Notation in Econometrics: A Proposal for a Standard

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Summary: This paper proposes a standard for notation in econometrics. It presents a fully integrated and internally consistent framework for notation and abbreviations, which is as close as possible to existing common practice and also obeys ISO regulations. The symbols used are instantly recognizable and interpretable, thus minimizing ambiguity and enhancing reading efficiency. The standard is designed in a flexible manner, thus allowing for future extensions.

Key words: Notation, Symbols, Econometrics, International Organization for Standardization (ISO).

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1 Introduction

Few things are as boring as questions of notation. Serious researchers should do serious research and not waste their time thinking about notation. The mathematician J.E. Littlewood said about Jordan that if he (Jordan) had four things on the same footing (such as $a$, $b$, $c$, $d$) they would appear as

$$a, \quad M_3^p, \quad \varepsilon_2, \quad \Pi_{1,2}^n;$$

see Bollobás (1986, p. 60).

On the other hand, many serious researchers did worry about notation. Jan Tinbergen propagated that ‘when you have an index to a certain variable you should use the capital letter as its upper limit.’ For example, $i = 1, \ldots, I$ and $j = 1, \ldots, J$, because this ‘was just a little detail that could help you a lot to see through things’ (Magnus and Morgan, 1987, p. 127).

In physics, engineering, and chemistry a serious attempt has been made to standardize symbols. The International Organization for Standardization (ISO) has published international regulations (ISO Standards Handbook, 1982) and the International Union of Pure and Applied Physics (IPU) has issued recommendations (CRC Handbook of Chemistry and Physics, 1988). These regulations are generally followed by the profession, with one major exception: the treatment of lowercase single-letter constants (such as the base of natural logarithms $e$ and the imaginary unit $i$—very often written as $e$ and $i$, contrary to ISO regulations) or operators (such as the derivative operator $d$—often written as $d$).\footnote{See Beccari (1997) for further discussion and some \LaTeX{} tricks for physicists and engineers.} It appears that the profession finds that single-letter lowercase mathematical symbols look odd. There are examples of this phenomenon in econometrics too: one often sees det($A$) for determinant, $E(x)$ for expectation, but $r(A)$ for rank.

Notation matters. A good and consistent notation helps in the understanding, communication and development of our profession. In the Renaissance, mathematics was written in a verbal style with $p$ for plus,
m for minus and R for square root. So, when Cardano (1501–1576) writes

\[ 5p : Rm : 15 \]
\[ 5m : Rm : 15 \]
\[ 25m : m : 15 \text{ qd est 40,} \]

he means \((5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40\), see Kline (1972, p. 260). There is no doubt that the development of good notation has been of great importance in the history of mathematics.

In this paper we attempt to harmonize the various practices in econometrics notation. It proposes a fully integrated and internally consistent framework for notation and abbreviations, which is as close as possible to existing common practice and also obeys ISO regulations. The symbols used are instantly recognizable and interpretable, thus minimizing ambiguity and enhancing reading efficiency. Using a common notation will save authors the effort to define their notation in every paper. Only special notation needs to be defined. We have tried to design our standard in a flexible manner, allowing for future extensions in specialized fields.

There are many problems in designing a consistent notation. Our hope is to provide a useful benchmark and starting point for an evolving process. The notation is \LaTeX{} oriented. Many \LaTeX{} definitions are provided, and the complete list of definitions can be downloaded from http://cwis.kub.nl/~few5/center/staff/magnus.

2 Vectors and matrices

Vectors are lowercase and matrices are uppercase symbols. Moreover, both vectors and matrices are written in bold-italic. The vectors \( \mathbf{a}, \mathbf{b}, \ldots, \mathbf{z} \) are produced by \( \backslash v\alpha, \backslash v\beta, \ldots, \backslash v\zeta \), and the matrices \( \mathbf{A}, \mathbf{B}, \ldots, \mathbf{Z} \) by \( \backslash m\alpha, \backslash m\beta, \ldots, \backslash m\zeta \).

Vectors can also be denoted by Greek lowercase letters: \( \alpha, \ldots, \omega \) (\( \backslash v\alpha, \ldots, \backslash v\omega \)), and matrices by Greek uppercase letters, such as \( \mathbf{\Gamma} (\backslash m\Gamma) \) or \( \mathbf{\Theta} (\backslash m\Theta) \).
We write

\[ a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \]

for an \( n \times 1 \) vector \( a \) and an \( m \times n \) matrix \( A \). If one has a choice, we recommend that \( m \geq n \).

