A Generalization of the Banach Contraction Principle
Voorneveld, M.

Publication date:
2000

Link to publication

Citation for published version (APA):
A Generalization of the Banach Contraction Principle

Mark Voorneveld
Department of Econometrics and CentER
Tilburg University
P.O.Box 90153
5000 LE Tilburg
The Netherlands
M.Voorneveld@kub.nl

Abstract: In a complete metric space \((X; d)\), we define contraction factor functions \(\alpha : X \times X \to [0, \infty)\) and \(\omega\)-distance functions \(\rho : X \times X \to [0, \infty)\), of which the distance function \(d\) is a special case, such that if

\[
\rho(Ax, Ay) \leq \alpha(x, y)\rho(x, y)
\]

for all \(x, y \in X\), then \(A : X \to X\) has a (unique) fixed point.

Keywords: Banach Contraction Principle, \(\omega\)-distance, fixed point.
1 Introduction

The Banach Contraction Principle states that if $A : X \rightarrow X$ is a mapping of a complete metric space $(X, d)$ into itself, and there exists a number $0 \leq \alpha < 1$ such that for every two points $x, y \in X$:

$$d(Ax, Ay) \leq \alpha d(x, y),$$  

(1)

then $A$ has a unique fixed point, i.e., there exists a unique $x \in X$ satisfying $Ax = x$. A function $A : X \rightarrow X$ satisfying (1) is a contraction; the number $\alpha \in [0, 1)$ is the contraction factor.

Rakotch (1962) considers the problem of defining contraction factor functions $\alpha : X \times X \rightarrow [0, \infty)$ such that the Banach Contraction Principle remains valid when the constant $\alpha$ in (1) is replaced by a function $\alpha(x, y)$. This note considers also other functions than the distance function $d$ in (1) under which the existence of a (unique) fixed point is guaranteed. More precisely, we define contraction factor functions $\alpha : X \times X \rightarrow [0, \infty)$, similar to those in Rakotch (1962), and functions $\rho : X \times X \rightarrow [0, \infty)$, of which the distance function $d$ is a special case, in such a way that if

$$\rho(Ax, Ay) \leq \alpha(x, y)\rho(x, y)$$

for all $x, y \in X$, then $A : X \rightarrow X$ has a (unique) fixed point. The functions $\rho$ are so-called $\omega$-distances, introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi (1996), Suzuki and Takahashi (1996), and Suzuki (1997).

The set-up of the note is as follows. Section 2 recalls the definition of $\omega$-distances and contains preliminary results. Section 3 presents the generalization of the Banach Contraction Principle. Section 4 contains concluding remarks.

2 Preliminaries

$\mathbb{N}$ denotes the set of positive integers. Let $X$ be a metric space with metric $d$. Following Kada et al. (1996, p. 381), we define an $\omega$-distance on $X$ to be a function $\rho : X \times X \rightarrow [0, \infty)$ such that:

- $\rho$ satisfies the triangle inequality, i.e., $\forall x, y, z \in X : \rho(x, z) \leq \rho(x, y) + \rho(y, z)$;
• \( \rho(x, \cdot) : M \to [0, \infty) \) is lower semicontinuous for every \( x \in X \), i.e., if \( y_m \to y \), then \( \rho(x, y) \leq \liminf_{m \to \infty} \rho(x, y_m) \);

• for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for each \( x, y, z \in X \): if \( \rho(z, x) \leq \delta \) and \( \rho(z, y) \leq \delta \), then \( d(x, y) \leq \varepsilon \).

The metric \( d \) is an \( \omega \)-distance. Examples of many other \( \omega \)-distances are found in Kada et al. (1996) and Suzuki and Takahashi (1996, Lemma 1). Kada et al. (1996, Lemma 1) prove:

**Lemma 2.1** Let \( (X, d) \) be a metric space and \( \rho \) an \( \omega \)-distance on \( X \). Consider points \( x, y, z \in X \), a sequence \( (x_n) \) in \( X \) such that \( x_n \to x \), sequences \( (\alpha_n) \) and \( (\beta_n) \) in \( [0, \infty) \) converging to zero. The following claims hold:

(a) If \( \rho(x_n, x_m) \leq \alpha_n \) for all \( n, m \in \mathbb{N} \) with \( m > n \), then \( (x_n) \) is a Cauchy sequence in \( (X, d) \).

(b) If \( \rho(x_n, y) \leq \alpha_n \) and \( \rho(x_n, z) \leq \beta_n \) for all \( n \in \mathbb{N} \), then \( y = z \). In particular, if \( \rho(x, y) = \rho(x, z) = 0 \), then \( y = z \).

Following Rakotch (1962), we define a family of functions that will take over the role of the contraction factors in the original statement of Banach’s Contraction Principle.

**Definition 2.2** Let \( (X, d) \) be a metric space and \( \rho \) an \( \omega \)-distance on \( X \). Denote by \( F(\rho) \) the family of functions \( \alpha \) on \( X \times X \) satisfying the following conditions:

(a) for each \( (x, y) \in X \times X \), \( \alpha(x, y) \) depends only on the \( \omega \)-distance \( \rho(x, y) \); with a slight abuse of notation, this allows us to write \( \alpha(\rho(x, y)) \) instead of \( \alpha(x, y) \);

(b) \( 0 \leq \alpha(d) < 1 \) for every \( d > 0 \);

(c) \( \alpha(d) \) is a decreasing function of \( d \): if \( d_1 \leq d_2 \), then \( \alpha(d_1) \geq \alpha(d_2) \).

