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**ON THE NUMERICAL INVERSION OF BUSY-PERIOD  
RELATED TRANSFORMS**

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**Discussion paper**

# ON THE NUMERICAL INVERSION OF BUSY-PERIOD RELATED TRANSFORMS

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## **Abstract**

Many quantities of interest in queueing theory can be determined in the form of transforms. Methods for numerically inverting those transforms are well developed. However, some transforms such as that of the busy period distribution can only be characterized implicitly via functional equations. Algorithms for inversion require the evaluation of transforms at many complex arguments. Although this is possible it may be computationally quite involved. In this paper, we show that the original contour integral with the implicitly determined transform as part of the integrand can be replaced by alternative contour integrals with only known transforms as part of the integrand via simple substitutions. The alternative contour integrals can be evaluated by the same techniques as the original ones.

*Keywords:* Numerical transform inversion; Busy period; Number served in busy period; Generating function; Laplace transform; M/G/1 queue; GI/M/1 queue.

*Jel-code:* 560

# 1 Introduction

The numerical inversion of transforms has been extensively developed during the last decade, see, e.g., Abate & Whitt [4, 5, 7] and Abate et al. [1, 2, 3]. A special case is formed by transforms that can only be characterized implicitly as solutions to functional equations. The most important examples are the Laplace-Stieltjes transform (LST) of the busy period distribution and the probability generating function (PGF) of the distribution of the number of customers served during a busy period, for instance, in an M/G/1 system. Many other quantities of interest, such as the Laplace transform (LT) of the time-dependent mean queue length or the time-dependent mean workload but also the LSTs of the waiting time distributions for various customer classes in priority systems, are determined in terms of the LST of a busy period distribution. Most algorithms for the numerical inversion of transforms require the values of the involved transforms at many complex arguments. This means for transforms which are only characterized implicitly that the related functional equation has to be numerically solved at many complex arguments. Abate & Whitt [6] discuss the solution of functional equations for complex arguments and provide conditions for iterative methods to converge. However, this approach is more involved than the basic methods for numerical inversion, and the iterative solution of a functional equation is an additional source of numerical inaccuracy. In Abate et al. [1] it is shown that the M/G/1 busy period density and the transient M/G/1 probability of emptiness can be computed without numerical solution of the functional equation. Alternative inversion formulas are derived on the basis of an infinite series representation of the busy period density in terms of the densities of the sums of numbers of i.i.d. service times. In this paper it will be shown that these and other alternative inversion formulas can be obtained by simple substitutions in the contour integrals, and possibly an integration by parts. We will now give a short review of the basic methods for numerical inversion of transforms. The reader is referred to the indicated references for more details and derivations. The terms of a sequence of real numbers  $\{g_n; n = 0, 1, 2, \dots\}$  with  $|g_n| \leq 1$  for all  $n$  can be recovered from its generating function (GF),

$$G(z) \doteq \sum_{n=0}^{\infty} g_n z^n, \quad |z| < 1, \quad (1)$$

by repeated differentiation or by means of a contour integral in the complex  $z$ -plane over a circle around the origin with radius  $r$ ,  $0 < r < 1$ :

$$g_n = \frac{1}{n!} \frac{d^n}{dz^n} G(z) \Big|_{z=0} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{G(z)}{z^{n+1}} dz, \quad n = 0, 1, 2, \dots \quad (2)$$

Repeated differentiation may be quite cumbersome if the GF  $G(z)$  does not possess a suitable structure. The contour integral can be converted into an integral over a real interval by means of the substitution  $z = re^{iu}$  and with the aid of some symmetry properties of the GF: for  $0 < r < 1$ ,

$$g_n = \frac{1}{\pi r^n} \int_0^\pi [\cos(nu) \Re G(re^{iu}) + \sin(nu) \Im G(re^{iu})] du, \quad n = 0, 1, 2, \dots; \quad (3)$$

here,  $i = \sqrt{-1}$  and  $\Re z$  ( $\Im z$ ) denotes the real (imaginary) part of a complex number  $z$ . The case  $n = 0$  is simple:  $g_0 = G(0)$ . For  $n > 0$ , Abate & Whitt [5] describe the following method for evaluating the above type of integrals with a prescribed accuracy of, say,  $\epsilon$ . Application of the trapezoidal rule with a step size of  $\pi/n$  to (3) yields

$$g_n \approx \frac{1}{nr^n} \left[ \frac{1}{2} \{G(r) + (-1)^n G(-r)\} + \sum_{k=1}^{n-1} (-1)^k \Re G(re^{ik\pi/n}) \right], \quad n = 1, 2, \dots, \quad (4)$$

while the prescribed accuracy and an upper bound on the discretization error lead to the choice of  $r = 2^n \bar{\epsilon}$ ,  $n = 1, 2, \dots$ ; to avoid roundoff problems, approximately  $\frac{3}{2}\gamma$ -digit precision is required to obtain  $\epsilon = 10^{-\gamma}$  accuracy.

