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Connection problems in mountains and monotonic allocation schemes[1]

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Abstract: Directed minimum cost spanning tree problems of a special kind are studied, namely those which show up in considering the problem of connecting units (houses) in mountains with a purifier. For such problems an easy method is described to obtain a minimum cost spanning tree. The related cost sharing problem is tackled by considering the corresponding cooperative cost game with the units as players and also the related connection games, for each unit one. The cores of the connection games have a simple structure and each core element can be extended to a population monotonic allocation scheme (pmas) and also to a bi-monotonic allocation scheme. These pmas-es for the connection games result in pmas-es for the cost game.

1 Introduction

Consider a group of persons whose homes in the mountains are not yet connected to a drainage where one has to empty their sewage. Obviously sewage has to be collected downhill in a water purifier where it has to be purified before introduction into the environment.

One solution for the houses is to get rid of the waste water immediately, so each one wants to connect his house with a drain pipe to the water purifier. However, it is possible but not necessary for everyone to be connected

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directly with the water purifier, being connected via others is sufficient. Assuming that pipes are large enough one pipe can serve more than one person.

On the other hand, employing pumps to send sewage from houses at lower heights to houses at upper heights could be too expensive. Also practical reasons due to the inhomogeneous consistency of the waste water could suggest not to employ pumps. Therefore, exploiting gravity, only connections from houses to strictly lower ones are allowed (connections between houses at the same height are not allowed in order to avoid dangerous stagnation). A possible situation is sketched in figure 1. The network drawn

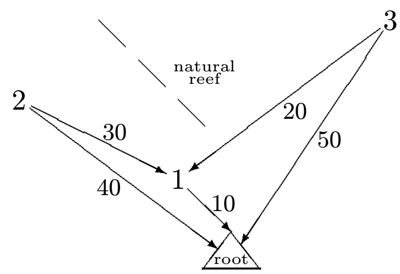

Figure 1: a possible mountain situation

in the picture is a directed weighted graph, whose vertices are the houses, whose root is the water purifier and whose edges are the drain pipes which are allowed to be built. The numbers indicate the cost of building the corresponding pipe. Sometimes connection from higher houses to lower houses is impossible (e.g. because of a natural reef between the two houses), as for example the connection from house 3 to house 2 in figure 1. However, it is always possible to connect a house directly with the root.

A mountain situation as described above leads to a connection problem of a directed graph without cycles and with some other properties. In section 2 we consider such connection problems in detail and describe a simple method to find a spanning tree with minimum costs. Section 3 tackles the cost sharing problem by introducing the cooperative cost game to a mountain situation. Interesting core elements of this cost game can be obtained by decomposing the cost game into connection games. These connection games have a zero- or one-dimensional core, for which the elements have a nice economic interpretation. Section 4 deals with a subset of the core of the cost game for which each element is extendable to a population monotonic allocation scheme or shortly a pmas (cf. Sprumont (1990), Thomson (1995)).

In section 5 it is shown that each core element of a connection game is extendable to a bi-monotonic allocation scheme. Section 6 deals with cost monotonic allocation rules (cf. Kent and Skorin-Kapov (1997)). The Bird allocation rule (cf. Bird (1976)) plays here a special role.

2 Connection problems on directed graphs without cycles

Consider a tuple given by $< N, \{0\}, A, w >$, where $N = \{1, 2, \ldots, n\}$, $< N \cup \{0\}, A >$ is a rooted directed graph with $N \cup \{0\}$ as set of points (vertices), $A \subset N \times (N \cup \{0\})$ as set of arcs and where 0 is the root. We assume also that the following conditions M.1 and M.2 hold.

M.1 (Direct connection possibility) For each $k \in N$, $(k, 0) \in A$.

M.2 (No cycles) For each $s \in \mathbb{N}$ and $v_1, v_2, \ldots, v_s \in N \cup \{0\}$ such that $(v_1, v_2) \in A$, $(v_2, v_3) \in A$, …, $(v_{s-1}, v_s) \in A$ we have $(v_s, v_1) \notin A$.

Further, $w : A \to \mathbb{R}_+$ is a non-negative function on the set of arcs. We call such a tuple $< N, \{0\}, A, w >$ with the properties M.1 and M.2 a *mountain situation* because of the following two reasons.

(i) Each mountain problem as described in section 1 leads to a mountain situation, where N corresponds to the set of agents (houses) in the mountain, 0 to the purifier, A to the set of allowed connections determined by the gravity condition

(2.1) $\quad (i, j) \in A \Rightarrow h(i) > h(j)$

(where $h(i)$ is the height of house i) and by reefs etc. Further $w(i, j)$ describes the cost of connecting i with j via a pipe line. M.1 is demanded and M.2 follows from (2.1).

