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*Publication date:*
2001

Center
for
Economic Research

No. 2001-05

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January 2001

ISSN 0924-7815

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January 15, 2001


### Abstract

Existing real options literature provides relatively little insight into the impact of structural changes of the economic environment on the investment decision of the firm. We propose a method to model the impact of a policy change on investment behavior in which, contrary to the earlier models based on Poisson processes, uncertainty concerning the moment of the change can be explicitly accounted for. Moreover, probabilities of the change depend on the state of the dynamic system, what ensures the consistency of the action of the policy maker. We model the policy change as an upward jump in the (net) investment cost, which is, for instance, caused by a reduction in the investment tax credit. The firm has an incomplete information concerning the trigger value of the process for which the jump occurs. We derive the optimal investment rule maximizing the value of the firm. It is shown that the impact of trigger value uncertainty is non-monotonic: the investment threshold decreases with the trigger value uncertainty for low levels of uncertainty, while the reverse is true for high uncertainty levels. Finally, we present policy implications for the authority that result from the firm's value-maximizing behavior.



Keywords: investment under uncertainty, policy change

JEL classification: C61, D81, G31


[*]This research was undertaken with support from the European Union's Phare ACE Programme 1997. The content of the publication is the sole responsibility of the authors and in no way represents the views of the Commission or its services. The authors would like to thank Kuno Huisman and Uli Hege for helpful comments and suggestions. All remaining errors are ours.

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# 1 Introduction

Corporate investment opportunities may be represented as a set of (real) options to acquire productive assets. In the literature it is widely assumed that the present values of cash ‡ows generated by these assets are uncertain and their evolution can be described by a stochastic process. Consequently, an appropriate identi…cation of the optimal exercise strategies for the real options plays a crucial role in capital budgeting and in the maximization of a …rm's value (cf. Lander and Pinches [13]). So far, existing real options literature provides relatively little insight into the impact of structural changes of the economic environment on the investment decision of the …rm. We propose a method to model such changes.

The modern theory of investment under uncertainty (Dixit and Pindyck [6]) provides tools for evaluating the …rm's investment opportunities and determining the value-maximizing investment rules. Within this theory, mainly use is made of two stochastic processes. The …rst process is called Brownian motion, or Wiener process, which can be applied to cases where an economic variable is continuous over time. Often, however, it is more realistic to model an economic variable as a process that makes infrequent but discrete jumps. Then use is made of Poisson (jump) processes. Under Poisson processes jumps always occur with a constant probability. Although it is convenient to model external shocks using a Poisson process, this technique has a major drawback. It does not allow for modeling situations, such as a policy change, where the shock occurs under speci…c circumstances.

We propose a new methodology that allows for modeling a structural change occurring after passing a certain trigger by an underlying variable. As an example consider a reduction of an investment tax credit which was previously analyzed by Hassett and Metcalf [11] (see also Dixit and Pindyck [6], Ch. 9) with the use of a Poisson process. Such a reduction is typically imposed by the authority when the economy is booming and an active investment policy is no longer needed or desired.[1] Hence, the moment of the reduction should depend on the state of the economy so that its probability should not be constant like under a Poisson process.[2]

Another example, where it is realistic to assume that the occurrence of the shock depends on the state of the economy, is a foreign direct investment (FDI) decision of a …rm that is aiming at the purchase of a local company from the government of a developing country (see Smets [18] and Cherian and Perotti [3] for a discussion of the e¤ects of strategic interactions and political risk). The government observes the performance of the local company and may decide to

[1] For example, after a period of fast economic growth, in 1999 Ireland announced an end to its special 10% rate for new foreign manufacturing and …nancial investors as one of the means to avoid "overheating" of the economy.

[2] Hassett and Metcalf [11] try to correct this by letting the arrival rate depend on the output price. But still it is then possible that an investment subsidy is reduced for low output prices, while the subsidy was maintained under high output prices. This kind of inconsistency in the authority's behavior is no longer possible under our approach.

increase the offering price for this company after it obtains higher profits.[3]

In the paper we introduce a possibility of an upward jump in the (net) investment cost. This jump will, for instance, be caused by a reduction of an investment tax credit. Rather than letting the probability of the jump to be constant, we propose to let the jump occur at the moment that an underlying variable reaches a certain trigger. Here, the underlying variable is the value of the investment project. The firm is not aware of the exact value of the trigger but it knows the probability distribution of the trigger instead. In the investment credit example, this corresponds to the situation where the firm has incomplete information about the authority's tax strategy. Taking into account consistent government behavior, the firm knows that a jump will not occur as long as the current value of the variable remains below the maximum that this variable has attained in the past. When the underlying variable reaches a new maximum and the jump does still not occur, the firm updates its conjecture about the value of the barrier.

Consequently, our objective is to determine the optimal timing of an irreversible investment when the investment cost is subject to change and the firm has incomplete information about the moment of the change. It is clear that the value of the project drops to zero at the moment that the investment cost jumps to infinity. However, we mainly consider scenarios where the cost of investment is still finite after the upward jump occurred. In this way this work generalizes Lambrecht and Perraudin [12] , Schwartz and Moon [17], and Berrada [2], where the value of the project drops to zero at the unknown point of time.

Our main results are the following. We derive an equation that implicitly determines the value of the project at which the firm is indifferent between investing and refraining from the investment. This value is the optimal investment threshold and it is shown that this threshold is decreasing in the hazard rate of the cost-increase trigger. For the most frequently used density functions it holds that, for a given value of the project, the hazard rate first increases and then decreases with trigger value uncertainty. This leads to the conclusion that the investment threshold decreases with the trigger value uncertainty when the uncertainty is low, while it increases with uncertainty for high uncertainty levels. Hence, for a policy maker interested in accelerating investment, an optimal level of uncertainty can be identified which is the level corresponding to

[3] The same idea can also be applied within the topic of technology adoption. In Farzin et al. [7] the arrival of a more efficient technology satisfies a Poisson process (this assumption is also adopted by Baudry [1] where the new technology has the advantage of being less polluting, and by Mauer and Ott [14] where maintenance and operation cost are lowered after the technological breakthrough). This way of modeling is satisfactory only when the firm has no insight at all in the innovation process of new technologies. If, instead, the firm could observe progress (but has no perfect information), a way to model it is to introduce a variable that stands for the state of technological progress. The firm is able to observe perfectly the realizations of this variable. As soon as the state of technological progress hits a certain barrier, which is ex ante unknown to the firm, the new technology is invented. This approach is similar to the one in Grenadier and Weiss [8] but there it was assumed that the value of the barrier is known beforehand.

the minimal investment threshold.