The inversion of the Laplace transform of a continuous real-valued function  $f(t)$  on the positive real line,

$$\Phi(\zeta) \doteq \int_0^\infty f(t) e^{-\zeta t} dt, \quad \Re \zeta > \Delta_f, \quad (5)$$

with  $\Delta_f$  the abscissa of convergence of the LT, can be based on the following integral

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta t} \Phi(\zeta) d\zeta \doteq \lim_{U \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iU}^{x+iU} e^{\zeta t} \Phi(\zeta) d\zeta, \quad t > 0; \quad (6)$$

the integral is taken over a line parallel to the imaginary axis through the real point  $x$  which has to be chosen such that the function  $\Phi(\zeta)$  has no singularities for  $\Re \zeta \geq x$ , that is,  $x > \Delta_f$ . For most queueing applications it holds that  $\Delta_f \leq 0$ . The line integral can be converted into an integral over a real interval by means of the substitution  $\zeta = x + iu$  and with the aid of some symmetry properties of the LT. It implies that the inversion of a LT can be performed by any of the following two real-valued integrals: for  $t > 0$ ,

$$f(t) = \frac{2e^{xt}}{\pi} \int_0^\infty \cos(ut) \Re \Phi(x + iu) du = -\frac{2e^{xt}}{\pi} \int_0^\infty \sin(ut) \Im \Phi(x + iu) du. \quad (7)$$

Abate & Whitt [4, 7] describe a method for evaluating such an integral based on the trapezoidal rule with a step size of  $\pi/(2t)$ . Since  $\cos(\frac{1}{2}k\pi) = 0$  for  $k$  odd, it follows from (7) that

$$f(t) \approx \frac{e^{xt}}{t} \left[ \frac{1}{2} \Phi(x) + \sum_{k=1}^{\infty} (-1)^k \Re \Phi(x + k\pi i/t) \right], \quad t > 0. \quad (8)$$

Further, they choose  $xt = \frac{1}{2}A$  and they let  $A = -\log \epsilon$  to achieve a discretization error associated with the trapezoidal rule of  $\epsilon$  when  $f(t)$  is a function bounded by 1. In this continuous case there is an additional source of inaccuracy caused by the necessary truncation of the infinite sum. Euler summation is recommended to accelerate the convergence of the series.

Consider an M/G/1 system with arrival rate  $\lambda$ , with  $\beta(\zeta)$  the LST and  $\beta_k$  the  $k$ th moment of the service time distribution, and with load  $\rho = \lambda\beta_1$ . The bivariate transform of the joint distribution of the length of a busy period,  $P$ , and the number of customers served in the same busy period,  $J$ , satisfies the following functional equation, cf. Cohen [10, Sect. II.4.4]:

$$E\{z^J e^{-\zeta P}\} = \nu(z, \zeta), \quad \nu(z, \zeta) = z\beta(\zeta + \lambda[1 - \nu(z, \zeta)]), \quad |z| \leq 1, \quad \Re\zeta \geq 0. \quad (9)$$

The probabilities  $\Pr\{J = n\}$ ,  $n = 1, 2, \dots$ , can be computed by applying (4) to the PGF  $\nu_0(z) \doteq \nu(z, 0)$ . Since the PGF  $\nu_0(z)$  can only be explicitly solved from the functional equation (9) in exceptional cases such as the M/M/1 system, direct application of (4) requires the numerical solution of the functional equation (9) for complex values of  $z$  (the number of these values increases linearly with  $n$ ). Similarly, direct application of (8) to functions involving the LST  $\nu_1(\zeta) \doteq \nu(1, \zeta)$  of the busy period distribution requires the numerical solution of the functional equation (9) for complex values of  $\zeta$  for most systems.

Next, consider an GI/M/1 system with  $\alpha(\zeta)$  the LST and  $\alpha_k$  the  $k$ th moment of the interarrival time distribution, with service rate  $\mu$ , and with load  $\rho = 1/(\mu\alpha_1)$ . The bivariate transform of the joint distribution of  $P$  and  $J$ , defined as above, is determined via the following functional equation, cf. Cohen [10, Sect. II.3.5]:

$$E\{z^J e^{-\zeta P}\} = \frac{\mu[z - \chi(z, \zeta)]}{\zeta + \mu[1 - \chi(z, \zeta)]}, \quad \chi(z, \zeta) = z\alpha(\zeta + \mu[1 - \chi(z, \zeta)]), \quad |z| \leq 1, \quad \Re\zeta \geq 0. \quad (10)$$

Again, direct application of (4) or (8) requires the computation of  $\chi_0(z) \doteq \chi(z, 0)$  for complex  $z$  or of  $\chi_1(\zeta) \doteq \chi(1, \zeta)$  for complex  $\zeta$ , respectively, by numerical solution of the functional equation in (10) which has the same structure as that in (9).

