 3\}) = v(N) = 1$, $v(\{1, 2, 4\}) = \frac{1}{2}$, $v(S) = 0$ otherwise. Then $< N, v >$ is a $\{1, 2\}$-simplex game, with $C(v) = \text{conv}\{e^1, e^2\}$. 

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Example 3.4. Now we show that $T$-simplex games do not necessarily form a cone by considering the two $5$-person $\{1, 2\}$-simplex games $< N, v >$ and $< N, w >$ with

$$v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = 1 = v(N), \ v(S) = 0 \quad \text{otherwise}$$

$$w(\{1, 2, 3\}) = w(\{1, 2, 5\}) = 1 = w(N), \ w(S) = 0 \quad \text{otherwise}.$$

Then

$$C(v) = \text{conv}\{f^1(v), f^2(v)\} = \text{conv}\{e^1, e^2\},$$

$$C(w) = \text{conv}\{f^1(w), f^2(w)\} = \text{conv}\{e^1, e^2\}.$$  

For the sum game $u = v + w$ we have

$$u(\{1, 2, 3\}) = 2 = u(N), \ u(\{1, 2, 4\}) = u(\{1, 2, 5\}) = 1, \ u(S) = 0 \text{ otherwise.}$$

Note that $u$ is not a simplex game, so it is certainly not an element of $SI^{(1, 2)}$.

Theorem 3.2. Let $< N, v >$ be a game with $v_0(N) > 0$. Suppose that

(i) $v_0(S) \leq v_0(N)$ for each $S \in 2^N$

(ii) $T : = \cap\{S \mid v_0(S) = v_0(N)\} \neq \emptyset$

(iii) $v_0(S) \leq 0$ for all $S$ with $T \setminus S \neq \emptyset$.

Then $C(v) = \Delta(T, v)$, $v \in SI^T$.

Proof. We have to show that $C(v) = \Delta(T, v)$.

(a) Suppose $x \in \Delta(T, v)$. Then for each $i \in N$ there is $\alpha_i \geq 0$ such that $x_i = v(\{i\}) + \alpha_i v_0(N)$ and $\sum_{i \in S} \alpha_i = 1, \alpha_i = 0$ for $i \in N \setminus T$. Then for $S \in 2^N : \sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \sum_{i \in S} \alpha_i.$

In case $T \subseteq S : \sum_{i \in S \cap T} \alpha_i = \sum_{i \in T} \alpha_i = \sum_{i \in N} \alpha_i = 1$, so $\sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \geq \sum_{i \in S} v(\{i\}) + v_0(S) = v(S)$, where the inequality follows from (i).

In case $T \setminus S \neq \emptyset : \sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \sum_{i \in S \cap T} \alpha_i \geq \sum_{i \in S} v(\{i\}) \geq v(S),$

where the last inequality follows from (iii). So $x \in C(v)$. We have proved that $\Delta(T, v) \subseteq C(v)$.

(b) For the converse inclusion, we show that $x \in I(v) \setminus \Delta(T, v)$ implies that $x \notin C(v)$. Take $x \in I(v) \setminus \Delta(T, v)$. Then there is a $k \in N \setminus T$ with $x_k = v(\{k\}) + \epsilon$ and $\epsilon > 0$. Further $x_i \geq v(\{i\})$ for all $i \in N$. By (ii) there is an $S$ with $v_0(S) = v_0(N)$ and $k \notin S$. This implies $x(S) = v(N) - \sum_{i \in N \setminus S} x_i \leq v(N) - \sum_{i \in N \setminus S} v(\{i\}) - \epsilon = v_0(N) + \sum_{i \in S} v(\{i\}) - \epsilon = v_0(N) + \sum_{i \in S} v(\{i\}) - \epsilon = v(S) - \epsilon.$

So we have proved that $\sum_{i \in S} x_i \leq v(S) - \epsilon$, hence $x \notin C(v)$. □
As a corollary of Theorem 3.2 we obtain the following well-known fact about simple games.

**Corollary 3.1.** Let \(<N, v> \) be a game with the properties:

(i) \(v(S) \in \{0, 1\} \) for each \(S \in 2^N\),

(ii) \(\Delta(N, v) = I(v) = \text{conv}\{e^1, e^2, \ldots, e^n\}\),

(iii) The set of veto players \(T = \{S \mid v(S) = 1\} \) is non-empty.

Then \(C(v) = \Delta(T, v)\).

**4 Characterization of dual simplex games**

Now, we focus on characterizing all games with the property that the core is a non-empty subsimplex of the dual imputation set \(I^*(v)\). Let us denote by \(SI_T^2\) the set of \(n\)-person games with \(\emptyset \neq T \subseteq N\), \(v^*(N) < \sum_{i=1}^{n} v^*(i)\) and \(C(v) = \text{conv}\{g^i(v) \mid i \in T\} = \Delta^*(T, v)\).