We denote the \( n \) columns of \( A \) by \( a_{1}, a_{2}, \ldots, a_{n} \), and the \( m \) rows by \( a'_{1}, a'_{2}, \ldots, a'_{m} \), where transpose is denoted by a prime. The symbol \( \cdot \) is produced by \( \backslash \text{cdot} \) since \( \text{cdot} \) (\( \cdot \)) is too small and \( \text{bullet} \) (\( \circ \)) is too large. Hence,

\[
A = (a_{1}, a_{2}, \ldots, a_{n}), \quad A' = (a'_{1}, a'_{2}, \ldots, a'_{m}).
\]

A vector \( a \) denotes a column and \( a' \) denotes a row. Special vectors are:

- \( 0, 0_n \) null vector \((0,0,\ldots,0)'\) \( \downarrow \text{zeros} \)
- \( e, e_n \) sum vector \((1,1,\ldots,1)'\) \( \downarrow \text{ones} \)
- \( e_i \) \( i \)-th column of \( I_n \) \( \downarrow \text{e}_i \)

Special matrices are:

- \( 0, \ 0_{mn} \) null matrix of order \( m \times n \) \( \downarrow \text{zeros} \)
- \( I, I_n \) identity matrix of order \( n \times n \) \( \downarrow \text{I} \)

Note that the null vector \( 0 \) is smaller than the null matrix \( 0 \). We say that two or more matrices (vectors) are \textit{conformable} if their sum or product is defined. For example, the equation \( Ax = b \) only makes sense if the dimension of \( x \) equals the number of columns of \( A \) and the dimension of \( b \) equals the number of its rows. If this is the case then \( A, x \) and \( b \) are conformable.

Two vectors \( a \) and \( b \) for which \( a'b = 0 \) are \textit{orthogonal}. We also write \( a \perp b \) (\( \backslash \text{bot} \)). The column space of \( A \) is denoted \( \text{col}(A) \) (\( \backslash \text{col} \)) and denotes the set \( \{ x : x = Ac \text{ for some } c \neq 0 \} \). The null space of \( A \) is the
set \( \{ x : A x = 0 \} \). The null space of \( A' \) is denoted \( \text{col}^{-1}(A) \) and is called the \textit{orthogonal complement} of \( \text{col}(A) \). It defines the set \( \{ x : A' x = 0 \} \), which can also be written as \( \{ x : x \perp A \} \).

3 \ Operations on matrix \( A \) and vector \( a \)

The following standard operations are proposed.

\[
\begin{align*}
A' & : \quad \text{transpose} \\
A^{-1} & : \quad \text{inverse} \\
A^+ & : \quad \text{Moore-Penrose inverse} \\
A^- & : \quad \text{generalized inverse} \\
dg A, dg(A) & : \quad \text{diagonal matrix containing the diagonal elements of } A \\
diag(a_1, \ldots, a_n) & : \quad \text{diagonal matrix containing } a_1, \ldots, a_n \\
diag(A_1, \ldots, A_n) & : \quad \text{block-diagonal matrix with } A_1, \ldots, A_n \text{ on the diagonal} \\
A^2 & : \quad AA \\
A^{1/2} & : \quad \text{(unique) square root of positive semidefinite matrix} \\
A^p & : \quad p\text{-th power} \\
A^\dag & : \quad \text{adjoint (matrix)} \\
A^* & : \quad \text{complex conjugate} \\
A_k & : \quad \text{principal submatrix of order } k \times k \\
(A, B), (A : B) & : \quad \text{partitioned matrix} \\
\text{vec } A, \text{vec}(A) & : \quad \text{vec operator} \\
\text{vech } A, \text{vech}(A) & : \quad \text{vector containing } a_{ij} (i \geq j) \\
\text{rk}(A) & : \quad \text{rank} \\
\lambda_i, \lambda_i(A) & : \quad i\text{-th eigenvalue (of } A) \\
\text{tr } A, \text{tr}(A) & : \quad \text{trace} \\
\text{etr } A, \text{etr}(A) & : \quad \exp(\text{tr } A) \\
|A|, \det A, \det(A) & : \quad \text{determinant} \\
\|A\| & : \quad \text{norm of matrix } (\sqrt{\text{tr } A^* A}) \\
\|a\| & : \quad \text{norm of vector } (\sqrt{(a^* a)}) \\
A \geq B, B \leq A & : \quad A - B \text{ positive semidefinite} \\
\end{align*}
\]
\[ A > B, \quad B < A \quad \text{positive definite} \quad >, < \]
\[ A \otimes B \quad \text{Kronecker product} \quad \times \]
\[ A \odot B \quad \text{Hadamard product} \quad \odot \]
\[ K_{mn} \quad \text{commutation matrix} \]
\[ K_n \quad K_{nn} \]
\[ N_n \quad \frac{1}{2}(I_{n^2} + K_n) \]
\[ D_n \quad \text{duplication matrix} \]
\[ J_k(\lambda) \quad \text{Jordan block of order } k \times k \]

Ambiguity can arise between the symbol | \cdot | for determinant and the same symbol for absolute value, for example in the multivariate transformation theorem. This ambiguity can be avoided by writing |det A| for the absolute value of a determinant.