In Section 2 we will present the derivation of an alternative contour integral for GFs which involve the PGF of the distribution of the number of customers served in a busy period. Section 3 is devoted to a similar transformation, but for LTs involving the LST of the distribution of the length of a busy period. Some other applications of the substitution method including the joint distribution of the busy period and the number of customers served in that busy period are discussed in Section 4. The main results are summarized in Section 5.

## 2 Number of customers in a busy period

First, consider the distribution of the number of customers,  $J$ , served during a busy period in an M/G/1 system. The PGF  $\nu_0(z)$  of this distribution is the unique solution inside the unit disk of the functional equation, cf. (9),

$$\nu_0(z) = z\beta(\lambda[1 - \nu_0(z)]), \quad |z| \leq 1. \quad (11)$$

Note that  $\Pr\{J = 0\} = \nu_0(0) = 0$  and that  $\Pr\{J = 1\} = \nu_0'(0) = \beta(\lambda) > 0$ . Direct application of (2) and (3) gives, for  $n = 2, 3, \dots$ :

$$\Pr\{J = n\} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\nu_0(z)}{z^{n+1}} dz \approx \frac{1}{nr^n} \left[ \frac{1}{2} \{ \nu_0(r) + (-1)^n \nu_0(-r) \} + \sum_{k=1}^{n-1} (-1)^k \Re \nu_0(re^{ik\pi/n}) \right]. \quad (12)$$

To avoid the computation of the PGF  $\nu_0(z)$  at the values  $re^{ik\pi/n}$  by iterative solution of (11) we propose to substitute  $w = \nu_0(z)$  in the contour integral in (12). Since  $\nu_0'(0) > 0$  this mapping has an inverse in a neighborhood of the origin. Moreover, it follows from (11) that this inverse is explicitly given by  $z = w/\beta(\lambda[1 - w])$ . This substitution leads to the representation, for  $n = 2, 3, \dots$ :

$$\Pr\{J = n\} = \frac{1}{2\pi i} \oint_{|w|=r} w \left[ \frac{\beta(\lambda[1 - w])}{w} \right]^{n+1} \frac{\beta(\lambda[1 - w]) + \lambda w \beta'(\lambda[1 - w])}{\beta^2(\lambda[1 - w])} dw. \quad (13)$$

The image of a circle  $|z| = r$  under the mapping  $w = \nu_0(z)$  is not a circle but a contour with the origin in its interior. Since the integrand in the  $w$ -plane has no singularities in  $\Re w < 1$  other than  $w = 0$  this contour can be replaced by a circle  $|w| = r$  by Cauchy's Theorem. The contour integral in (13) already has the property that it can be evaluated without solving the functional equation (11). In this particular case, integration by parts leads to an even simpler contour integral. Here, we use that for two functions  $h_1(w)$  and  $h_2(w)$  that are regular on a circle  $|w| = r$  we have (since the "endpoints" of a circle coincide):

$$\frac{1}{2\pi i} \oint_{|w|=r} h_1(w) h_2'(w) dw = \frac{-1}{2\pi i} \oint_{|w|=r} h_1'(w) h_2(w) dw. \quad (14)$$

Applying the rule to the contour integral in (13) with  $h_1(w) = w$ ,  $h_1'(w) = 1$ , and

$$h_2(w) = \frac{-1}{n} \left[ \frac{\beta(\lambda[1 - w])}{w} \right]^n, \quad h_2'(w) = \left[ \frac{\beta(\lambda[1 - w])}{w} \right]^{n-1} \frac{\beta(\lambda[1 - w]) + \lambda w \beta'(\lambda[1 - w])}{w^2},$$

we obtain the contour integral representation

$$\Pr\{J = n\} = \frac{1}{2\pi i n} \oint_{|w|=r} \frac{\beta^n(\lambda[1 - w])}{w^n} dw, \quad n = 2, 3, \dots \quad (15)$$

This alternative contour integral also avoids the determination of a set of complex roots. It can be treated in a similar way as the original contour integral (12) with the trapezoidal rule: for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} \approx \frac{1}{(n-1)nr^{n-1}} \times \left[ \frac{1}{2} \{ \beta^n(\lambda[1-r]) + (-1)^{n-1} \beta^n(\lambda[1+r]) \} + \sum_{k=1}^{n-2} (-1)^k \Re \beta^n(\lambda[1 - r e^{ik\pi/(n-1)}]) \right]. \quad (16)$$