# 2 Value Maximizing Investment Rule

We apply the value-maximization criterion to determine the optimal investment rule of the firm. First, we consider the basic model of investment under uncertainty with no change in the investment cost. Subsequently, we develop a model which allows for an increase in the investment cost after the value of the project reaches a certain trigger.

## 2.1 Basic Model

We start by considering the basic model of investment under uncertainty. The model was first developed by McDonald and Siegel [15] and is extensively analyzed in Dixit and Pindyck [6], Ch. 5. The general problem is to find the optimal timing of an irreversible investment, $I$, given that the value of investment project, $V$, follows a geometric Brownian motion:

$$dV_t = \alpha V_t dt + \sigma V_t dw_t. \tag{1}$$

The parameter $\alpha$ denotes the deterministic drift parameter, $\sigma$ is an instantaneous standard deviation and $dw_t$ is an increment of a Wiener process. Within this setting a firm's investment opportunity is a perpetual American option with an exercise price equal to $I$ and where $V$ is the value of the underlying asset of $V$.[4] In our case $I$ denotes a lump sum payment needed to undertake the project. The firm is assumed to be risk-neutral.[5] After making the payment, the firm owns the project, which generates a present value of cash flows $V$. The firm maximizes the expected present value of cash flow by choosing the optimal $V$ at which the project is undertaken. A well-described procedure (see Dixit and Pindyck [6]), involving the use of Ito's lemma and solving a differential equation under the corresponding value-matching and smooth-pasting conditions, yields the value of the optimal investment threshold, $V_m$:

$$V_m = \frac{\beta}{\beta - 1} I; \tag{2}$$

where

$$\beta = -\frac{\alpha}{\sigma^2} + \frac{1}{2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1; \tag{3}$$

and $r$ is an instantaneous interest rate.[6] For the value of the investment opportunity, $W(V)$; it holds that

$$W(V) = E\left[\int_{T_m}^{\infty} V_t (r - \alpha) e^{-rt} dt - I e^{-rT_m}\right] = (V_m - I)\left(\frac{V}{V_m}\right)^{\beta}; \tag{4}$$

[4] To simplify notation, we skip the time subscripts whenever it does not yield ambiguity.

[5] Alternatively, we can apply the replicating portfolio argument.

[6] The problem has a finite solution for $\alpha < r$.

where $T_m$ is the first passage time corresponding to the threshold $V_m$.[7] By following Dixit [4] and substituting variables, we obtain

$$E[T_m] = -\frac{1}{\alpha - \frac{1}{2}\sigma^2} \ln\left(\frac{V}{V_m}\right). \tag{5}$$

There are two factors determining the value of the investment opportunity. The first factor, $V_m - I$, corresponds to the net payoff realized at the time of the optimal exercise. The second one, often referred to as a probability-weighted discount factor, $\left(\frac{V}{V_m}\right)^{\beta}$, allows for translating the future payoff from the investment opportunity into its present value.

The value of the optimal investment threshold is positively related to the volatility of the project's value (the higher it is, the higher $V$ must be reached for the project to be undertaken) and negatively related to its growth rate. $W(V)$ increases in the volatility of the value of the project ($\beta$ is a decreasing function of $\sigma$ and $W$ is decreasing in $\beta$) what results from the convex payoff of the investment opportunity. $W$ is increasing in the growth rate, $\alpha$, since the effective discount rate of future cash flows decreases linearly in $\alpha$.

## 2.2 Model with Switching Cost

We introduce an upward change in the investment cost that occurs if the project becomes more valuable.[8] This change may be attributed to an action of the authority such as reducing the investment tax credit or increasing the offering price for a privatized enterprise. Moreover, it can be a result of an arrival of a competitive firm offering a higher bid for a particular project (as soon as its value is sufficiently high).

The investment cost jumps upwards at the moment that the project value becomes sufficiently large. In case of the investment tax credit it is no longer needed to stimulate investment policy, in case of FDI the government simply requires a higher price for an asset that is worth more, and in case of a competitive firm it holds that often potential buyers are attracted when the project value increases.

Let us denote by $V^*$ such a realization of the process for which the (net) investment cost changes with probability 1 from $I_l$ to $I_h$, where $I_h > I_l$. At this stage we assume that $I_h$ is deterministic. Later we consider a straightforward extension to the stochastic case and discuss its implications. We assume that the firm knows only the distribution, $F(V)$, of the cost-increase trigger.[9] The

[7] The expected timing of reaching the threshold can only be calculated under the condition $2\alpha > \sigma^2$ (cf. Harrison [9]).

[8] The obtained results also hold if, instead, a downward jump in $V$ is considered.

[9] This assumption is consistent with Lambrecht and Perraudin [12] who analyze a duopoly game under incomplete information. In their model the investment opportunity deteriorates completely for some realization of the process and the firms know only the density function generating the arrival. In the Lambrecht and Perraudin framework the deterioration of the investment opportunity results from a competitive entry. Our model allows for a partial deterioration of the value of the investment opportunity what is captured by setting $I_h < \infty$.

key assumption here is that the distribution function governing the change is stationary over time. If by time $\hat{t}$ the investment cost has not increased for $V \le \bar{V}$, where $\bar{V}$ is the highest realization of the process so far, the cost will not increase at any $t > \hat{t}$ for which $V(s) \le \bar{V}$ for all $s \le t$. Hence, the probability of the jump in investment cost is a function of $V$ alone.

In order to restrict our analysis to the most relevant case, we impose the following assumptions on the values of the variables used in the model:

$$\begin{cases} V_0 < I_l; & \text{(i)} \\ I_l < V^*; & \text{(ii)} \\ V^* < \frac{\beta}{\beta - 1} I_l; & \text{(iii)} \\ (V_h - I_h)\left(\frac{V^*}{V_h}\right)^{\beta} < V^* - I_l; & \text{(iv)} \end{cases} \tag{6}$$

where $V_0$ denotes the initial value of the project and $V_h$ is the unconditional optimal investment threshold corresponding to the cost $I_h$. The assumption (i) means that the initial value of the project is sufficiently low to exclude an immediate investment. Violating either (ii) or (iii) would imply that the investment would be undertaken either at a high cost or at a low cost, respectively, with probability one.[10] Finally, (iv) ensures that ex post it is never optimal to wait with investing until the upward change in cost occurs.