If we have a symmetric matrix \( A \) of order \( n \times n \), then the eigenvalues are real and can be ordered. We recommend the ordering

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \]

since there are many cases where it is desirable that \( \lambda_1 \) denotes the largest eigenvalue.

4 The linear regression model

We write the linear regression model \( y = X\beta + \varepsilon \) as

\[ y = \sum_{h=1}^{k} \beta_h x_{ih} + \varepsilon \]

or as

\[ y_i = x_{i1}^T \beta + \varepsilon_i \quad (i = 1, 2, \ldots, n) \]

or as

\[ y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{i k} + \varepsilon_i \quad (i = 1, 2, \ldots, n). \]

If there is a constant term this specializes to

\[ y_i = \beta_1 + \beta_2 x_{i2} + \cdots + \beta_k x_{i k} + \varepsilon_i \quad (i = 1, 2, \ldots, n). \]
In the two-variable case one can write

\[ y_i = \beta_1 + \beta_2 x_i + \varepsilon_i \quad \text{or} \quad y_i = \alpha + \beta x_i + \varepsilon_i, \]

but not \( y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \), since \( \beta_0 \) is often used for other purposes, in particular as the value of the parameter \( \beta \) under the null hypothesis.

The observations are typically indexed \( i = 1, \ldots, n \) (in cross sections) or \( t = 1, \ldots, T \) (in time series). If there are two cross sections one can use \( i \) and \( j \); if there are two time series one uses \( t \) and \( s \). There are \( k \) regressors (not \( K \)) indexed by \( h = 1, \ldots, k \). Acronyms and special symbols take precedence over index labels. For example, in defining the \( t \)-statistic one should not use \( t \) as a summation index, and in formulae involving the imaginary unit \( i \) confusion can be avoided by not using \( i \) as an index.

This formulation is not without controversy. Some authors write \( X_{ht} \) instead of \( x_{ih} \), which is unsatisfactory, since \( X \) is an \( n \times k \) matrix and hence in their formulation \( X_{ht} \) is the \( th \)-th element of \( X \). Some write \( \beta_0 \) for the first element of \( \beta \), if the regression contains a constant term, and then let \( k \) denote the number of ‘real’ regressors (so that \( X \) has \( k + 1 \) columns). We prefer to avoid this formulation for many reasons. It is convenient to always have \( k \) regressors independent of whether there is a constant term or not. Also, the inclusion of a constant does make an important difference, for example in potentially non-stationary time series, and it can translate into a ‘real’ variable such as a drift, which alters distributions and time paths.

Another issue is the disturbance term. We denote this by \( \varepsilon \) (\( \varepsilon \) for a scalar, \( \varepsilon \) for a vector) if the disturbances (or errors) are spherically distributed.\(^2\) If the errors are not spherical, we denote them by \( u \).

Estimators are random variables which say something about a fixed but unknown quantity, called a parameter. They are denoted by ‘hats’, such as \( \hat{\beta} \) (\( \hat{\beta} \)). If we have a second estimator of \( \beta \) this is denoted by a ‘tilde’: \( \tilde{\beta} \). The realization of an estimator is an estimate.\(^3\) The vector \( \varepsilon \) is spherically distributed if \( \varepsilon \) and \( H\varepsilon \) have identical distributions for every orthogonal matrix \( H \).

\(^2\)In general, we recommend to use \( \hat{\beta} \) and \( \tilde{\beta} \) as the default. Typing \( \hat{\beta} \) produces \( \hat{\beta} \), while \( \tilde{\beta} \) produces \( \tilde{\beta} \).
Predictors are like estimators, except that they say something about a random variable. They are also denoted by ‘hats’ (\(\hat{y}, \hat{\varepsilon}\)) or tildes (\(\tilde{y}, \tilde{\varepsilon}\)). The realization of a predictor is the ‘predicted value’.

The symbols \(R^2\) and \(\overline{R}^2\) denote the coefficient of determination and the adjusted coefficient of determination, respectively.

In the case of OLS (ordinary least squares), it is tradition to write \(b\) instead of \(\hat{\beta}\) for the OLS estimator \((X'X)^{-1} X'y\), \(e\) instead of \(\hat{\varepsilon}\) for the residuals, and \(s^2\) instead of \(\tilde{\sigma}^2\) for the OLS estimator of \(\sigma^2\). We prefer not to do so, in order to stress the randomness of the estimators (one often thinks of \(b\) as a vector of constants).

If \(a\) is a vector, say of order \(n\), then \(\overline{a}\) (\(\text{\textbar}\)) denotes the average of its components: \(\overline{a} = \bar{a}/\bar{a}\).