In comparison with (12),  $n$  is replaced by  $n - 1$  in the approximation formula (16). This is due to the facts that  $\nu_0(0) = 0$  and that this zero has been absorbed by the performed substitution (note that the factor  $z^{n+1}$  in the denominator of the integrand is replaced by  $w^n$ ). The additional factor  $n$  in the denominator of (16) stems from the integration by parts. In contrast with the direct numerical inversion formulas (12) with PGF  $\nu_0(z)$  the new inversion functions depend on  $n$ . For instance, it reads  $\frac{1}{n} \beta^n(\lambda[1-w])$  in (15) and (16). As a consequence, the error analysis performed in Abate & Whitt [4, 5] is not applicable with a single function. However, the same type of error analysis can be applied with a different function for each  $n$ , namely, with the PGF  $\beta^n(\lambda[1-w])$ . In fact, the representation (15) means that

$$\Pr\{J = n\} = \frac{1}{n} \Pr\{Y_1 + \dots + Y_n = n - 1\} < \frac{1}{n}, \quad n = 1, 2, \dots; \quad (17)$$

here,  $Y_1, Y_2, \dots$  are i.i.d. random variables with the distribution of the number of arrivals during a service time. Observe that this upper bound of  $\frac{1}{n}$  on  $\Pr\{J = n\}$  holds uniformly in the arrival rate and the service time distribution. Applying the above mentioned error analysis implies that the discretization error  $e_d(n)$  for the approximation of  $\Pr\{J = n\}$  in (16) is bounded by

$$|e_d(n)| \leq \frac{1}{n} \frac{r^{2(n-1)}}{1 - r^{2(n-1)}}, \quad n = 2, 3, \dots \quad (18)$$

Hence, choosing  $r = 10^{-\gamma/2(n-1)}$  in (16) yields an accuracy of about  $\frac{1}{n} \cdot 10^{-\gamma}$ ,  $n = 2, 3, \dots$

A similar substitution can be applied in the computation of the excess probabilities  $\Pr\{J > n\}$ .

The direct numerical inversion formula reads

$$\Pr\{J > n\} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{1 - \nu_0(z)}{1 - z} \frac{1}{z^{n+1}} dz, \quad n = 0, 1, 2, \dots \quad (19)$$

Applying the same substitution as above leads to: for  $n = 1, 2, \dots$ ,

$$\Pr\{J > n\} = \frac{1}{2\pi i} \oint_{|w|=r} \frac{(1-w)\beta(\lambda[1-w])}{\beta(\lambda[1-w]) - w} \left[ \frac{\beta(\lambda[1-w])}{w} \right]^{n+1} \frac{\beta(\lambda[1-w]) + \lambda w \beta'(\lambda[1-w])}{\beta^2(\lambda[1-w])} dw. \quad (20)$$



Performing an integration by parts, cf. (14), we obtain

$$\Pr\{J > n\} = \frac{1}{2\pi i n} \oint_{|w|=r} \left[ \frac{(1-w)\beta(\lambda[1-w])}{\beta(\lambda[1-w]) - w} \right]' \frac{\beta^n(\lambda[1-w])}{w^n} dw, \quad n = 1, 2, \dots \quad (21)$$

Again,  $\beta^n(\lambda[1-w])$  is a PGF. Further, we recognize that the function of which the derivative is taken is the PGF of the stationary queue-length distribution, apart from a factor  $1-\rho$ , cf. Cohen [10, Sect. II.4.3]. Generally, the derivative of a PGF divided by the mean of the distribution represents again a PGF, and the product of two PGFs is a PGF. This implies the following upper bounds: for  $n = 1, 2, \dots$ ,

$$\Pr\{J > n\} < \frac{1}{n} \frac{1}{1-\rho} \left[ \rho + \frac{\rho^2}{1-\rho} \frac{\beta_2}{2\beta_1^2} \right], \quad |e_d(n)| \leq \frac{1}{n} \frac{r^{2(n-1)}}{1-r^{2(n-1)}} \frac{1}{1-\rho} \left[ \rho + \frac{\rho^2}{1-\rho} \frac{\beta_2}{2\beta_1^2} \right]; \quad (22)$$

here,  $e_d(n)$  stands for the discretization error when the trapezoidal rule is applied to the contour integral in (21). Note that these upper bounds only hold in the stable case  $\rho < 1$ , in contrast with those in (17) and (18).

Next, consider the distribution of the number of customers,  $J$ , served during a busy period in an GI/M/1 system. It follows from (10) that  $\chi_0(0) = 0$  and  $\chi_0'(0) = \alpha(\mu)$  so that  $\Pr\{J = 1\} = 1 - \alpha(\mu)$ . Direct application of (2) gives, for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{z - \chi_0(z)}{1 - \chi_0(z)} \frac{1}{z^{n+1}} dz; \quad \chi_0(z) = z\alpha(\mu[1 - \chi_0(z)]), \quad |z| < 1. \quad (23)$$

In contrast with the solution  $\nu_0(z)$  to (11) which satisfies  $\nu_0(1) = 1$  for  $\rho < 1$ , the solution  $\chi_0(z)$  to the above equation has the property  $\chi_0(1) < 1$  for  $\rho < 1$ ; in fact,  $\chi_0(1)$  is equal to the probability that a customer has to wait. Substitution of  $w = \chi_0(z)$ , with inverse  $z = w/\alpha(\mu[1-w])$  for  $|w| \leq \chi_0(1)$ , leads to the alternative contour integral representation, for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} = \frac{1}{2\pi i} \oint_{|w|=r} \frac{1 - \alpha(\mu[1-w])}{1-w} \left[ \frac{\alpha(\mu[1-w])}{w} \right]^n \frac{\alpha(\mu[1-w]) + \mu w \alpha'(\mu[1-w])}{\alpha^2(\mu[1-w])} dw. \quad (24)$$