(ii) On the other hand, given a mountain situation $< N, \{0\}, A, w >$ with the properties M.1 and M.2, there exists an intrinsic height function $h_0 : N \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that $(i, j) \in A$ implies $h_0(i) > h_0(j)$. One defines h_0 as follows: for $i \in N \cup \{0\}$, $h_0(i)$ is the length of a longest path from i to 0.

The interesting problems related to such a mountain situation are

Q.1 How to find a 0-connecting subtree $< N \cup \{0\}, T >$ of $< N \cup \{0\}, A >$, i.e. a subtree connecting each $i \in N$ with 0, with minimum cost?

Q.2 How to allocate the connection costs in such a tree among the agents?

In this section we will solve Q.1 and the next sections deal with Q.2.

To avoid too many technicalities we will assume in the following that $< N, \{0\}, A, w >$ does not only satisfy M.1 and M.2, but also M.3:

M.3 (Genericity condition) For each $k \in N$ and all $i, j \in N \cup \{0\}, i \neq j$: $(k, i) \in A, (k, j) \in A \Rightarrow w(k, i) \neq w(k, j)$.

We invite the reader to adjust our results for situations where M.3 does not hold. M.3 gives us the possibility to speak of <u>the</u> best connection $b(k)$ of $k \in N$. Here

$$b(k) = \underset{i \in N \cup \{0\}:(k,i) \in A}{\operatorname{argmin}} w(k, i).$$

Given a mountain situation $< N, \{0\}, A, w >$ (with property M.3!), the next theorem shows that there is a unique optimal tree (with minimum costs), connecting all players in N with the root 0. This tree corresponds to the situation where each agent $k \in N$ connects himself with his best connection point $b(k) \in N \cup \{0\}$.

Theorem 2.1 *Let $< N, \{0\}, A, w >$ be a mountain situation (satisfying, beside M.1 and M.2, also M.3). Let $T = \{(k, b(k)) \mid k \in N\}$. Then*

(i) $< N \cup \{0\}, T >$ is a 0-connecting subtree of $< N \cup \{0\}, A >$.

(ii) The tree $< N \cup \{0\}, T >$ is the unique 0-connecting subtree with minimum cost.

Proof (i) Since $T \subset A$, clearly T does not contain cycles. That T is a tree connecting each point $i \in N$ via a path with 0 follows from the claim that for each $s \in \{1, \ldots, L\}$, where $L = \max\{h_0(i) \mid i \in N \cup \{0\}\}$, the next property $P(s)$ holds:

> $P(s)$: for each $k \in N$ with $h_0(k) = s$ there is a $t(k) \in \mathbb{N}$ and a sequence $v_0, v_1, \ldots, v_{t(k)}$ such that $v_0 = k$, $v_{r+1} = b(v_r)$ for $r = 0, 1, \ldots, t(k) - 1$, and $v_{t(k)} = 0$.

We prove the claim by induction to s. $P(1)$ holds because for each $k \in N$ with $h_0(k) = 1$ we take $t(k) = 1$, $v_0 = k$ and $v_1 = 0$. Suppose now that

$P(s)$ holds for each $s < m$ with $m \in \{2, \ldots, L\}$. Let $k \in N$ be such that $h_0(k) = m$. Then $h_0(b(k)) < m$. If $h_0(b(k)) = 0$, then $b(k) = 0$ and we take $t(k) = 1$, $v_0 = k$, $v_1 = 0$. Suppose $h_0(b(k)) \neq 0$. Then, by the induction hypothesis, there is a $t(b(k))$ and a sequence $v_0, v_1, \ldots, v_{t(b(k))}$ determining a path in A from $b(k)$ to 0 with $v_{r+1} = b(v_r)$ for $r \in \{0, 1, \ldots, t(b(k)) - 1\}$. Then $w_0, w_1, \ldots, w_{t(k)}$ is a desired path for k, where $t(k) = t(b(k)) + 1$, $w_0 = k$, $w_i = v_{i-1}$ for $i \in \{1, \ldots, t(k)\}$. So $P(m)$ holds.
(ii) Let $< N \cup \{0\}, G >$ be a 0-connecting tree unequal to $< N \cup \{0\}, T >$. Then for each point $k \in N$, there is a $\pi(k) \in N \cup \{0\}$ such that $(k, \pi(k)) \in G$. Moreover, since $G \neq T$ we can choose $\pi : N \to N \cup \{0\}$ such that there is a $k^* \in N$ with $\pi(k^*) \neq b(k^*)$, implying $w(k^*, \pi(k^*)) > w(k^*, b(k^*))$ by M.3. Then

$$\sum_{(i,j) \in G} w(i,j) \geq \sum_{k \in N} w(k, \pi(k)) > \sum_{k \in N} w(k, b(k)).$$