It is customary to write
\[
P_X = X(X'X)^+ X', \quad M_X = I_n - P_X
\]
where \(X\) has \(n\) rows. If there is no possibility of confusion, we can write \(M\) and \(P\) instead of \(M_X\) and \(P_X\). The matrix which puts a vector in deviation form is thus
\[
M_t = I_n - (1/n)a',
\]
and the vector \(M_t a\) denotes the vector \(a\) in deviation from its mean.

We denote a null hypothesis as \(H_0\) (\(\text{\textbar}\)) and an alternative as \(H_A\) (not \(H_a\) since \(a\) may be a scalar or may refer to ‘asymptotic’). The statement of \(H_0: R'\beta = c\) is preferred over \(R\beta = r\). In the latter formulation, the single-hypothesis case is usually written as \(w'\beta = r\) or \(r'\beta = r\), neither of which is ideal. However, if one writes \(R'\beta = c\), this specializes to \(r'\beta = c\) in the one-dimensional case. This has the additional advantage that we can use \(r\) to denote the number of restrictions (dimension of \(c\)). In the special case where \(R = I\), or where \(R\) is square and invertible, we usually write \(\beta = \beta_0\) rather than \(\beta = c\).

\[\text{In line with current practice, we write the estimator for } \sigma^2 \text{ as } \hat{\sigma}^2 \text{ (\(\hat{\sigma}\)) and not as } \tilde{\sigma}^2, \text{ although strictly speaking the latter is the correct notation.}\]
The GLS model is written

\[ y = X\beta + u, \quad u \sim N(0, \Omega). \]

We prefer the use of \( \Omega \) over \( \Sigma \), which can be confused with the summation symbol.

For the simultaneous equations model our starting point is the (univariate) linear regression model

\[ y_i = x_i' \beta + u_i \quad (i = 1, 2, \ldots, n). \]

This can be generalized to the multivariate linear regression model:

\[ y_i' = x_i' B + u_i' \quad (i = 1, 2, \ldots, n), \]

where \( y_i \) and \( u_i \) are random \( m \times 1 \) vectors and \( B \) is a \( k \times m \) matrix. The univariate case is obtained as a special case when \( m = 1 \). The simultaneous equations model provides a further generalization:

\[ y_i' \Gamma = x_i' B + u_i' \quad (i = 1, 2, \ldots, n), \]

where \( \Gamma \) is an \( m \times m \) matrix. This is the structural form of the simultaneous equations model. In matrix notation this becomes \( Y \Gamma = X B + U \). If \( \Gamma \) is invertible, we obtain the reduced form \( Y = X \Pi + V \), where \( \Pi = B \Gamma^{-1} \) and \( V = U \Gamma^{-1} \).

5 Greek symbols

Some Greek lowercase letters have variant forms and these can be used to mean different things than the usual letter. We have:

\[
\begin{align*}
\epsilon & \quad \text{\textbackslash epsilon, \textbackslash eps} \quad \varepsilon \quad \text{\textbackslash varepsilon, \textbackslash epsi} \\
\theta & \quad \text{\textbackslash theta} \quad \vartheta \quad \text{\textbackslash vartheta} \\
\pi & \quad \text{\textbackslash pi} \quad \varpi \quad \text{\textbackslash varpi} \\
\rho & \quad \text{\textbackslash rho} \quad \varrho \quad \text{\textbackslash varrho} \\
\sigma & \quad \text{\textbackslash sigma} \quad \varsigma \quad \text{\textbackslash varsigma} \\
\phi & \quad \text{\textbackslash phi} \quad \varphi \quad \text{\textbackslash varphi}
\end{align*}
\]
We shall use \( \varepsilon \) (\( \varepsilon \) for a scalar, \( \boldsymbol{\varepsilon} \) for a vector) for a disturbance term and \( \epsilon \) (\( \varepsilon \)) for an arbitrarily small positive number. Also, we use \( \theta \) (\( \theta \)) to denote a variable and \( \vartheta \) (\( \vartheta \)) for a function.

6 Mathematical symbols, functions and operators

Definitions, implications, convergence, and transformations are denoted by

\[ \equiv \quad \text{identity, equivalence} \quad \equiv \]
\[ a := b \quad \text{defines } a \text{ in terms of } b \]
\[ \Longrightarrow \quad \text{implies} \quad \Rightarrow \]
\[ \iff \quad \text{if and only if} \quad \iff \]
\[ \to, \longrightarrow \quad \text{converges to} \quad \to, \longto \]
\[ x \mapsto y \quad \text{transformation from } x \text{ to } y \quad \mapsto \]