Again, we obtain a contour integral which can be evaluated without solving the functional equation in (23). Observe that the integrand is regular for  $|w| \leq 1$ . In this case, integration by parts does not lead to a really simpler representation but still to a representation that allows a simpler error analysis: for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} = \frac{1}{2\pi i(n-1)} \oint_{|w|=r} \left[ \frac{1 - \alpha(\mu[1-w])}{1-w} \right]' \left[ \frac{\alpha(\mu[1-w])}{w} \right]^{n-1} dw. \quad (25)$$

Here,  $\alpha^{n-1}(\mu[1-w])$  is a PGF and  $[1 - \alpha(\mu[1-w])]/(1-w)$  is a PGF, apart from a factor  $\rho$ , of a distribution with mean  $\frac{1}{2}\alpha_2\mu^2\rho$ . This implies as before, cf. (22), that for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} < \frac{1}{(n-1)\rho^2} \frac{\alpha_2}{2\alpha_1^2}, \quad |e_d(n)| \leq \frac{1}{(n-1)\rho^2} \frac{r^{2(n-2)}}{1-r^{2(n-2)}} \frac{\alpha_2}{2\alpha_1^2}. \quad (26)$$

These bounds are not useful if the load  $\rho$  is small. Replacing  $w$  by  $\chi_0(1)w$  in (25) yields, for  $n = 2, 3, \dots$ ,

$$\Pr\{J = n\} = \frac{1}{2\pi i(n-1)} \oint_{|w|=r} \left[ \frac{1 - \alpha(\mu[1 - \chi_0(1)w])}{1 - \chi_0(1)w} \right]' \left[ \frac{\alpha(\mu[1 - \chi_0(1)w])}{\chi_0(1)w} \right]^{n-1} dw. \quad (27)$$

Here,  $[\alpha(\mu[1 - \chi_0(1)w])/\chi_0(1)]^{n-1}$  is still a PGF since  $\chi_0(1) = \alpha(\mu[1 - \chi_0(1)])$ , and also the function  $[1 - \alpha(\mu[1 - \chi_0(1)w])]/(1 - \chi_0(1)w)$  is a PGF. In this way, we obtain the following upper bound for the discretization error  $e_d(n)$  when the trapezoidal rule is applied to (27): for  $n = 2, 3, \dots$ ,

$$|e_d(n)| \leq \frac{1}{n-1} \frac{r^{2(n-2)}}{1 - r^{2(n-2)}} \frac{\chi_0(1)}{1 - \chi_0(1)} [1 + \mu\alpha'(\mu[1 - \chi_0(1)])]. \quad (28)$$

This bound requires the numerical solution of the functional equation in (23), but only for  $z = 1$ . For M/M/1 systems this bound simplifies with  $\chi_0(1) = \rho$  to

$$|e_d(n)| \leq \frac{\rho}{n-1} \frac{r^{2(n-2)}}{1 - r^{2(n-2)}}, \quad n = 2, 3, \dots$$

For the excess probabilities the substitution  $w = \chi_0(z)/\chi_0(1)$  leads to: for  $n = 1, 2, \dots$ ,

$$\Pr\{J > n\} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{1}{1 - \chi_0(z)} \frac{1}{z^{n+1}} dz = \frac{1}{2\pi i n} \oint_{|w|=r} \frac{\chi_0(1)}{(1 - \chi_0(1)w)^2} \left[ \frac{\alpha(\mu[1 - \chi_0(1)w])}{\chi_0(1)w} \right]^n dw. \quad (29)$$

This implies the following upper bound for the discretization error  $e_d(n)$  with the trapezoidal rule:

$$|e_d(n)| \leq \frac{1}{n} \frac{r^{2(n-1)}}{1 - r^{2(n-1)}} \frac{\chi_0(1)}{[1 - \chi_0(1)]^2}, \quad n = 1, 2, \dots \quad (30)$$

### 3 Busy period distribution

In this section, we will apply the method of substitution to derive integral representations for the complementary distribution function of the busy period,  $\Pr\{P > y\}$ , which avoid the numerical solution of functional equations. First, consider the M/G/1 system. The LST  $\nu_1(\zeta)$  of the busy period distribution in an M/G/1 system is the unique solution inside the unit disk of the functional equation, cf. (9),

$$\nu_1(\zeta) = \beta(\zeta + \lambda[1 - \nu_1(\zeta)]), \quad \Re\zeta \geq 0. \quad (31)$$

The inverse of the LT of the complementary busy period distribution is according to (6): for  $x > 0$ ,

$$\Pr\{P > y\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta y} \frac{1 - \nu_1(\zeta)}{\zeta} d\zeta, \quad y > 0. \quad (32)$$