### 2.2.1 Value of Investment Opportunity

Since the cost-increase trigger is not known beforehand, two scenarios are possible. In the first scenario, the investment occurs before the change in the price of the asset, and, in the second scenario, the investment takes place after the upward change. Consequently, the value of the investment opportunity reflects the structure of the expected payoffs:

$$\begin{aligned} W_s(V; \bar{V} | I = I_l) &= p_s(\bar{V}) E\left[\int_{T_s}^{\infty} V_t (r - \alpha) e^{-rt} dt - I_l e^{-rT_s}\right] + \\ &+ \left(1 - p_s(\bar{V})\right) E\left[\int_{T_h}^{\infty} V_t (r - \alpha) e^{-rt} dt - I_h e^{-rT_h}\right]; \end{aligned} \tag{7}$$

where $p_s(\bar{V})$ is a conditional (on the highest realization of $V$, $\bar{V}$) probability that the investment cost will increase after the investment is made optimally, and $T_s$ and $T_h$ denote first passage time corresponding to the optimal investment threshold at the low and at the high cost, respectively. Their expected values can be calculated in a similar fashion as (5). After rearranging and including these expected values, we obtain the following maximization problem allowing

[10] This would degenerate the problem to the basic McDonald and Siegel [15] model.

for finding the optimal investment threshold:

$$
\begin{aligned}
W_s(V;\bar{V} | I = I_l) &= \max_{V_s} (V_s - I_l) \left( \frac{V}{V_s} \right)^{\beta} \frac{1 - F(V_s)}{1 - F(\bar{V})} + \\
&+ (V_h - I_h) \left( \frac{V}{V_h} \right)^{\beta} \left( 1 - \frac{1 - F(V_s)}{1 - F(\bar{V})} \right) . \qquad (8)
\end{aligned}
$$

$V_s$ is the optimal investment threshold in case the investment takes place before the change in cost, $\bar{V}$ is the highest realization of the process so far and $F(.)$ denotes the cumulative density function of the cost-increase trigger. Hence, $\frac{1-F(V_s)}{1-F(\bar{V})}$ is the probability that the jump in the investment cost will not occur by the moment $V$ is equal to $V_s$, given that the jump has not occurred for $V$ smaller than $\bar{V}$. Equation (8) is interpreted as follows: the value of the investment opportunity is equal to the weighted average of the values of two investment opportunities. They correspond to the investment cost $I_l$ and $I_h$, respectively, given that investment is made optimally (at $V_s$ if the price is still equal to $I_l$ and at $V_h$ if the upward change has already occurred).[11]

The value of the investment opportunity depends on the highest realization of the process, $\bar{V}$. A higher $\bar{V}$ (thus a one closer to $V_s$) implies a lower probability of the trigger falling into the interval $(\bar{V}; V_s)$ and, as a consequence, a higher probability of making the investment at the lower cost, $I_l$. In order to calculate the value of the investment opportunity, first we need to establish the value of $V_s$ by solving the maximization problem.

### 2.2.2 Optimal Timing of Investment

The optimal investment threshold, $V_s$, is determined by maximizing the value of the investment opportunity or the RHS of the Equation (8).

**Proposition 1** Under the sufficient condition that

$$h'(V_s)V_s + h(V_s) \geq 0; \qquad (11)$$

[11] It is worth pointing out that for $I_h \to \infty$ the value of the investment opportunity boils down to:

$$W_s(V;\bar{V} | I = I_1) = \max_{V_s} (V_s - I_1) \left( \frac{V}{V_s} \right)^{\beta} \frac{1 - F(V_s)}{1 - F(\bar{V})}; \qquad (9)$$

what directly corresponds to the result of Lambrecht and Perraudin [12]. In the other limiting case, i.e. for $I_h \to I_l$, the value of investment opportunity converges to

$$W_s(V;\bar{V} | I = I_l) = (V_l - I_l) \left( \frac{V}{V_l} \right)^{\beta} \qquad (10)$$

which is the formula obtained by McDonald and Siegel [15].

the investment is made optimally at $V_s$ which is a solution to the following equation:

$$h(V_s)V_s^2 + (\beta - 1)V_s - (V_s h(V_s) + \beta)I_l - h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^{\beta}}\frac{V_s^{\beta+1}}{I_h^{\beta-1}} = 0; \qquad (12)$$

where $h(x) = \frac{F'(x)}{1-F(x)}$ denotes the hazard rate.[12]

Proof. See Appendix. ■

A sufficient condition for (11) to hold is that the hazard rate has to be non-decreasing.[13] This condition (11) is satisfied for most of the common density functions as, e.g., exponential, uniform and Pareto.[14]

# 3 Solution Characteristics

In this section we analyze the sensitivity of the optimal threshold with respect to the changes in the parameters characterizing the dynamics of the project value. Moreover, we determine the direction of the impact of the changes in the investment costs in both scenarios. Subsequently, we examine how the uncertainty concerning the moment of imposing the change influences the firm's optimal investment rule.

## 3.1 Changing Parameters of Investment Opportunity

We are interested in how potential changes in the characteristics of the investment opportunity influence the optimal investment rule. Therefore we establish the sign of relationships between the value of $V_s$ and the parameters capturing such features of the project (via $\beta$) as its growth rate, $\alpha$, volatility, $\sigma$, and the interest rate, $r$. Moreover, we analyze the impact of the changes in a current investment cost, $I_l$, as well as of a shift in the expectations about its future value, $I_h$. For this purpose we formulate the following proposition.

Proposition 2 The effects on the investment threshold level of the changes in

[12] In our case, the hazard rate has the following interpretation. The probability of the upward change in the investment cost during the nearest increment of the value of the project, $dV$, (given that the cost-increase has not occurred by now) is equal to the appropriate hazard rate multiplied by the size of the value increment, i.e. to $h(x;\cdot)dV$.

[13] More precisely, the relevant hazard rate has to be "not too fastly decreasing" so that the component with a negative derivative is not greater (in absolute terms) than the value of the function.