So $< N \cup \{0\}, G >$ is not optimal. ■

Example 2.1 Figure 2.1 corresponds to a mountain situation $< N, \{0\}, A, w >$, where $N = \{1, 2, 3\}$, $A = \{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2)\}$ and $w(i,j) = 10i - 5j$ for each $(i,j) \in A$. Then the intrinsic height function h_0 is described by $h_0(i) = i$ for each $i \in N$. Since $b(1) = 0$, $b(2) = 1$, $b(3) = 2$, the tree $< N \cup \{0\}, T >$ with $T = \{(1,0), (2,1), (3,2)\}$ is an optimal 0-connecting tree with costs $10 + 15 + 20 = 45$. The payoff vector $B(N, \{0\}, A, w) = (10, 15, 20)$ corresponding to the situation where each

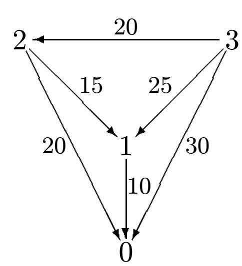

Figure 2: the mountain situation of example 2.1.

player i pays $w(i, b(i))$ will be called the Bird allocation (cf. Bird (1976)). In the next section we will see that the Bird allocation is a special core element of the cost game corresponding to the mountain situation.

3 Mountain situations and cooperative cost games

Recall that a cooperative cost game is an ordered pair $< N, c >$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $c : 2^N \to I\!R$ is the characteristic function, which assigns to each coalition $S \in 2^N$ a real number $c(S)$, and where $c(\emptyset) = 0$. The subgame of $< N, c >$ with player set $T \in 2^N \backslash \{\emptyset\}$ is the cooperative cost game $< T, c >$, where $c : 2^T \to I\!R$ is the restriction of $c : 2^N \to I\!R$. A core allocation of $< N, c >$ is a vector $x \in I\!R^N$ satisfying

(3.1) efficiency: $\sum_{i=1}^{n} x_i = c(N)$,

(3.2) stability: $\sum_{i \in S} x_i \leq c(S)$ for each $S \in 2^N$.

The core of $< N, c >$ (cf. Gillies (1953)) is denoted by $\text{Core}(N, c)$ and consists of all core allocations.

Sprumont (1990) introduced population monotonic allocation schemes (see also Thomson (1995)). A population monotonic allocation scheme (pmas) for a cost game $< N, c >$ is a scheme $[a_{S,i}]_{S \in 2^N \backslash \{\emptyset\}, i \in S}$, where $(a_{S,i})_{i \in S} \in \text{Core}(S, c)$ for each $S \in 2^N \backslash \{\emptyset\}$ and where the following monotonicity condition holds:

(3.3) $a_{S,i} \geq a_{T,i}$ for all $S, T \in 2^N$ and $i \in N$ with $i \in S \subset T$.

For further use we recall bounds for core elements:

(3.4) $M_i(N, c) \leq x_i \leq c(\{i\})$ for all $x \in \text{Core}(N, c)$ and all $i \in N$.

Here $M_i(N, c) = c(N) - c(N \backslash \{i\})$, the marginal contribution to the costs of N by player $i \in N$. Note that the second inequality in (3.4) is one of the stability inequalities in (3.2). For the first inequality in (3.4) note that

$$x_i = \sum_{k=1}^{n} x_k - \sum_{k \in N \backslash \{i\}} x_k = c(N) - \sum_{k \in N \backslash \{i\}} x_k \geq c(N) - c(N \backslash \{i\}) = M_i(N, c)$$

where the second equality follows from (3.1) and the inequality from (3.2) with $N \backslash \{i\}$ in the role of S. In general the core and the pmas-set $\text{PMAS}(N, c)$ may be empty. For the games to be introduced for mountain situations these sets will be non-empty as we will see.

Let $< N, \{0\}, A, w >$ be a mountain situation. Then the corresponding cooperative cost game $< N, c >$ is given by $c(\emptyset) = 0$ and for $T \in 2^N \backslash \{\emptyset\}$ the cost $c(T)$ of coalition T is the cost of the optimal 0-connecting tree in the mountain problem $< T, \{0\}, A(T), w_T >$, where $A(T) = \{(i, j) \in A \,|\, i \in$