We write \( f(x) \approx g(x) \) (\( \text{\textasciitilde{approx}} \)) if the two functions are approximately equal in some sense depending on the context. If \( f(x) \) is proportional to \( g(x) \) we write \( f(x) \propto g(x) \) (\( \text{\textasciitilde{propto}} \)). We say that ‘\( f(x) \) is at most of order \( g(x) \)’ and write \( f(x) = O(g(x)) \), if \( |f(x)/g(x)| \) is bounded above in some neighborhood of \( c \) (possibly \( \pm \infty \)), and we say that ‘\( f(x) \) is of order less than \( g(x) \)’ and write \( f(x) = o(g(x)) \), if \( f(x)/g(x) \to 0 \) when \( x \to c \). Finally, we write \( f(x) \sim g(x) \) (\( \text{\textasciitilde{sim}} \)) if \( f(x)/g(x) \to 1 \) when \( x \to c \). The two functions are then said to be ‘asymptotically equal’. Notice that when \( f(x) \) and \( g(x) \) are asymptotically equal, then \( f(x) \approx g(x) \) and also \( f(x) = O(g(x)) \), but not vice versa.

For example, when \( \phi \) and \( \Phi \) denote the p.d.f. and c.d.f. of the standard-normal distribution, respectively, we write the leading term (first term) of the asymptotic expansion

\[
\frac{\Phi(x)}{\phi(x)} \sim \frac{1}{|x|} \quad \text{as } x \to -\infty.
\]

\(^{5}\text{The ISO prescribes the symbol } \simeq \text{ (\( \text{\textasciitilde{simeq}} \)) for asymptotic equality, but } \sim \text{ is common practice in econometrics and statistics, even though the same symbol is also used for ‘is distributed as’.}\)
However, there are many good local approximations of this ratio which
are not necessarily asymptotically equal to it.

The usual sets are denoted as follows:

| N  | natural numbers 1, 2, ... |
| Z  | integers ..., −2, −1, 0, 1, 2, ... |
| Q  | rational numbers |
| R  | real numbers |
| C  | complex numbers |

Superscripts denote the dimension and subscripts the relevant subset.
For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ denotes the real plane, $\mathbb{R}^n$ the set of real $n \times 1$
>vectors, and $\mathbb{R}^{m \times n}$ the set of real $m \times n$ matrices. The set $\mathbb{R}_+^n$ denotes
>the positive orthant of $\mathbb{R}^n$, while $\mathbb{Z}_+$ denotes the set of positive integers
>(hence, $\mathbb{Z}_+ = \mathbb{N}$) and $\mathbb{Z}_{0,+}$ denotes the non-negative integers. Finally,
>$\mathbb{C}^{n \times n}$ denotes the set of complex $n \times n$ matrices.

Set differences are denoted by a backslash (\backslash). For example, $\mathbb{N} = \mathbb{Z}_{0,+}\setminus\{0\}$. Real-line intervals defined by $x$ in $a \leq x < b$ are
denoted by $[a, b)$. Occasionally it might be unclear whether $(a, b)$ indicates a real-line interval or a point in $\mathbb{R}^2$. In that case the interval
$a < x < b$ can alternatively be written as $[a, b]$.

Sequences are special ordered sets. They are delimited, as usual, by
braces (curly brackets). It is often convenient to write $\{Z_j\}_{j=1}^n$ (or simply
$\{Z_j\}$) for the sequence of matrices $Z_1, Z_2, \ldots, Z_n$.

Other symbols used are:

- $\in$ belongs to \in
- $\notin$ does not belong to \notin
- $\{x : x \in S, x \text{satisfies } P\}$ set of all elements of $S$ with property $P$
- $\subseteq$ is a subset of \subseteqq
- $\subset$ is a proper subset of \subset
- $\cup$ union \cup
- $\cap$ intersection \cap
- $\emptyset$ empty set \emptyset
- $A^c$ complement of $A$
\begin{align*}
\setminus A & \quad B \setminus A^c \\
\text{\textcircled{\textbullet}} S & \quad \text{interior of } S \\
S' & \quad \text{derived set of } S \\
\overline{S} & \quad \text{closure of } S \\
\partial S & \quad \text{boundary of } S
\end{align*}

We denote functions by
\begin{align*}
f : S \to T & \quad \text{function defined on } S \text{ with values in } T \\
f, g, \varphi, \psi, \vartheta & \quad \text{scalar-valued function} \\
f, g & \quad \text{vector-valued function} \\
F, G & \quad \text{matrix-valued function} \\
g \circ f, G \circ F & \quad \text{composite function} \\
g \ast f & \quad \text{convolution } (g \ast f)(x) = \int_{-\infty}^{\infty} g(y) f(x - y) \, dy
\end{align*}