[14] In fact, the hazard rate based on the Pareto function is decreasing at an order of $1/x$ and the property (11) is still met.

the different parameters are as follows:

$$\frac{\partial V_s}{\partial I_l} > 0; \quad (13)$$

$$\frac{\partial V_s}{\partial I_h} < 0; \quad (14)$$

$$\frac{\partial V_s}{\partial \beta} < 0; \quad (15)$$

$\forall I_l; I_h$ satisfying $0 < I_l < I_h$; $\forall \beta \in (1; r/\alpha)$ if $\alpha > 0$ and $\forall \beta \in (1; \infty)$ if $\alpha \leq 0$:

**Proof.** See Appendix. ■

Consequently, the optimal threshold (*ceteris paribus*) increases in the initial investment cost and decreases in the size of the potential cost-increase as well as in the parameter $\beta$. The latter implies that the threshold increases with uncertainty of the value of the project and decreases with the wedge between interest rate and the project's growth rate. All these results are intuitively plausible.

## 3.2 Impact of Policy Change

The optimal investment rule depends not only on the characteristics of the project itself but also on the firm's conjecture about the probability distribution underlying the expected policy change. The parameters of this distribution can be influenced by actions of the authority. For instance, an information campaign about the expected changes in investment tax credit will lead to a reduction of the variance of the distribution underlying the value triggering the change. Therefore, it is important to know how the changes in the uncertainty related to the project value triggering the jump in the investment cost influence the firm's optimal investment rule. Knowing that the firms are going to act optimally, the authority can implement a desired policy, which is, for instance, accelerating the investment expenditure by changing the level of the firms' uncertainty about the tax strategy.

### 3.2.1 Hazard Rate

The hazard rate of the arrival of the cost-increase trigger is one of the basic inputs for calculating the optimal investment threshold. Although it is exogenous to the firm, it may well be controlled by another party such as the authority. Here, we determine the impact of its change on the firm's investment rule. Later, we discuss some of the policy implications of the obtained result.

After applying the envelope theorem (see Appendix) to the LHS of (12), we can formulate the following proposition.

**Proposition 3** *The optimal investment threshold is decreasing in the corresponding hazard rate, i.e. the following inequality holds:*

$$\frac{\partial V_s}{\partial h} < 0; \quad (16)$$

$\forall h \in (0;\infty)$:

This result implies that an increasing risk of the switch leads to an earlier optimal exercise. The intuition is quite simple: an increasing risk of a partial deterioration of the investment opportunity after a small appreciation in the project value decreases the value of waiting. Consequently, (16) implies that for any parameter of the density function underlying the jump, denoted by a, the following condition holds:

$$\forall a \quad \operatorname{sgn} \frac{\partial h}{\partial a} = -\operatorname{sgn} \frac{\partial V_s}{\partial a}. \tag{17}$$

Using (17) we can establish how the hazard rate is affected by changes in the parameters of the distribution underlying the occurrence of the jump. It is easy to show that, in the relevant interval, the hazard rate is monotonic in $V^*$ which denotes the mean of the corresponding density function.[15]

### 3.2.2 Trigger Value Uncertainty

We analyze the impact of the uncertainty related to the value of the cost-increase trigger on the optimal investment threshold. For this purpose, the concept of a mean-preserving spread (see Rotschild and Stiglitz [16]) is applied. Following Proposition 3, we know that the optimal investment threshold is monotonic in the hazard rate corresponding to the trigger. Hence, what is left is to establish the sign of the relationship between the hazard rate and the uncertainty related to the value of the trigger.

If the cost-increase trigger is known with certainty, the investment is made optimally at an infinitesimal instant before $V^*$ is reached. At this point, the hazard rate is zero (there is no risk that the cost increases before the optimal threshold is reached). As the uncertainty marginally increases, the hazard rate is affected by: 1) an increase in the value of the density function, $f(V)$, underlying the trigger, and 2) a change in the value of the survival function, $1 - F(V)$. It is easy to verify that, for the most frequently used density functions, such as normal, uniform, exponential and Pareto, the value of the hazard rate, for any $V \in [V_0; V^*)$, first increases and then decreases in the mean-preserving spread. An example for the normal density function is shown on Figure 1.[16]

[15] This property holds for the most frequently used density functions, such as normal, uniform, exponential and Pareto.

[16] Although the concepts of the mean-preserving spread and increased standard deviation are, in general, not equivalent, they may be treated as such for the types of density functions referred to in this paper.

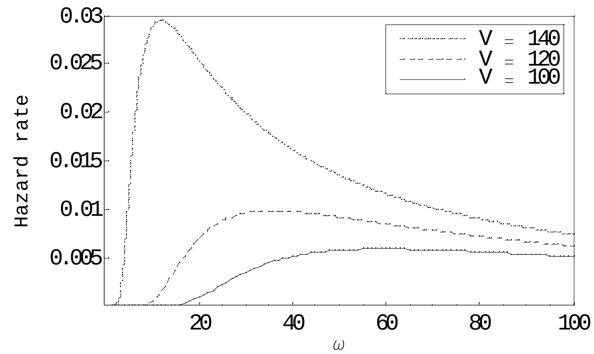


Figure 1

Figure 1. The relationship between the hazard rate and standard deviation of a normal density function with a mean equal to 150. Hazard rates are plotted for $V = 100$, 120 and 140.

Moreover, for each degree of the trigger value uncertainty, there exists such a value of $V < V^*$, say $\tilde{V}$, that for $V \in [V_0; \tilde{V})$ the hazard rate increases, and for $V \in (\tilde{V}; V^*)$ decreases, in this uncertainty. This form of the relationship between the hazard rate and the uncertainty implies (via Proposition 3) that $V_s$ decreases in the uncertainty if it falls into the interval $[V_0; \tilde{V})$ and increases otherwise, as depicted in Figure 2.

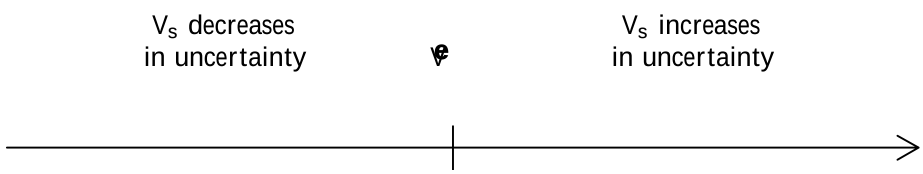


Figure 2

Figure 2. The relationship between trigger value uncertainty and the optimal investment threshold.