$T, j \in T \cup \{0\}\}$, and $w_T : A(T) \to I\!R_+$ is the restriction of $w : A \to I\!R_+$ to $A(T)$. Note that for each $T \in 2^N \backslash \{\emptyset\}$,

$$c(T) = \sum_{k \in T} w(k, b_T(k)),$$

where

$$b_T(k) = \underset{l \in T \cup \{0\}:(k,l) \in A}{\operatorname{argmin}} w(k,l),$$

the cheapest connection point of k in $T \cup \{0\}$. The introduced number $b(k)$ in section 2 is equal to $b_N(k)$. It is easy to describe one core element of $< N, c >$. Take the Bird allocation (cf. Bird (1976)) $B \in I\!R^N$ with $B_k = w(k, b_N(k))$. Then B is a core element of $< N, c >$, since $c(N) = \sum_{k \in N} w(k, b_N(k)) = \sum_{k \in N} B_k$ by theorem 2.1. Further

$$c(T) = \sum_{k \in T} w(k, b_T(k)) \geq \sum_{k \in T} w(k, b_N(k)) = \sum_{k \in T} B_k$$

for each $T \in 2^N \backslash \{\emptyset\}$. This core element corresponds to the situation where the player $b_N(k)$ to which k connects himself does not ask a compensation for this service to k. But there are interesting other core allocations in general, corresponding to situations where compensation plays a role. In the description of these core elements the second cheapest connection point of k in $T \cup \{0\}$,

$$s_T(k) = \begin{cases} \underset{l \in (T \cup \{0\}) \backslash \{b_T(k)\}:(k,l) \in A}{\operatorname{argmin}} w(k,l) & \text{if } b_T(k) \neq 0 \\ 0 & \text{if } b_T(k) = 0, \end{cases}$$

plays a role.

Suppose player k wants to connect to $b_N(k) \neq 0$ and player $b_N(k)$ wants to ask a price $p_k \geq 0$ from k for connecting k. Which price can $b_N(k)$ ask for his service to k such that k connects with $b_N(k)$ and does not go e.g. to the second best connection point $s_N(k)$ for a connection? The price should be an element of the closed interval $[0, w(k, s_N(k)) - w(k, b_N(k))]$. A price p_k larger than $w(k, s_N(k)) - w(k, b_N(k))$ can lead to a connection to $s_N(k)$ and if $s_N(k) \neq 0$ even to a positive compensation for $s_N(k)$, e.g. $\frac{1}{2}(p_k - w(k, s_N(k)) + w(k, b_N(k)))$ and then both players k and $s_N(k)$ are better off. The allocations $(x_1, \ldots, x_n)$ corresponding to such competitive prices in the given closed interval turn out to be just the core allocations of the k-connection game $< N, c_k >$ to be introduced now.

The k-connection game $< N, c_k >$ is the cooperative cost game with $c_k(S) = 0$ if $k \notin S$ and $c_k(S) = w(k, b_S(k))$ otherwise. Note that, if $b_N(k) \neq 0$, then $M_{b_N(k)}(N, c_k) = c_k(N) - c_k(N \backslash \{b_N(k)\}) = w(k, b_N(k)) - w(k, s_N(k))$.

Theorem 3.1 *Let $< N, c_1 >, \ldots, < N, c_n >$ be the connection games corresponding to the mountain situation $< N, \{0\}, A, w >$ and $< N, c >$ the corresponding cost game. Then*

(i) $c = \sum_{k=1}^{n} c_k$

(ii) *Core*$(N, c) \supset P(N, c)$ *where* $P(N, c) = \sum_{k=1}^{n}$ *Core*(N, c_k)

(iii) *for every* $T \in 2^N \backslash \{\emptyset\}$ *we have Core*$(T, c_k) = \{0\}$ *if* $k \notin T$,

$$\text{Core}(T, c_k) = \{w(k, b_T(k))e^k - p(e^{b_T(k)} - e^k) \mid 0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))\}$$

if $k \in T, b_T(k) \neq 0$, *and Core*$(T, c_k) = \{w(k, 0)e^k\}$ *if* $k \in T, b_T(k) = 0$. *[Here* $e^k \in I\!R^T$ *is the k-th standard basis vector with k-th coordinate 1 and the other coordinates 0.]*

Proof (i) is a direct consequence of the definitions of $c, c_1, \ldots, c_n$.
(ii) follows from (i) because Core$(N, \cdot)$ is a superadditive correspondence.
(iii) Note that if $k \notin T$ then $< T, c_k >$ is the zero game and hence Core(T, c_k)= $\{0\}$. If $k \in T$ and $b_T(k) \neq 0$ then $M_i(T, c_k) = c_k(i) = 0$ if $i \in T \backslash \{k, b_T(k)\}$. For $x \in$ Core(T, c_k) we have, by (3.4), $x_i = 0$ for each $i \in T \backslash \{k, b_T(k)\}$. Further, by (3.1), $x_k + x_{b_T(k)} = c_k(T) = w(k, b_T(k))$, and, by (3.4), $w(k, b_T(k)) - w(k, s_T(k)) = M_{b_T(k)}(T, c_k) \leq x_{b_T(k)} \leq 0$. This implies that

$$\text{Core}(T, c_k) \subset \{w(k, b_T(k))e^k - p(e^{b_T(k)} - e^k) \mid 0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))\}.$$