For their differentials, derivatives and differences, we write
\begin{align*}
d & \quad \text{differential} \\
d^n & \quad n-\text{th order differential} \\
D_j \varphi(x) & \quad \text{partial derivative}, \frac{\partial \varphi(x)}{\partial x_j} \\
D_j f_i(x) & \quad \text{partial derivative}, \frac{\partial f_i(x)}{\partial x_j} \\
D_{jk} \varphi(x) & \quad \text{second-order partial derivative}, \frac{\partial^2 \varphi(x)}{\partial x_k \partial x_j} \\
D_{jk} f_i(x) & \quad \text{second-order partial derivative}, \frac{\partial^2 f_i(x)}{\partial x_k \partial x_j} \\
\varphi^{(n)}(x) & \quad n-\text{th order derivative of } \varphi(x) \\
D \varphi(x), \frac{\partial \varphi(x)}{\partial x'} & \quad \text{derivative of } \varphi(x) \\
D f(x), \frac{\partial f(x)}{\partial x'} & \quad \text{derivative (Jacobian matrix) of } f(x) \\
D F(X) & \quad \text{derivative (Jacobian matrix) of } F(X) \\
\partial \text{vec } F(X)/\partial (\text{vec } X)' & \quad \text{derivative of } F(X), \text{ alternative notation} \\
\nabla \varphi, \nabla f, \nabla F & \quad \text{gradient (transpose of derivative)} \\
H \varphi(x), \frac{\partial^2 \varphi(x)}{\partial x \partial x'} & \quad \text{second derivative (Hessian matrix) of } \varphi(x) \\
L, B & \quad \text{backward shift operator: } L x_t = x_{t-1} \quad (\text{\textbackslash L}, \text{\textbackslash B}) \\
\nabla & \quad \text{(backward) difference operator: } \nabla x_t = x_t - x_{t-1} \quad (\text{\textbackslash diff}) \\
\Delta & \quad \text{forward difference operator: }
\end{align*}
\[ \Delta x_t = x_{t+1} - x_t \]
\[
[f(x)]_a^b, \quad f(x)\big|_a^b \quad f(b) - f(a)
\]

Instead of \( \varphi^{(1)}(x) \) and \( \varphi^{(2)}(x) \), one can write the more common \( \varphi'(x) \) and \( \varphi''(x) \), but otherwise we prefer to reserve the prime for matrix transposes rather than derivatives. Notice the difference between the differencing operator \( \text{\textbackslash diff}(\forall) \) and the gradient \( \text{\textbackslash nabla}(\nabla) \).

We use \( L \) (or \( B \)) rather than \( \mathcal{L} \) for the lag operator in order to avoid confusion with the Laplace transform. This and other useful transforms are defined by

\[
\begin{align*}
\mathcal{F}\{\cdot\} & \quad \text{Fourier transform} \\
\mathcal{F}^{-1}\{\cdot\} & \quad \text{inverse Fourier transform} \\
\mathcal{L}\{\cdot\} & \quad \text{Laplace transform} \\
\mathcal{L}^{-1}\{\cdot\} & \quad \text{inverse Laplace transform} \\
\mathcal{M}\{\cdot\} & \quad \text{Mellin transform} \\
\mathcal{M}^{-1}\{\cdot\} & \quad \text{inverse Mellin transform}
\end{align*}
\]

Finally, various other symbols in common use are

\[
\begin{align*}
i & \quad \text{imaginary unit (\textbackslash iu)} \\
e, \exp & \quad \text{exponential (\textbackslash eu, \textbackslash exp)} \\
\log & \quad \text{natural logarithm (\textbackslash log)} \\
\log_a & \quad \text{logarithm to the base } a \\
! & \quad \text{factorial} \\
\delta_{ij} & \quad \text{Kronecker delta} \\
\sgn(x) & \quad \text{sign of } x (\textbackslash sgn) \\
\lfloor x \rfloor, \text{int}(x) & \quad \text{integer part of } x, \text{that is, largest integer } \leq x (\textbackslash floor, \textbackslash rfloor, \textbackslash i\textbackslash p) \\
|x| & \quad \text{absolute value (modulus) of scalar } x \in \mathbb{C} \\
x^* & \quad \text{complex conjugate of scalar } x \in \mathbb{C} \\
\Re(x) & \quad \text{real part of } x (\textbackslash Re) \\
\Im(x) & \quad \text{imaginary part of } x (\textbackslash Im) \\
\Gamma(x) & \quad \text{gamma (generalized factorial) function},
\end{align*}
\]
satisfying $\Gamma(x + 1) = x \Gamma(x)$

$B(x, y)$ beta function, $\Gamma(x)\Gamma(y)/\Gamma(x + y)$

$\mathbb{1}_K$ indicator function (use 1, not $I$): equals 1 if condition $K$ is satisfied, 0 otherwise

$B(c), B(c; r), B(C; r)$ neighborhood (ball) with center $c$ ($C$) and radius $r$