Consequently, in order to determine the sign of the effect of uncertainty on $V_s$, we need to establish the relative position of $V_s$ with respect to $\tilde{V}$. Let us denote the standard deviation of a density function underlying the cost-increase trigger as $\sigma$. Since the expression for $V_s$ is already known (see (12)), all we have to calculate is $\tilde{V}$ as a function of $\sigma$, such that, for each pair $(V; \sigma)$, the following condition holds:[17]

$$\frac{\partial h}{\partial \sigma} = 0. \tag{18}$$

[17] Although $\tilde{V}(\sigma)$ cannot be written explicitly in a general form, its values corresponding to a given density function may be easily found numerically.

For most frequently used density functions it can be shown that $\tilde{V}$ decreases in uncertainty. For a relatively low degree of uncertainty, it holds that $V_s < \tilde{V}$. Since for $V < \tilde{V}$ the hazard rate increases in $\omega$, $V_s$ is moving to the left when the uncertainty rises. After the uncertainty reaches a critical level, say $\omega^e$, at which $V_s = \tilde{V}$; the hazard rate at $V_s$ is decreasing in $\omega$ and the optimal threshold begins to increase. This implies that optimal investment threshold attains the minimum for $\omega = \omega^e$. Now, we are able to formulate the following proposition.

**Proposition 4** For density functions such that

$$\begin{aligned} &\lim_{\omega \to \infty} f(V;\cdot) = 0; \forall V \quad \text{and} \\ &f(V;\cdot) \text{ is unimodal,} \end{aligned} \tag{19}$$

there exists a non-monotonic relationship between the optimal investment threshold and the trigger value uncertainty. At a low degree of uncertainty, the marginal increase in uncertainty leads to an earlier optimal investment. The reverse is true for a high degree of uncertainty. There exists a unique point $\omega^e$, such that $V_s(\omega^e) = \tilde{V}(\omega^e)$; which separates the areas of low and high uncertainty levels.

Figures 3 and 4 show the relationship between the uncertainty, $\omega$, and the optimal investment threshold.

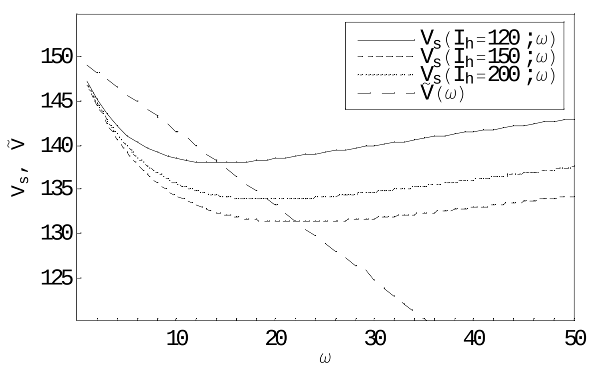


Figure 3

Figure 3. The relationship between the uncertainty, $\omega$, and the optimal investment threshold, $V_s$, for different sizes of the high investment cost ($I_h = 120; 150$ and $200$). The values are calculated for a normal density function with mean 150. An intersection of $V_s$ and $\tilde{V}$ corresponds to the minimal investment threshold, $V_s(\omega^e)$. The parameters of the underlying process are: $\alpha = 0$; $r = 0.025$ and $\sigma = 0.1$.[18]

[18] This set of parameters is used in Dixit [5], and Lambrecht and Perraudin [12] (we rescale the investment cost with the factor 100).

In Figure 3 it can be seen that the optimal investment threshold is first decreasing and then increasing in the uncertainty concerning the value of the trigger. The minimum is always reached when $V_s(\omega)$ intersects $\hat{V}(\omega)$. The hazard rate increases in $\omega$ in the area located to the south-west from $\hat{V}(\omega)$ and decreases in the north-eastern region. The opposite holds for $V_s$. Moreover, the optimal threshold is higher if the expected change in the investment cost is smaller (cf. Proposition 2).

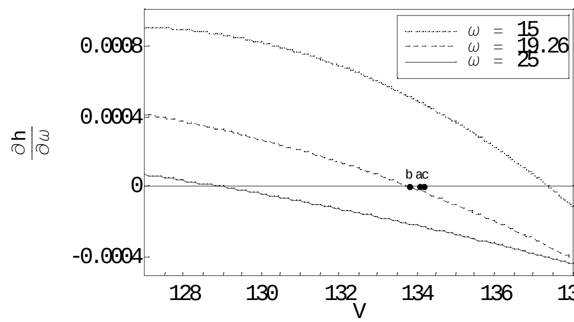


Figure 4

Figure 4. The relationship between $V$ and the derivative of the hazard rate with respect to the trigger value uncertainty. The optimal investment thresholds for $I_h = 150$ and different uncertainty levels are shown on the horizontal axis (Point a corresponds to $V_s(15)$, b to $V_s(\omega^e = 19.26)$ and c to $V_s(25)$. The values are calculated for a normal density function with mean 150. The parameters of the underlying process are: $\alpha = 0$, $r = 0.025$ and $\sigma = 0.1$.

In Figure 4 it can be noticed that the point, $\hat{V}$, at which the derivative of the hazard rate is equal to zero moves to the left when the trigger uncertainty increases. As long as $V_s < \hat{V}$, the optimal threshold also moves to the left (cf. the location of $V_s(15)$). When the standard deviation is equal to 19.26, $V_s$ equals $\hat{V}$. After a further increase in the uncertainty, $\hat{V}$ continues moving to the left and $V_s$ starts moving to the right (cf. $V_s(25)$). For a sufficiently high degree of uncertainty, $V_s$ exceeds $V^*$ and for the uncertainty tending to infinity, $V_s$ converges to the unconditional threshold $V_I$.[19] This fact has implications for the optimal investment tax credit policy, discussed as an example in the subsequent section.

[19] The necessary and sufficient condition for $\lim_{\omega\to\infty} V_s = V_s$ is $\lim_{\omega\to\infty} h(V_s;\cdot) = 0$.

# 4 Implications for the Investment Credit Tax Policy

In our setting, a policy implemented by the authority may be expressed as a triple $\left(\frac{I_h}{I_l}; V^*; \omega\right)$. In the example we assume that the ratio $\frac{I_h}{I_l}$ is predetermined by the current amount of the tax credit (and is *a priori* a common knowledge). The variables $V^*$ and $\omega$ are the authority's decision variables.