For the reverse inclusion, note that for $x^p = w(k, b_T(k))e^k - p(e^{b_T(k)} - e^k)$ with $0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k))$ we have $x^p(T) = \sum_{i \in T} x_i^p =$

$w(k, b_T(k)) = c_k(T)$ and for $S \subset T$:

$$
\begin{array}{rcll}
x^p(S) & = & w(k, b_T(k)) & \\
 & = & c_k(S) & \text{if } \{k, b_T(k)\} \subset S \\
x^p(S) & = & 0 & \\
 & = & c_k(S) & \text{if } \{k, b_T(k)\} \cap S = \emptyset \\
x^p(S) & = & w(k, b_T(k)) + p & \\
 & \leq & w(k, s_T(k)) & \\
 & \leq & c_k(S) & \text{if } k \in S, b_T(k) \notin S, \text{ and} \\
x^p(S) & = & -p & \\
 & \leq & 0 & \\
 & = & c_k(S) & \text{if } k \notin S, b_T(k) \in S.
\end{array}
$$

So $x^p \in \mathrm{Core}(T, c_k)$.
If $k \in T$ and $b_T(k) = 0$ the statement can be proved in a similar way. ■

The subset $P(N, c)$ of $\mathrm{Core}(N, c)$ is the set of price supported core elements. In the next section we will show that elements x of $P(N, c)$ are pmas-extendable i.e. there exists a population monotonic allocation scheme $[a_{T,i}]_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ such that $a_{N,i} = x_i$ for each $i \in N$.

Example 3.1 Consider again the mountain situation of example 2.1. The cost game $< N, c >$ corresponding to this situation and the k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	(1, 2)	(1, 3)	(2, 3)	(1, 2, 3)
$c(S) =$	10	20	30	25	35	40	45
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	30	0	25	20	20

Note that $c = c_1 + c_2 + c_3$, $\mathrm{Core}(N, c_1) = \{(10, 0, 0)\}$, $\mathrm{Core}(N, c_2) = \mathrm{conv}\{(0, 15, 0), (-5, 20, 0)\}$, and $\mathrm{Core}(N, c_3) = \mathrm{conv}\{(0, 0, 20), (0, -5, 25)\}$.

4 Population monotonic cost allocation schemes

In mountain situations the set of agents involved in cooperation can vary. One can think of unoccupied houses, agents who want to stay alone, etc. Therefore it is interesting to know how to solve for each population $T \subset N$ the optimization problem and to have available a cost distribution vector for each $< T, c >$. The optimization problems are simple. In problem $< T, \{0\}, A(T), w_T >$ each player $i \in T$ connects to $b_T(i)$, the element in $T \cup \{0\}$ with the lowest connection cost. This leads to a cheapest connection of all members of T to the root 0.

For the cost problems stable allocation schemes $A = [a_{T,i}]_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ are interesting, where 'row' $(a_{T,i})_{i \in T} \in \text{Core}(T, c)$. We are especially interested in population monotonic allocation schemes (pmas), that is in stable allocation schemes with the monotonicity condition (3.3). If such a pmas is used larger coalitions are more interesting than smaller coalitions for everybody. The Bird allocation scheme $A^0 = [w(i, b_T(i))]_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ is an example of a pmas. To find other pmas-es it is interesting to note that $\sum_{k=1}^n \text{PMAS}(N, c_k) \subset \text{PMAS}(N, c)$, i.e. if $A^k \in \text{PMAS}(N, c_k)$ for each $k \in N$, then $\sum_{k=1}^n A^k \in \text{PMAS}(N, c)$. This motivates us to concentrate on $\text{PMAS}(N, c_k)$.

If $k \notin T$ then $< T, c_k >$ is the zero game and hence $\text{Core}(T, c_k) = \{0\}$. If $k \in T$ then it follows from theorem 3.1 (iii) that

$$\text{Core}(T, c_k) = \{x_T^\alpha \in I\!R^T \mid \alpha \in [0, 1]\},$$

where

$$x_T^\alpha = w(k, b_T(k))e^k + \alpha(w(k, b_T(k)) - w(k, s_T(k)))(e^{b_T(k)} - e^k)$$

if $b_T(k) \neq 0$, and $x_T^\alpha = w(k, 0)e^k$ if $b_T(k) = 0$. Note that the core has a unique element if $b_T(k) = 0$. The next theorem 4.1 shows that each core element x_N^α of $< N, c_k >$ can be extended to a pmas, namely A^α. Here $A^\alpha = [a_{T,i}^\alpha]_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ is the allocation scheme, where, for every $T \in 2^N \setminus \{\emptyset\}$,

$$(a_{T,i}^\alpha)_{i \in T} = \begin{cases} 0 & \text{if } k \notin T; \\ (x_N^\alpha)_{i \in T} & \text{if } k \in T \text{ and } b_N(k) \in T; \\ x_T^0 & \text{if } k \in T \text{ and } b_N(k) \notin T. \end{cases}$$