**Example 4.1.** Let \(<N, v> \) be the 3-person game with \(v(\{i\}) = 0\) for each \(i \in N\), \(v(\{1, 2\}) = 1\), \(v(\{1, 3\}) = 2\), \(v(\{2, 3\}) = v(N) = 6\). Then \(v^*(N \setminus \{i\}) = v^*(\{i\}) = 6\) for each \(i \in N\), \(v^*(\{1\}) = 0\), \(v^*(\{2\}) = 4\) and \(v^*(\{3\}) = 5\). Here \(C(v) = I(v) \cap I^*(v) = \text{conv}\{6e^1, 6e^2, 6e^3\} \cap \text{conv}\{-3, 4, 5\}, (0, 1, 5), (0, 4, 2)\} = \text{conv}\{(0, 1, 5), (0, 4, 2)\} = \Delta^*(\{2, 3\}, v)\), so \(v \in SI_T^{2, 3}\).

**Example 4.2.** Let \(<N, v> \) be the 3-person unanimity game based on \(\{1, 2\}\), so \(v(\{1, 2\}) = v(\{1, 2, 3\}) = 1\), \(v(S) = 0\) otherwise. Then \(v^*(\{3\}) = 0\) and \(v^*(S) = 1\) otherwise. The core \(C(v)\) equals \(\text{conv}\{e^1, e^2\} = \text{conv}\{f^1(v), f^2(v)\} = \text{conv}\{g^2(v), g^1(v)\}\). So \(C(v) = \Delta(\{1, 2\}, v) = \Delta^*(\{1, 2\}, v)\), hence \(v \in SI_{1, 2}^1\) and \(v \in SI_T^{1, 2}\).

To solve the characterization problem for dual simplex games, we can use our characterization result in Section 3 for simple games. For that purpose we develop some duality relations for cooperative games in the next lemma.

**Lemma 4.1.** For each \(v \in G^N\) and all \(k \in N\), \(T \subseteq N\), \(T \neq \emptyset\) we have

(i) \((v^*)^* = v\)

(ii) \(-f^k(v) = g^k(-v^*)\)

(iii) \(\Delta^*(T, v) = -\Delta(T, -v^*)\)

(iv) \(C(-v^*) = -C(v)\)

(v) \(C(-v^*) = \Delta(T, -v^*)\) iff \(C(v) = \Delta^*(T, v)\), which is equivalent to \(-v^* \in SI_T^T \) iff \(v \in SI_T^*\).
Proof. We only prove (iv) and leave the other proofs to the readers.

\[
C(-v^*) = \\
= \left\{ x \in \mathbb{R}^n \mid \sum_{i \in S} x_i = -v^*(N), \sum_{i \in S} x_i \leq -v^*(S) \text{ for each } S \in 2^N \right\} = \\
= \left\{ y \in \mathbb{R}^n \mid \sum_{i \in N \setminus S} y_i = v(N), \sum_{i \in N \setminus S} y_i \geq v(N \setminus S) \text{ for each } S \in 2^N \right\} = \\
= -\left\{ y \in \mathbb{R}^n \mid \sum_{i \in T} y_i = v(N), \sum_{i \in T} y_i \geq v(T) \text{ for each } T \in 2^N \right\} = \\
= -C(v).
\]

\[\Box\]

The key of finding the characterization of dual simplex games lies now in Lemma 4.1 (v): \( v \in SI^T \) iff \( -v^* \in SI^T \). So we can use the characterization of \( v \in SI^T \) of Section 3 but with \( -v^* \) in the role of \( v \) and obtain

Theorem 4.2. Let \( \emptyset \neq T \subset N \) and let \( v_0(N) > 0 \). Then \( v \in SI^T \) iff the following three properties hold:

(i) Dual \((N, 0)\)-monotonicity property: \( (v^*)_0(S) \geq (v^*)_0(N) \) for all \( S \in 2^N \)

(ii) Dual veto player property: \( \cap \{ S \in 2^N \mid (v^*)_0(S) = (v^*)_0(N) \} = T \neq \emptyset \)

(iii) Dual losing property: \( (v^*)_0(S) \geq 0 \) for all \( S \in 2^N \) with \( T \setminus S \neq \emptyset \).

5 Concluding remark

It could be interesting to study for simplex games and also for dual simplex games the relations between different existing solution concepts such as the \( \tau \)-value, the nucleolus, the Shapley value, CIS etc.

References


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