As we already know, a decrease in $V^*$ results in a lower optimal threshold. Consequently, in case of a single firm a reduction in the trigger value is going to accelerate investment. However, in case of multiple heterogenous firms, lowering the trigger has two opposite effects. First, as in the single-firm case, it triggers an early investment for those firms for which Assumption iv (6) is still satisfied. On the other hand, it results in the other firms waiting longer and investing at a high cost (if Assumption iv (6)) no longer holds). Therefore, if the firms are sufficiently heterogenous, reducing $V^*$ does not yield a desired effect.

Therefore, the authority may prefer to resort to another instrument, such as $\omega$. An appropriately designed threat of abandoning the investment tax credit can trigger an early investment (see Dixit and Pindyck [6], Ch. 9). Since the firm's optimal investment threshold reaches a minimum for a certain degree of uncertainty, $\omega^e$, the objective of the authority interested in accelerating the investment should be to set the standard deviation of the density function underlying the cost-increase trigger equal to $\omega^e$. Such a policy allows for Assumption iv (6) to be satisfied for a larger fraction of firms that in case of reducing $V^*$.

Although a relatively small deviation from the optimal policy would result in a small delay in the aggregate investment, a sufficiently large deviation would have much more severe adverse effects. There exists a critical level of $\omega$, say $\hat{\omega}$, above which the optimal threshold is greater than $V^*$. In such a situation, the change in the cost is occurs before the investment is made and the project of a benchmark firm is undertaken at $V_h$. Therefore, increasing $\omega$ beyond $\hat{\omega}$ leads to a discontinuous change in the investment threshold what results in a considerable delay of the investment.[20] According to Proposition 1, $\hat{\omega}$ satisfies the following equation[21]

$$\begin{aligned} 0 \;=\; & h(V^*; \hat{\omega}; \Delta)\,(V^*)^2 + (\beta - 1)V^* - (V^* h(V^*; \hat{\omega}; \Delta) + \beta) I_l \;- \\ & - h(V^*; \hat{\omega}; \Delta) \frac{(\beta - 1)^{\beta - 1}}{\beta^\beta} \frac{(V^*)^{\beta+1}}{I_h^{\beta - 1}}. \end{aligned} \tag{20}$$

If the uncertainty related to the timing of imposing the trigger for a given firm is higher than $\hat{\omega}$, then the optimal threshold, $V_s$, is larger than $V^*$:

[20] The expected delay, $\Delta T$, can be calculated from a direct application of the first passage time. In this case $\Delta T = -\frac{1}{\alpha - \frac{1}{2}\sigma^2} \ln \frac{V_h}{V_s}$.

[21] Equation (20) is also satisfied for $\omega = 0$, since the optimal threshold in the deterministic case is equal to $V^*$.

Consequently, the impact of uncertainty associated with the timing of the change may be presented in the following way:

$$\omega \in \begin{cases} [0; \hat{\omega}) \setminus \omega^e & : \text{ feasible policy,} \\ \omega^e & : \text{ most effective policy,} \\ [\hat{\omega}; \infty) & : \text{ policy resulting in the investment delay.} \end{cases}$$

The threat of the increased investment cost is used as an investment stimulus most effectively when there exists a positive degree of informational noise concerning the timing of imposing the measure. The level of noise corresponds to $\omega^e$. Perfect information allows investors to wait until $V^*$ is reached. Excessive noise (above $\hat{\omega}$) results in the threat of imposing the trigger being lowered too much to trigger an early investment. As an effect, the change in the cost occurs before the investment is made. The optimal degree of uncertainty results in the optimal investment threshold being lower than $V^*$ and, at the same time, does not excessively dilute the threat. In such a case the preemption is most significant.

## 5 Extension: Stochastic Jump Size

Now we introduce a stochastic size of the upward change in the investment cost. Similarly to the previous case, the value of the investment opportunity, $W_s$, reflects the structure of the expected payoffs maximized with respect to the optimal investment threshold, $V_s$. Allowing for a stochastic $I_h$, distributed according to the cumulative density function $G(I_h)$, the value of investment opportunity becomes (cf. (8)):

$$\begin{aligned} W_s(V; \bar{V} | I = I_l) &= \max_{V_s} (V_s - I_l) \left( \frac{V}{V_s} \right)^{\beta} \frac{1 - F(V_s)}{1 - F(\bar{V})} + \\ &+ \int_{\underline{I_h}}^{\overline{I_h}} (V_h - I_h) \left( \frac{V}{V_h} \right)^{\beta} \left( 1 - \frac{1 - F(V_s)}{1 - F(\bar{V})} \right) dG(I_h) : \quad (21) \end{aligned}$$

$\underline{I_h}$ and $\overline{I_h}$ denote the lower and the higher bound of $I_h$, respectively. Equation (21) is interpreted analogously to (8), and the second component is the expected value of the option to wait after the upward switch. We prove in the Appendix that the following proposition holds.

**Proposition 5** In case the size of the jump is stochastic, the optimal investment rule can be determined by replacing $I_h$ by

$$I_h^* = \sqrt[1-\beta]{\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta} dG(I_h)} \qquad (22)$$

in the expression for the optimal threshold (12).

Formula (22) can be interpreted as a certainty equivalent of the high investment cost.[22]. This allows for a relatively simple analysis of the impact on the optimal investment timing of the uncertainty of the size of the jump.

The impact of the uncertainty concerning the size of the jump can be analyzed by directly comparing (12) and (14). By Jensen's inequality it holds that

$$\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1-\beta} dG(I_h) > \left( \int_{\underline{I_h}}^{\overline{I_h}} I_h dG(I_h) \right)^{1-\beta}; \qquad (23)$$

since the function $f(x) = x^a; a < 0;$ is convex for all $x > 0$. The RHS of Equation (23) is an inversely monotonic transformation of the expected value of $I_h$. Since, by (14), $\frac{\partial V_s}{\partial I_h} < 0;$ the threshold increases in $I_h^{1-\beta}$: Consequently, the threshold is higher for LHS than for RHS. In other words, the uncertainty in the size of the jump of the investment cost leads to the higher optimal investment threshold.

This result may be explained in the following way. The optimal timing is a convex function of the new investment cost, $I_h$. Therefore, the gains from below average realizations of the jump are assigned a larger weight by the firm than the symmetric losses from above-the-average realizations. Consequently, the firm is going to wait longer if the realizations are random than in the case when all of them are equal to the average.