This cost allocation scheme corresponds to the situation where $k \in T$ pays his connection cost $w(k, b_T(k))$ and also as compensation α times the marginal contribution of $b_N(k)$ in T to $b_N(k)$ if $b_N(k) \in T$, and no compensation if $b_N(k) \notin T$. Note that 'column' k of A^0 equals 'column' k of the Bird allocation scheme. Note that in the rows T with $k \notin T$ we have a core element since 0 is the unique core element of $< T, c_k >$. Note moreover that in the rows T with $k \in T$ and $b_N(k) \notin T$ we also have core elements. It follows from the following lemma that also the rows with $k \in T$ and $b_N(k) \in T$ contain core elements. So A^α is a stable monotonic allocation scheme.

Lemma 4.1 *Let* $T \in 2^N$ *be such that* $k \in T$ *and* $b_N(k) \in T$. *Then* $(a^\alpha_{T,i})_{i\in T} = (x^\alpha_N)_{i\in T} \in Core(T, c_k)$.

Proof The only thing to show is that $-\alpha(w(k, b_N(k)) - w(k, s_N(k))) \in [0, w(k, s_T(k)) - w(k, b_T(k))]$. Note that

$$\begin{aligned} 0 &\leq -\alpha(w(k, b_N(k)) - w(k, s_N(k))) \\ &= \alpha(w(k, s_N(k)) - w(k, b_N(k))) \\ &\leq w(k, s_N(k)) - w(k, b_N(k)) \\ &= w(k, s_N(k)) - w(k, b_T(k)) \\ &\leq w(k, s_T(k)) - w(k, b_T(k)). \end{aligned}$$

At the last equality we used the fact that $b_N(k) = b_T(k)$ and at the last inequality that

$$\begin{aligned} w(k, s_N(k)) &= \min\{w(k,i) \mid i \in (N \cup \{0\}) \backslash \{b_N(k)\}, (k,i) \in A\} \\ &\leq \min\{w(k,i) \mid i \in (T \cup \{0\}) \backslash \{b_T(k)\}, (k,i) \in A\} \\ &= w(k, s_T(k)). \end{aligned}$$

■

Theorem 4.1 *For each* $\alpha \in [0, 1]$, A^α *is a pmas for* $< N, c_k >$.

Proof We noted above already that A^α is a stable allocation scheme. So, we only have to prove (3.3). Take $i \in N$, $S, T \in 2^N$ such that $i \in S \subset T$. We consider 3 cases.
(i) Suppose that $i \in S\backslash\{k, b_N(k)\}$. Then $a^\alpha_{S,i} = 0 \geq 0 = a^\alpha_{T,i}$ since the column $(a_{U,i})_{U\in 2^N\backslash\{\emptyset\}:i\in U}$ is a zero column.
(ii) Suppose that $i = b_N(k) \in S \subset T$. Then $a^\alpha_{S,b_N(k)} = a^\alpha_{T,b_N(k)} = \alpha(w(k, b_N(k)) - w(k, s_N(k)))$ if $k \in S$, $a^\alpha_{S,b_N(k)} = a^\alpha_{T,b_N(k)} = 0$ if $k \notin T$. If $k \notin S$ and $k \in T$ then $a^\alpha_{S,b_N(k)} = 0 \geq a^\alpha_{T,b_N(k)} = \alpha(w(k, b_N(k)) - w(k, s_N(k)))$.
(iii) Suppose that $i = k \in S \subset T$. Then $a^\alpha_{S,k} = a^\alpha_{T,k} = (x^\alpha_N)_k$ if $b_N(k) \in S$, and $a^\alpha_{S,k} = w(k, b_S(k)) \geq w(k, b_T(k)) = a^\alpha_{T,k}$ if $b_N(k) \notin T$. If $b_N(k) \notin S$ and $b_N(k) \in T$ then $a^\alpha_{S,k} = w(k, b_S(k)) \geq w(k, s_N(k)) \geq (1-\alpha)w(k, b_N(k)) + \alpha w(k, s_N(k)) = (x^\alpha_N)_k = a^\alpha_{T,k}$. ■

Theorem 4.2 *Each core element* $x \in P(N, c)$ *can be extended to a pmas of* $< N, c >$.