$\mathcal{V}^{n \times k}$ Stiefel manifold: set of real $n \times k$ matrices $X$ such that $X'X = I_k$ ($k \leq n$) ($\calV$)

$\mathcal{O}^n$ $\mathcal{V}^{n \times n}$, orthogonal group of dimension $n$ ($\calO$)

$\mathcal{O}_+^n$ proper orthogonal group of dimension $n$

($\text{orthogonal } n \times n \text{ matrices with determinant } +1$)

$\mathcal{S}^n$ $\mathcal{V}^{n \times 1}$, unit sphere in $\mathbb{R}^n$ ($\calS$)

The Stiefel manifold $\mathcal{V}^{n \times k}$ is also denoted as $\mathcal{V}^{k \times n}$ in the literature. We recommend the former notation which is in line with $\mathbb{R}^{n \times k}$.

7 Statistical symbols, functions and operators

The following symbols are commonly used.

$\sim$ is distributed as $\distr$

$\mathcal{L}$ is asymptotically distributed as $\text{adistr}$

$\Pr()$ probability $\Pr$

$\mathbb{E}()$ expectation $\mathbb{E}$

$\mathbb{E}(-|\cdot)$ conditional expectation

$\text{var}()$ variance (matrix) $\text{var}$

$\text{cov}(\cdot, \cdot)$ covariance (matrix) $\text{cov}$

$\text{corr}(\cdot, \cdot)$ correlation (matrix) $\text{corr}$

$L(\cdot)$ likelihood function

$l(\cdot)$ log-likelihood function $\ell$

$\varsigma$ score vector $\text{score}$

$\mathcal{H}$ Hessian matrix $\text{Hessian}$

$I$ (Fisher) information matrix $\text{Infmat}$

$\mathcal{F}_t$ filtration at time $t$ $\text{calF}$

$t$ $t$-statistic, $t$-value
\( \rightarrow, \rightarrow \) converges a.s. to, long to
\( \xrightarrow{p} \) converges in probabilitypto
\( \xrightarrow{d} \) converges in distributiondto
\( \xrightarrow{w} \) converges weaklyuto
\( \text{plim} \) probability limitplim
\( O_p(g(x)) \) at most of probabilistic order \( g(x) \)
\( o_p(g(x)) \) of probabilistic order less than \( g(x) \)

Notice that the symbol \( \rightarrow \) (\( \xrightarrow{} \)) indicates both convergence and a.s. convergence. The symbol \( \xrightarrow{w} \) for weak convergence is preferred to \( \Rightarrow \), which denotes logical implication. The matrix \( -\mathbf{H} \) is also called the observed information matrix, while its expectation \( \mathbf{I} := -\mathbb{E}(\mathbf{H}) \) is the expected information matrix.

The main distributions in statistics are denoted as follows.

\begin{align*}
\text{bin}(n, p) & \quad \text{binomial distribution (} \text{bin} \text{)} \\
\text{Po}(\mu) & \quad \text{Poisson distribution (} \text{Po} \text{)} \\
U(a, b) & \quad \text{uniform distribution (} \text{rU} \text{)} \\
N_m(\mu, \Omega) & \quad m\text{-dimensional normal distribution (} \text{rN} \text{)} \\
\text{LN}(\mu, \sigma^2) & \quad \text{lognormal distribution (} \text{LN} \text{)} \\
\phi(\cdot) & \quad \text{standard-normal p.d.f. (} \text{phi} \text{)} \\
\Phi(\cdot) & \quad \text{standard-normal c.d.f.} \\
\text{IN}_m(\mu_i, \Omega_i) & \quad \text{sequence } i = 1, 2, \ldots \text{ of independent} \\
& \quad m\text{-dimensional normal distributions} \\
\chi^2_n(\delta) & \quad \text{chi-squared distribution with } n \text{ d.f. and non-centrality parameter } \delta. \\
\chi^2_n & \quad \text{central chi-squared } (\delta = 0) \\
t_n(\delta) & \quad \text{Student distribution with } n \text{ d.f. and noncentrality } \delta \ (\text{rt}) \\
t_n & \quad \text{central } t \ (\delta = 0) \\
C(a, b) & \quad \text{Cauchy distribution (} \text{rC} \text{)} \\
F_{m,n}(\delta) & \quad \text{Fisher distribution with } m \text{ (numerator) and } n \text{ (denominator) d.f. and non-centrality } \delta \ (\text{rF}) \\
F_{m,n} & \quad \text{central } F \ (\delta = 0) \\
\Gamma(\alpha, \lambda) & \quad \text{gamma distribution}
\end{align*}
We use the word ‘expectation’ to denote mathematical expectation of a random vector \( x \), denoted \( E(x) \). The word ‘average’ refers to taking the average of some numbers: \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \). The word ‘mean’ which could indicate either is best avoided. Like ‘expectation’, the words ‘variance’ (\( \text{var} \)), ‘covariance’ (\( \text{cov} \)), and ‘correlation’ (\( \text{corr} \)) indicate population parameters. The corresponding sample parameters are called ‘sample variance’, ‘sample covariance’ and ‘sample correlation’.