Compared to the basic model where investment cost is constant, the threat of an upward change in the investment cost reduces the optimal investment threshold. Now, we can see that the uncertainty in the size of the jump mitigates this reduction of the threshold value. Again, it holds that the increased uncertainty raises the option value of waiting.

Apart from the overall difference between the uncertain and deterministic outcome, we are interested in a marginal impact of uncertainty on the optimal investment strategy. In other words, we aim at establishing how the investment threshold behaves for the different degrees of uncertainty concerning the size of the jump. Therefore, we compare the investment triggers corresponding to a relatively small and a high degree of uncertainty. For this purpose, we use the concept of mean preserving spread (Rotschild and Stiglitz [16]). In this setting, the effect of increasing uncertainty is examined by replacing the original random variable $I_h$ ('low uncertainty' case) by a new random variable $I_h + \varepsilon$ ('high uncertainty' case), where $E[\varepsilon] = 0$ and $\sigma_\varepsilon \in (0; \infty)$: By applying Jensen's inequality it can be proven that the expected value of a convex function (in our case $f(I_h) = I_h^{1-\beta}$) increases as its argument undergoes a mean preserving spread (cf. Hartman [10]). Consequently, an increase in the uncertainty leads to the higher expected value of $I_h^{1-\beta}$ what corresponds to the lower $I_h^*$, and a higher (or less distant from the basic case) investment threshold.

[22] Using the term certainty equivalent is a simplification since the firm is assumed to be risk-neutral. In our sense, (22) corresponds to such a value of a certain investment cost (within the high regime) that yields an identical optimal investment rule as when uncertain costs are distributed according to $G(I_H)$:

The impact on the optimal investment rule of uncertainty related to the magnitude of the change in the cost is monotonic. Therefore, in the investment credit example, increasing this type of uncertainty has the same e¤ect on the investment as the reduction of the magnitude of the change. Furthermore, (13) implies that a lower potential increase in the investment cost is associated with a higher optimal investment threshold. Therefore, a higher degree of uncertainty associated with the magnitude of the potential cost-increase results in a later investment. An e¤ective policy triggering an early investment should, therefore, be associated with minimizing the investors' uncertainty about the size of the expected change.

# 6 Conclusions

In the paper we consider an investment opportunity of a …rm. The investment cost is irreversible and subject to an increase resulting from a policy change. The value of the cost-increase trigger is unknown to the …rm and the …rm knows the underlying density function instead. This corresponds to the situation where the …rm has some information concerning the authority's future policy and this information is incomplete. Moreover, it is taken into account that policy changes are more likely to occur under certain economic conditions.

Recent tax debates across Europe introduce a signi…cant source of uncertainty for potential investors. Although some of the changes result from the need to unify the EU tax systems, in many cases the policy change can be attributed to the pace of economic growth. The booming Irish economy will face an increase in corporate tax from a special 10% rate for new foreign manufacturing and …nancial investors to 12.5%.[23] Other proposals include abandoning corporate tax exemptions in Germany and withdrawing approximately seventy tax reliefs used by EU governments use to draw investment.[24]

We show that the threat of a policy change resulting in a higher (net) investment cost leads to a reduction in the option value to wait. Consequently, the …rm invests earlier than in the case of the constant investment cost. The optimal investment threshold decreases in the magnitude of the change in investment cost and increases in the market volatility (the latter result also hold for the Dixit and Pindyck [6] framework). The impact of the trigger value uncertainty on the optimal investment threshold is non-monotonic. If the uncertainty is su¢ ciently low, then the investment threshold is negatively related to the trigger value uncertainty. However, a rise in the uncertainty beyond a certain critical point reverses this relationship and leads to an increase of the optimal investment threshold.

We use our …ndings to determine the optimal design of a policy change

[23] "Ireland: Burning Too Bright; Can Ireland control its rapid growth?", Businessweek, 10 Apr., 2000.

[24] The tax reliefs subject to change range from Belgian exemptions on multinational headquarters to incentives made by Spain for investors in the Basque region. For the details, see "Hey, Let's All Get Together and Raise Taxes!", Businessweek, 25 Nov., 1998.

that accelerates investment expenditures. There exists a certain (strictly positive) level of the uncertainty concerning the policy change that triggers the earliest investment. Hence, a policy maker interested in accelerating investment should aim at achieving the level of uncertainty, corresponding to this minimal investment threshold.

Finally, we extend the analysis by considering a case when the size of the change is stochastic. The uncertainty in the magnitude of the change appears to mitigate the degree of 'preemption' so it leads to the outcome which is closer to the unconditional optimal level.

# 7 Appendix

Proof of Proposition 1. The implicit solution for the optimal investment threshold is found by calculating the first order condition of (8). Consequently, by differentiating (8) with respect to $V_s$; and equalizing to zero, we obtain:

$$
\begin{aligned}
0 \quad = \quad & -\frac{V^{\beta}}{V_s^{\beta+1}}\left(V_s - \beta V_s + \beta I_l\right)\frac{1-F(V_s)}{1-F(\bar{V})} + \left(V_s - I_l\right)\left(\frac{V}{V_s}\right)^{\beta}\left(\frac{f(V_s)}{1-F(\bar{V})}\right) \\
& -\left(V_h - I_h\right)\left(\frac{V}{V_h}\right)^{\beta}\frac{f(V_s)}{1-F(\bar{V})}; \qquad (24)
\end{aligned}
$$

where $f(x) = \frac{\partial F(x)}{\partial x}$. Further simplification yields:

$$
\begin{aligned}
0 \quad = \quad & \frac{1}{V_s^{\beta+1}}\left(V_s - \beta V_s + \beta I_l\right)\left(1 - F(V_s)\right) - \left(V_s - I_l\right)\left(\frac{1}{V_s}\right)^{\beta} f(V_s) \\
& + \left(V_h - I_h\right)\left(\frac{1}{V_h}\right)^{\beta} f(V_s);
\end{aligned}
$$

thus

$$
(\beta - 1)V_s - \beta I_l + h(V_s)(V_s - I_l)V_s - (V_h - I_h)V_s^{\beta+1}\left(\frac{1}{V_h}\right)^{\beta} h(V_s) = 0.
$$

Since $V_h = \frac{\beta}{\beta-1} I_h$ (after the jump the McDonald-Siegel problem is left), this is equal to

$$
h(V_s)V_s^2 + (\beta - 1)V_s - \left(h(V_s)V_s + \beta\right)I_l - V_s^{\beta+1}h(V_s)\frac{I_h}{\beta - 1}\left(\frac{\beta - 1}{\beta I_h}\right)^{\beta} = 0;
$$

what in a straightforward way leads to (12).