For functional equations of the form (31) the following one-to-one transformation will be used:

$$\theta = \zeta + \lambda[1 - \nu_1(\zeta)], \quad \zeta = \theta - \lambda[1 - \beta(\theta)]. \quad (33)$$

Notice that the first relation together with (31) implies that  $\beta(\theta) = \nu_1(\zeta)$  under this mapping. The latter equality leads with the first relation to the second relation of (33). Substitution of  $\theta$  for  $\zeta$  in the integral (32) yields

$$\Pr\{P > y\} = \frac{1}{2\pi i} \int_{C_x} e^{[\theta - \lambda + \lambda\beta(\theta)]y} \frac{1 - \beta(\theta)}{\theta - \lambda + \lambda\beta(\theta)} [1 + \lambda\beta'(\theta)] d\theta, \quad y > 0. \quad (34)$$

The curve  $C_x$  is the image of the line  $\Re\zeta = x$  under the transformation (33). Since  $|\nu_1(\zeta)| < 1$  on this line, the curve  $C_x$  is contained in the strip  $x < \Re\theta < x + 2\lambda$ . However, the curve  $C_x$  can be replaced by the line  $\Re\theta = x$  because the integrand has no singularities in the right half plane  $\Re\theta > 0$ . Again, the trapezoidal rule can be applied when we let  $e^{\theta y}$  in (34) play the role of  $e^{\zeta t}$  in (6). This yields

$$\Pr\{P > y\} \approx \frac{e^{xy}}{y} \left[ \frac{1}{2}\Phi(x, y) + \sum_{k=1}^{\infty} (-1)^k \Re\Phi(x + k\pi i/y, y) \right], \quad (35)$$

with  $x = \frac{1}{2}A/y$  and

$$\Phi(\theta, y) \doteq e^{-\lambda[1-\beta(\theta)]y} \frac{1 - \beta(\theta)}{\theta - \lambda + \lambda\beta(\theta)} [1 + \lambda\beta'(\theta)]. \quad (36)$$

Observe that the function  $\Phi(\theta, y)$  depends on  $y$ , in contrast with the LT  $(1 - \nu_1(\zeta))/\zeta$  in (32). An alternative approximation is obtained by performing an integration by parts in (34). It leads to an approximation of the type (35) but with

$$\Phi(\theta, y) \doteq \frac{1}{y} e^{-\lambda[1-\beta(\theta)]y} \frac{1 - \beta(\theta) + \theta\beta'(\theta)}{[\theta - \lambda + \lambda\beta(\theta)]^2}. \quad (37)$$

The error analysis is not so simple because a bound on the discretization error requires knowledge of the (maximum of the) density related to the LST in the integrand, cf. Abate & Whitt [4]. Note that the function  $e^{-\lambda[1-\beta(\theta)]y}$  is the LST of the distribution of a random number of service time where this number has a Poisson distribution with mean  $\lambda y$ . The last factor is minus the derivative of the LST of the waiting time distribution,  $E\{W e^{-\theta W}\}$ , apart from a factor  $(1 - \rho)\lambda$ .

We have implemented the trapezoidal rule as indicated in (35) with  $A = \log(10^8) \approx 18.42$  and truncation at 100 terms, and with Euler summation applied to the last 15 partial sums, with the function (36) as well as with the function (37). The numerical results compare well with results for M/M/1 systems obtained by applying the same method to (32) with the explicitly solved LST, and with the results for M/E<sub>4</sub>/1 and M/Γ<sub>1/2</sub>/1 systems presented in [8].

The substitution (33) is not new. It has been used with the purpose of asymptotic expansion of the transient probability of emptiness by the saddle point technique in Wishart [13]. It does not seem to have been used with the aim of simplifying numerical transform inversion.

The same substitution (33) can be used to simplify the computation of the busy period density  $f_P(y)$ ,  $y > 0$ , for nonlattice service time distributions. This leads to: for  $y > 0$ ,

$$f_P(y) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta y} \nu_1(\zeta) d\zeta = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{[\theta-\lambda+\lambda\beta(\theta)]y} \beta(\theta) [1 + \lambda\beta'(\theta)] d\theta. \quad (38)$$

Performing an integration by parts gives, provided that  $|\beta(x \pm iU)| \rightarrow 0$  as  $U \rightarrow \infty$ :

$$f_P(y) = \frac{-1}{2\pi i y} \int_{x-i\infty}^{x+i\infty} e^{[\theta-\lambda+\lambda\beta(\theta)]y} \beta'(\theta) d\theta, \quad y > 0. \quad (39)$$

This representation has been derived in Abate et al. [1] by considering the conditional density of a busy period given the service time of the first customer. These authors show that this representation can be converted into the representation

$$f_P(y) = \frac{1}{2\pi i \lambda y} \int_{x-i\infty}^{x+i\infty} e^{[\theta-\lambda+\lambda\beta(\theta)]y} d\theta, \quad y > 0, \quad (40)$$

which also follows via a series representation of the busy period density. Numerical experiments have taught us that application of the trapezoidal rule to (38), (39) and (40) gives comparably good results.