Proof Since $P(N, c) = \sum_{k=1}^n \text{Core}(N, c_k)$ in view of theorem 3.1 one can find $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ such that $x = \sum_{k=1}^n x_N^{k,\alpha_k}$ with $x_N^{k,\alpha_k} \in \text{Core}(N, c_k)$ for every $k \in \{1, \ldots, n\}$. Each x_N^{k,α_k} has a pmas extension A^{k,α_k} by theorem 4.1. Then $A = \sum_{k=1}^n A^{k,\alpha_k} \in \text{PMAS}(N, c)$. ■

Example 4.1 Reconsider the situation of example 3.1. Then $(10, 0, 0)$ is the unique core element of (N, c_1), the core element $(-2\frac{1}{2}, 17\frac{1}{2}, 0)$ in (N, c_2) is the midpoint of the core of (N, c_2), and $(0, -2\frac{1}{2}, 22\frac{1}{2})$ is the midpoint of the core of (N, c_3). So $x = (7\frac{1}{2}, 15, 22\frac{1}{2}) = (10, 0, 0) + (-2\frac{1}{2}, 17\frac{1}{2}, 0) + (0, -2\frac{1}{2}, 22\frac{1}{2}) \in P(N, c)$. Then $A^{1,\frac{1}{2}} + A^{2,\frac{1}{2}} + A^{3,\frac{1}{2}}$ is a pmas extending x. In matrix notation

$$A^{1,\frac{1}{2}} + A^{2,\frac{1}{2}} + A^{3,\frac{1}{2}} =$$

$$\begin{array}{c} \\ N \\ (12) \\ (13) \\ (23) \\ (1) \\ (2) \\ (3) \end{array}
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 10 & 0 & 0 \\ 10 & 0 & * \\ 10 & * & 0 \\ * & 0 & 0 \\ 10 & * & * \\ * & 0 & * \\ * & * & 0 \\ \hline \end{array}
+
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline -2\frac{1}{2} & 17\frac{1}{2} & 0 \\ -2\frac{1}{2} & 17\frac{1}{2} & * \\ 0 & * & 0 \\ * & 20 & 0 \\ 0 & * & * \\ * & 20 & * \\ * & * & 0 \\ \hline \end{array}
+
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 0 & -2\frac{1}{2} & 22\frac{1}{2} \\ 0 & 0 & * \\ 0 & * & 25 \\ * & -2\frac{1}{2} & 22\frac{1}{2} \\ 0 & * & * \\ * & 0 & * \\ * & * & 30 \\ \hline \end{array}
=
\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ \hline 7\frac{1}{2} & 15 & 22\frac{1}{2} \\ 7\frac{1}{2} & 17\frac{1}{2} & * \\ 10 & * & 25 \\ * & 17\frac{1}{2} & 22\frac{1}{2} \\ 10 & * & * \\ * & 20 & * \\ * & * & 30 \\ \hline \end{array},$$

a pmas extension of $(7\frac{1}{2}, 15, 22\frac{1}{2})$.

5 Bi-monotonic allocation schemes for connection games

A connection game (N, c_k) has the property that k is a veto player because $c_k(S) = 0$ for all S not containing k. For such games bi-monotonic allocation schemes (bi-mas) are introduced in Brânzei et al. (2000) (see also Voorneveld et al. (2000)). A bi-mas for such a game with a veto player is a stable allocation scheme with the property that the veto player is weakly better off and the other players weakly worse off in larger coalitions. Let us be more specific. An allocation scheme $B = [b_{T,i}]_{T \in 2^N \setminus \{\emptyset\}, i \in T}$ is a bi-monotonic allocation scheme for (N, c_k) if

(5.1) each row $(b_{T,i})_{i \in T} \in \text{Core}(T, c_k)$,

and for all $S, T \in 2^N$ with $k \in S \subset T$

(5.2) $b_{T,k} \leq b_{S,k}$

(5.3) $b_{T,i} \geq b_{S,i}$ for all $i \in S \setminus \{k\}$.

It turns out that for connection games bi-monotonic allocation schemes exist. Moreover, each core element of (N, c_k) can be extended to a bi-mas, as theorem 5.1 shows. For $\alpha \in [0,1]$, let $B^\alpha = [b^\alpha_{T,i}]_{T\in 2^N\setminus\{\emptyset\}, i\in T}$ be the allocation scheme with

$$(b^\alpha_{T,i})_{i\in T} = \begin{cases} x^\alpha_T & \text{if } k \in T, \\ 0 & \text{if } k \notin T. \end{cases}$$

Theorem 5.1 *For every $\alpha \in [0,1]$, B^α is a bi-mas extending x^α_N.*

Proof (i) In view of theorem 3.1 row T in B^α is a core element for each $T \subset N$ and row N equals x^α_N. So (5.1) holds.
(ii) To prove (5.2) note that for $S \subset T$ and $k \in S$ we have