In order to prove that (12) is the expression for the maximal value of the project, we calculate the second order condition, which is equal to the following derivative:

$$\frac{\partial}{\partial V_s}\left(-h(V_s)V_s^2-(\beta-1)V_s+(V_sh(V_s)+\beta)I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}\right).$$

After differentiating, we obtain expression for the second order condition of (8):

$$\begin{aligned}\frac{\partial^2 W_s}{\partial V_s^2} &= -\left(h'(V_s)V_s+h(V_s)\right)\left(V_s-I_l-\frac{V_s}{\beta}\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1}\right)\\ &\quad -h(V_s)V_s\left(1-\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1}\right)-(\beta-1). \end{aligned} \tag{25}$$

The sign of the second component can be proven to be negative by observing that:

$$1-\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)^{\beta-1} > 1-\left(\frac{\beta-1}{\beta}\frac{V_h}{I_h}\right)^{\beta-1}=0. \tag{26}$$

The sign of the first component can be determined by notifying that the lower bound of $V_s$, denoted by $\underline{V_s}$, is a solution to the following equation:

$$\underline{V_s}-I_l=(V_h-I_h)\left(\frac{\underline{V_s}}{V_h}\right)^{\beta}. \tag{27}$$

For $V_s=\underline{V_s}$ the second factor in the first component of (25) is equal to zero and for $V_s>\underline{V_s}$ it is positive. Therefore the whole expression is surely negative if (11) holds. ■

Proof of Proposition 2. Let us define the LHS of (12) as a function:

$$\begin{aligned}&H(V_s,I_l,I_h,\beta)\\ =\ & h(V_s)V_s^2+(\beta-1)V_s-(V_sh(V_s)+\beta)I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}.\end{aligned} \tag{28}$$

Differentiating (28) with respect to $I_l$, $I_h$ and $\beta$, respectively, yields:

$$\begin{aligned}\frac{\partial H}{\partial I_l} &= -(V_sh(V_s)+\beta)<0;\\ \frac{\partial H}{\partial I_h} &= (\beta-1)h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_H^{\beta}}>0;\\ \frac{\partial H}{\partial \beta} &= V_s-I_l-h(V_s)\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\frac{V_s^{\beta+1}}{I_h^{\beta-1}}\ln\left(\frac{\beta-1}{\beta}\frac{V_s}{I_h}\right)>0;\end{aligned} \tag{29}$$

$\forall I_l, I_h$ satisfying $0 < I_l < I_h$; $\forall \beta \in (1; r/\alpha)$ if $\alpha > 0$ and $\forall \beta \in (1; \infty)$ if $\alpha \le 0$. Furthermore, differentiating (28) with respect to $V_s$ gives:

$$\begin{aligned} \frac{\partial H}{\partial V_s} &= (h'(V_s)V_s + h(V_s))\left(V_s - I_l - (V_h - I_h)\left(\frac{V_s}{V_h}\right)^{\beta}\right) + \\ &\quad + h(V_s)V_s\left(1 - \left(\frac{\beta - 1}{\beta}\frac{V_s}{I_h}\right)^{\beta - 1}\right) + (\beta - 1). \end{aligned} \tag{30}$$

From the proof of Proposition 1 it is known that under condition (11) $\frac{\partial H}{\partial V_s}$ is positive. Finally, by (30), we know that

$$\forall z \quad \operatorname{sgn}\frac{\partial V_s}{\partial z} = -\operatorname{sgn}\frac{\partial H}{\partial z}; \tag{31}$$

what completes the proof. ■

Proof of Proposition 3. By differentiating (28) with respect to the hazard rate, while taking into account that $V_h = \frac{\beta}{\beta - 1} I_h$, we obtain:

$$\frac{\partial H}{\partial h} = V_s\left(V_s - I_l - (V_h - I_h)\left(\frac{V_s}{V_h}\right)^{\beta}\right) > 0. \tag{32}$$

The inequality holds since the both factors are positive (cf. (27) and the proof of (30)). Since $\frac{\partial H}{\partial V_s}$ is also positive, from the envelope theorem we directly obtain the sign of (16). ■

Proof of Proposition 5. Equation (22) requires the optimal investment threshold with a deterministic size of the jump be equal to the threshold with a jump with a stochastic size distributed according to $G(I_h)$. Since the maximization problem with a stochastic size of the jump can be expressed as follows:

$$\begin{aligned} W_s(V; \bar{V} | I &= I_l) = \max_{V_s}(V_s - I_l)\left(\frac{V}{V_s}\right)^{\beta}\frac{1 - F(V_s)}{1 - F(\bar{V})} + \\ &\quad + \left(1 - \frac{1 - F(V_s)}{1 - F(\bar{V})}\right)\frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta}}\frac{V^{\beta}}{\int_{\underline{I_h}}^{\overline{I_h}} I_h^{\beta - 1} dG(I_h)}; \end{aligned} \tag{33}$$

the expression for the optimal investment threshold is a slight modification of (12):

$$\begin{aligned} 0 &= h(V_s)V_s^2 + (\beta - 1)V_s - (V_s h(V_s) + \beta)I_l - \\ &\quad - V_s^{\beta + 1} h(V_s)\frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta}}\int_{\underline{I_h}}^{\overline{I_h}} I_h^{1 - \beta} dG(I_h). \end{aligned} \tag{34}$$

Comparing (34) with (12) allows for observing that the threshold values are equal if:

$$h(V_s)\frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta}}\frac{V_s^{\beta + 1}}{I_h^{*\,\beta - 1}} = \int_{\underline{I_h}}^{\overline{I_h}} h(V_s)\frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta}}\frac{V_s^{\beta + 1}}{I_h^{\beta - 1}} dG(I_h). \tag{35}$$

A simple algebraic manipulation yields:

$$I_h^{*1-\beta} = \int_{\underline{I}_h}^{\overline{I}_h} I_h^{1-\beta} \, dG(I_h): \tag{36}$$

what in a straightforward way leads to (22). ■

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