Next, consider the GI/M/1 system. The inverse of the LT of the complementary busy period distribution is according to (6), cf. (10): for  $x > 0$ ,  $y > 0$ ,

$$\Pr\{P > y\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta y} \frac{1}{\zeta + \mu[1 - \chi_1(\zeta)]} d\zeta; \quad \chi_1(\zeta) = \alpha(\zeta + \mu[1 - \chi_1(\zeta)]), \quad \Re\zeta \geq 0. \quad (41)$$

We use the analog transformation to (33), where now the point  $\zeta = 0$  corresponds to the point  $\theta = \mu[1 - \chi_1(0)] > 0$ :

$$\theta = \zeta + \mu[1 - \chi_1(\zeta)], \quad \zeta = \theta - \mu[1 - \alpha(\theta)]. \quad (42)$$

This substitution leads to the following line integral

$$\Pr\{P > y\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{[\theta-\mu+\mu\alpha(\theta)]y} \frac{1}{\theta} [1 + \mu\alpha'(\theta)] d\theta, \quad y > 0. \quad (43)$$

In this case, integration by parts gives a simplification:

$$\Pr\{P > y\} = \frac{1}{2\pi i y} \int_{x-i\infty}^{x+i\infty} e^{[\theta-\mu+\mu\alpha(\theta)]y} \frac{1}{\theta^2} d\theta, \quad y > 0. \quad (44)$$

This leads to a similar relation as (35) but with

$$\Phi(\theta, y) \doteq \frac{1}{y\theta^2} e^{-\mu[1-\alpha(\theta)]y}. \quad (45)$$

Table 1: Excess probabilities  $\Pr\{P > y\}$  for M/G/1 systems with load  $\rho = 0.8$  ( $\lambda = 0.8, \beta_1 = 1$ ).

$y$	M/E <sub>8</sub> /1	M/E <sub>2</sub> /1	M/M/1	M/Γ <sub>1/2</sub> /1	M/Γ <sub>1/8</sub> /1
0.1	1.0000	0.9834	0.9084	0.7573	0.3893
0.5	0.9633	0.7896	0.6615	0.5204	0.2650
1	0.6943	0.5844	0.4967	0.3970	0.2107
2	0.4390	0.3876	0.3407	0.2815	0.1596
5	0.2197	0.2034	0.1862	0.1618	0.1021
10	0.1158	0.1131	0.1080	0.0986	0.0685
25	0.0380	0.0418	0.0439	0.0445	0.0370
50	0.0113	0.0147	0.0176	0.0205	0.0212
100	0.0019	0.0033	0.0050	0.0073	0.0108

The same substitution (42) can be used to simplify the computation of the busy period density  $f_P(y), y > 0$ . This leads to: for  $y > 0$ ,

$$f_P(y) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta y} \frac{\mu[1 - \chi_1(\zeta)]}{\zeta + \mu[1 - \chi_1(\zeta)]} d\zeta = \frac{\mu}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{[\theta - \mu + \mu\alpha(\theta)]y} \frac{1 - \alpha(\theta)}{\theta} [1 + \mu\alpha'(\theta)] d\theta. \quad (46)$$

Performing an integration by parts gives:

$$f_P(y) = \frac{\mu}{2\pi i y} \int_{x-i\infty}^{x+i\infty} e^{[\theta - \mu + \mu\alpha(\theta)]y} \frac{1 - \alpha(\theta) + \theta\alpha'(\theta)}{\theta^2} d\theta, \quad y > 0. \quad (47)$$

Observe that the integrands in (43), (44), (46) and (47) are all regular functions for  $\Re\theta > 0$ . Still, one may replace  $\theta$  by  $\theta + \mu[1 - \chi_1(0)]$  in all these integrals. In this way, the integrands in (37) and (44), and the integrands in (39) and (47), agree for the case of M/M/1 systems. Both versions yield accurate results.

This section is concluded with some numerical examples. Table 1 contains the excess probabilities of the busy period for some M/G/1 systems with a service time distribution from the family of gamma distributions (including Erlang distributions). Table 2 contains the excess probabilities of the busy period for some GI/M/1 systems with gamma distributed interarrival times. In all instances, the mean interarrival time is 1.25 and the mean service time is 1. For the considered GI/M/1 systems,  $\Pr\{P > y\}$  is increasing in the variance of the interarrival time distributions for every fixed  $y$ . A similar ordering only holds for rather large values  $y$  in M/G/1 systems, while for small values  $y$  the ordering is reversed. This phenomenon is related to the fact that the mean busy period  $E\{P\} = \beta_1/(1 - \rho)$  does not depend on second and higher moments of the service time

Table 2: Excess probabilities  $\Pr\{P > y\}$  for GI/M/1 systems with load  $\rho = 0.8$  ( $\alpha_1 = 1.25$ ,  $\mu = 1$ ).