(5.4) $\quad w(k, b_S(k)) \geq w(k, b_T(k)),$

(5.5) $\quad w(k, s_S(k)) \geq w(k, s_T(k)).$

Using (5.4) and (5.5) we obtain (5.2) as follows:

$$\begin{aligned} b^\alpha_{T,k} &= (1-\alpha)w(k, b_T(k)) + \alpha w(k, s_T(k)) \\ &\leq (1-\alpha)w(k, b_S(k)) + \alpha w(k, s_S(k)) \\ &= b^\alpha_{S,k}. \end{aligned}$$

(iii) To prove (5.3) for S, T with $i, k \in S \subset T$, $i \neq k$, we consider 3 cases: $i \neq b_S(k)$; $i = b_T(k)$; $i = b_S(k)$ and $i \neq b_T(k)$.
If $i \neq b_S(k)$, then $i \neq b_T(k)$, so $b^\alpha_{S,i} = b^\alpha_{T,i} = 0$.
If $i = b_T(k)$, then $i = b_S(k)$ and then

$$\begin{aligned} b^\alpha_{T,i} &= \alpha(w(k,i) - w(k, s_T(k))) \\ &\geq \alpha(w(k,i) - w(k, s_S(k))) \\ &= b^\alpha_{S,i}, \end{aligned}$$

where the inequality follows from (5.5).
If $i = b_S(k)$ and $i \neq b_T(k)$, then $b^\alpha_{S,i} = \alpha(w(k, b_S(k)) - w(k, s_S(k))) \leq 0 = b^\alpha_{T,i}$. ■

Example 5.1 Take the game of example 3.1. Then for $k = 3$ the bi-mas, corresponding to $\alpha = \frac{1}{2}$, is given by

	1	2	3
(123)	0	$-2\frac{1}{2}$	$22\frac{1}{2}$
(12)	0	0	*
(13)	$-2\frac{1}{2}$	*	$27\frac{1}{2}$
(23)	*	-5	25
(1)	0	*	*
(2)	*	0	*
(3)	*	*	30

6 Cost monotonicity

The Bird rule, which assigns to each mountain situation the corresponding Bird allocation, has an interesting monotonicity property, called cost monotonicity (cf. Kent and Skorin-Kapov (1997)). Here a cost allocation rule is called cost monotonic if the decrease (or increase) in the cost of any arc does not increase (or decrease) the cost of any player. Suppose a mountain situation $< N, \{0\}, A, w >$ changes to $< N, \{0\}, A, w' >$, where $w'(i,j) = w(i,j)$ for all $(i,j) \in A \backslash \{(k,l)\}$ and $w'(k,l) > w(k,l)$. Suppose that B and B' are the corresponding Bird allocations. Then, obviously, $B_i = B'_i$ for all $i \in N \backslash \{k\}$, and $B_k = w(k, b(k)) = B'_k$ if $b(k) \neq l$, while $B'_k > B_k$ if $b(k) = l$. So the Bird rule is cost monotonic. Allocation rules, where compensations for connections play a role do not have this cost monotonicity property. The reason is that if an arc increases so much in costs that there is a change of best connection points, the new connection point profits from the compensation and is better off.

Example 6.1 Consider again the mountain situation of example 2.1. Consider the Bird rule B and the rule E, where compensations of half of the marginal contribution take place. The Bird rule assigns to the mountain situation $(10, 15, 20)$ and E assigns the allocation $(7\frac{1}{2}, 15, 22\frac{1}{2})$. If we change the mountain situation such that the cost of $(3,2)$ raises to 40 then we obtain as allocations for B and E respectively $(10, 15, 25)$ and $(5, 17\frac{1}{2}, 27\frac{1}{2})$. In the rule E player 1 is better off in the second situation because of compensations from 2 players now.

7 Concluding remarks

We studied optimal connection problems and related cost sharing problems for mountain situations with the properties M.1, M.2 and M.3. Interesting results were that finding optimal connections was easy as well as giving one core element, the Bird allocation. Insight in core elements with compensations was obtained and also stable allocation schemes were given for situations where the involved houses vary.

If we drop M.3, then it is still easy to find an optimal 0-connecting tree, but in these situations there may be more. Again population monotonic allocations exist. If we consider mountain situations with more purifiers, then we get special minimum cost 0-connecting forest problems, where all houses are connected with at least one purifier, with properties M.1', M.2, and M.3. Here M.1' tells that each house can be connected directly with at least one purifier. Roughly speaking the obtained results can be extended to the forest situation.

Finally, we want to note that for general directed connection problems we cannot expect that a pmas exists. In fact in Norde et al. (2001) a 6-person directed connection situation is given without a pmas. So we were lucky to find a class of directed connection problems for which pmas-es exist. For undirected connection problems pmas-es exist (cf. Kent and Skorin-Kapov (1997) and Norde et al. (2001)).

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