The ‘standard deviation’ is the positive square root of the variance. If \( \theta \) is a parameter which we estimate by \( \hat{\theta} \), then this estimator is a random variable with a variance \( \text{var}(\hat{\theta}) \) and a standard deviation \( \sqrt{\text{var}(\hat{\theta})} \). In general, this standard deviation depends on unknown parameters. Both the estimator of the standard deviation and its realization are called the ‘standard error’. The \( t \)-statistic is a random variable (not necessarily Student distributed); its realization is the \( t \)-value.

## 8 Abbreviations and acronyms

- 2SLS: two-stage least squares
- 3SLS: three-stage least squares
- AR(\( p \)): autoregressive process of order \( p \)
- ARCH: autoregressive conditional heteroskedasticity
- ARIMA(\( p, d, q \)): autoregressive integrated moving-average process
- ARMA(\( p, q \)): autoregressive moving-average process
- a.s.: almost surely
- BAN: best asymptotically normal
- c.d.f.: cumulative distribution function
- c.f.: characteristic function
- c.g.f.: cumulant-generating function
- CLT: central limit theorem
- CUAN: consistent uniformly asymptotically normal
- d.f.: degrees of freedom
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
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<tbody>
<tr>
<td>DW</td>
<td>Durbin-Watson</td>
</tr>
<tr>
<td>FCLT</td>
<td>functional CLT (invariance principle)</td>
</tr>
<tr>
<td>FGLS</td>
<td>feasible generalized least squares</td>
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<tr>
<td>FIML</td>
<td>full-information maximum likelihood</td>
</tr>
<tr>
<td>f.m.g.f.</td>
<td>factorial moment-generating function</td>
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<tr>
<td>GLS</td>
<td>generalized least squares</td>
</tr>
<tr>
<td>GMM</td>
<td>generalized method of moments</td>
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<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>ILS</td>
<td>indirect least squares</td>
</tr>
<tr>
<td>I(d)</td>
<td>(fractionally) integrated process of order (d)</td>
</tr>
<tr>
<td>IV</td>
<td>instrumental variable</td>
</tr>
<tr>
<td>LAD</td>
<td>least absolute deviations</td>
</tr>
<tr>
<td>LIL</td>
<td>law of iterated logarithm</td>
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<tr>
<td>LIML</td>
<td>limited-information maximum likelihood</td>
</tr>
<tr>
<td>LLN</td>
<td>law of large numbers</td>
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<tr>
<td>LM</td>
<td>Lagrange multiplier</td>
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<tr>
<td>LR</td>
<td>likelihood ratio</td>
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<tr>
<td>LS[E]</td>
<td>least squares [estimator]; see also 2SLS, 3SLS, FGLS, GLS, ILS, NLS, OLS, RLS</td>
</tr>
<tr>
<td>MA(q)</td>
<td>moving-average process of order (q)</td>
</tr>
<tr>
<td>m.g.f.</td>
<td>moment-generating function</td>
</tr>
<tr>
<td>ML[E]</td>
<td>maximum likelihood [estimator]; see also FIML, LIML, QML</td>
</tr>
<tr>
<td>MSE</td>
<td>mean squared error</td>
</tr>
<tr>
<td>NLS</td>
<td>nonlinear least squares</td>
</tr>
<tr>
<td>OLS</td>
<td>ordinary least squares</td>
</tr>
<tr>
<td>p.d.f.</td>
<td>probability density function</td>
</tr>
<tr>
<td>QML[E]</td>
<td>quasi-maximum likelihood [estimator]</td>
</tr>
<tr>
<td>RLS</td>
<td>restricted least squares</td>
</tr>
<tr>
<td>r.v.</td>
<td>random variable</td>
</tr>
<tr>
<td>s.e.</td>
<td>standard error</td>
</tr>
<tr>
<td>SUR</td>
<td>seemingly unrelated regression</td>
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<tr>
<td>UMP</td>
<td>uniformly most powerful</td>
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<tr>
<td>W</td>
<td>Wald</td>
</tr>
</tbody>
</table>
9 Hopes, fears and expectations

Our hope is that this paper may contribute towards the establishment of a common notation in econometrics. Our fear is that it will not. We realize that it will be difficult to get consensus. The = sign for equality was first proposed in the middle of the 16th century, but 150 years later Bernoulli still used \(\alpha\) (stylized \(\alpha\), short for \(aequalis\)) in his \textit{Ars Conjectandi}. Thus, our expectation is that it could take 150 years before a common notation is adopted.

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References