$y$	$E_8/M/1$	$E_2/M/1$	$M/M/1$	$\Gamma_{1/2}/M/1$	$\Gamma_{1/8}/M/1$
0.1	0.9048	0.9052	0.9084	0.9184	0.9541
0.5	0.6073	0.6287	0.6615	0.7121	0.8379
1	0.3925	0.4467	0.4967	0.5667	0.7420
2	0.2475	0.2923	0.3407	0.4130	0.6170
5	0.1236	0.1527	0.1862	0.2412	0.4303
10	0.0656	0.0850	0.1080	0.1475	0.2993
25	0.0220	0.0316	0.0439	0.0664	0.1648
50	0.0068	0.0113	0.0176	0.0305	0.0948
100	0.0012	0.0026	0.0050	0.0108	0.0481

distribution for M/G/1 systems, so that a relatively heavy tail must be compensated by a relatively high probability of a small busy period. The mean busy period is increasing with the variance of gamma interarrival time distributions in GI/M/1 systems.

## 4 Other applications

An important application of the substitution method is in the computation of time-dependent moments and probabilities. The Laplace transforms of many of such time-dependent quantities involve the LST of the busy period distribution. For instance, the transient empty probability in an M/G/1 queue given that the system starts empty at time 0 satisfies

$$\Pr\{V(t) = 0 \mid V(0) = 0\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\zeta t} \frac{1}{\zeta + \lambda[1 - \nu_1(\zeta)]} d\zeta, \quad t > 0. \quad (48)$$

Substitution according to (33) gives:

$$\Pr\{V(t) = 0 \mid V(0) = 0\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{[\theta - \lambda + \lambda\beta(\theta)]t} \frac{1}{\theta} [1 + \lambda\beta'(\theta)] d\theta, \quad t > 0, \quad (49)$$

which reduces by an integration by parts to

$$\Pr\{V(t) = 0 \mid V(0) = 0\} = \frac{1}{2\pi i t} \int_{x-i\infty}^{x+i\infty} e^{[\theta - \lambda + \lambda\beta(\theta)]t} \frac{1}{\theta^2} d\theta, \quad t > 0, \quad (50)$$

a relation derived by Beneš [9] in a different way, see also Abate et al. [1]. There are various other applications of this procedure such as to the just mentioned Laplace transform of the transient mean or higher moments of the workload distribution, but also to generating functions of autocorrelations of the successive waiting times and interdeparture times of the M/G/1 queue. For instance, Daley

[11] has shown that the GF of the series of autocorrelations of successive interdeparture times in M/G/1 systems can be expressed in terms of the PGF of the distribution of the number of customers served in a busy period and its derivative. This case will be elaborated in a future paper. Further applications include the LSTs of waiting time distributions in systems with non-FCFS disciplines. The LST of the waiting time distribution in an M/G/1 system with nonpreemptive LCFS discipline involves the LST of the busy period distribution, and is, in fact, closely connected to the probability of emptiness (48), cf. Abate et al. [1]. Also, in an M/G/1 system with several classes of customers and nonpreemptive priority the LSTs of the waiting time distributions of all but the highest priority class involve the LST of the busy period in an M/G/1 system with only the higher priority classes than the considered class present.

We conclude this section with a discussion of the joint distribution of a busy period and the number of customers served in that busy period. For  $n = 1, 2, \dots$ , and for  $y > 0$ , we have, cf. (9),

$$\Pr\{J = n, P < y\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \frac{e^{\zeta y}}{\zeta} \left[ \frac{1}{2\pi i} \oint_{|z|=r} \frac{\nu(z, \zeta)}{z^{n+1}} dz \right] d\zeta. \quad (51)$$

Substitution of  $w = \nu(z, \zeta)$ , with inverse  $z = w/\beta(\zeta + \lambda[1 - w])$ , leads after an integration by parts to: for  $n = 1, 2, \dots$ , for  $y > 0$ ,

$$\Pr\{J = n, P < y\} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \frac{e^{\zeta y}}{\zeta} \left[ \frac{1}{2\pi i n} \oint_{|w|=r} \frac{\beta^n(\zeta + \lambda[1 - w])}{w^n} dw \right] d\zeta. \quad (52)$$

Alternatively, the order of integration can be changed and the substitution  $\theta = \zeta + \lambda[1 - \nu(z, \zeta)]$ , with inverse  $\zeta = \theta - \lambda[1 - z\beta(\theta)]$ , cf. (33), can be performed. Both substitutions allow the numerical evaluation of the joint distribution without solving the functional equation (9).

Finally, we remark that Rogers [12] applies a double substitution to circumvent numerical solution of a nonlinear equation in the numerical inversion of a bivariate transform.

## 5 Conclusion

This paper has shown that the numerical inversion of busy period related transforms can be simplified by performing a substitution of the variable in the contour or line integral. An integration by parts often simplifies the integrand. The iterative solution of functional equations at complex values is avoided. Typically, the resulting integrand contains the LST of the service time distribution and its derivative for M/G/1 systems or the LST of the interarrival time distribution and its derivative for GI/M/1 systems.

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