### 3 A Contraction Principle

This section contains the statement and proof of our generalization of the Banach Contraction Principle.
Theorem 3.1 Let \((X, d)\) be a complete metric space, \(\rho\) an \(\omega\)-distance on \(X\), and \(A : X \to X\) a function. If there exists an \(\alpha \in F(\rho)\) such that

\[
\forall x, y \in X : \quad \rho(Ax, Ay) \leq \alpha(x, y)\rho(x, y),
\]

(2)

then \(A\) has a unique fixed point \(x\). This fixed point satisfies \(\rho(x, x) = 0\).

**Proof.** Let \(x_0 \in X\) and define for each \(n \in \mathbb{N}\): \(x_n = A^n x_0\). A simple inductive argument based on (2) and property (b) in Definition 2.2 yields that

\[
\forall n \in \mathbb{N} : \quad \rho(x_{n+1}, x_n) \leq \rho(x_1, x_0),
\]

(3)

and

\[
\forall k, \ell, m \in \mathbb{N} : \quad \text{if } k > m, \text{ then } \rho(x_k, x_{k+\ell}) \leq \rho(x_m, x_{m+\ell}).
\]

(4)

Let \(\varepsilon > 0\). Define \(R := \max \{\varepsilon, \frac{\rho(x_0, x_1) + \rho(x_1, x_0)}{1 - \alpha(\varepsilon)}\}\). By property (b) in Definition 2.2, \(\alpha(\varepsilon) < 1\), so \(R\) is well-defined. We claim that

\[
\forall n \in \mathbb{N} : \quad \rho(x_0, x_n) \leq R.
\]

(5)

Let \(n \in \mathbb{N}\). Inequality (5) trivially holds if \(\rho(x_0, x_n) \leq \varepsilon\), so assume that \(\rho(x_0, x_n) > \varepsilon\). Consecutively using

- the triangle inequality for \(\rho\),
- inequalities (2) and (3),
- the fact that \(0 \leq \alpha(x_0, x_n) \leq \alpha(\varepsilon) < 1\), which follows from the assumption that \(\rho(x_0, x_n) > \varepsilon\) and properties (b) and (c) in Definition 2.2,

the following chain of inequalities holds:

\[
\rho(x_0, x_n) \leq \rho(x_0, x_1) + \rho(x_1, x_{n+1}) + \rho(x_{n+1}, x_n) \\
\leq \rho(x_0, x_1) + \alpha(x_0, x_n)\rho(x_0, x_n) + \rho(x_1, x_0) \\
\leq \rho(x_0, x_1) + \alpha(\varepsilon)\rho(x_0, x_n) + \rho(x_1, x_0).
\]
Hence $\rho(x_0, x_n) \leq \frac{\rho(x_0, x_1) + \rho(x_1, x_0)}{1 - \alpha(\varepsilon)} \leq R$, finishing the proof of (5). We proceed to prove that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall p \in \mathbb{N}: \quad \rho(x_N, x_{N+p}) < \varepsilon. \tag{6}$$

Let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that

$$R(\alpha(\varepsilon))^N < \varepsilon. \tag{7}$$

This is possible, since $0 \leq \alpha(\varepsilon) < 1$ by property (b) of Definition 2.2. Let $p \in \mathbb{N}$. For each $k = 0, \ldots, N - 1$, inequality (2) implies

$$\rho(x_{k+1}, x_{k+p+1}) \leq \alpha(x_k, x_{k+p}) \rho(x_k, x_{k+p}).$$

Taking the product from $k = 0$ to $k = N - 1$ and dividing both sides by the common term $\prod_{k=0}^{N-2} \rho(x_{k+1}, x_{k+p+1})$ yields

$$\rho(x_N, x_{N+p}) \leq \rho(x_0, x_p) \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}). \tag{8}$$

Division by the common term $\prod_{k=0}^{N-2} \rho(x_{k+1}, x_{k+p+1})$ is correct by definition if $\rho(x_{k+1}, x_{k+p+1}) > 0$ for every $k \in \{0, \ldots, N - 2\}$, but also if $\rho(x_{k+1}, x_{k+p+1}) = 0$ for a specific $k \in \{0, \ldots, N - 2\}$, inequality (8) remains valid. In this case, namely, inequality (4) yields that $0 \leq \rho(x_N, x_{N+p}) \leq \rho(x_{k+1}, x_{k+p+1}) = 0$, i.e., the left-hand side of (8) equals zero, whereas its right-hand side is always nonnegative. Combining (5) and (8):

$$\rho(x_N, x_{N+p}) \leq R \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}). \tag{9}$$

Discern two cases:

**Case 1:** If $\rho(x_k, x_{k+p}) < \varepsilon$ for some $k \in \{0, \ldots, N - 1\}$, then (4) yields that $\rho(x_N, x_{N+p}) \leq \rho(x_k, x_{k+p}) < \varepsilon$.

**Case 2:** If $\rho(x_k, x_{k+p}) \geq \varepsilon$ for every $k \in \{0, \ldots, N - 1\}$, then property (c) in Definition 2.2 implies that $\alpha(x_k, x_{k+p}) \leq \alpha(\varepsilon)$ for every $k \in \{0, \ldots, N - 1\}$. Using (7) and (9) yields

$$\rho(x_N, x_{N+p}) \leq R \prod_{k=0}^{N-1} \alpha(x_k, x_{k+p}) \leq R(\alpha(\varepsilon))^N < \varepsilon.$$
This proves (6). Statements (4) and (6) immediately imply

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \forall p \in \mathbb{N}: \ \rho(x_n, x_{n+p}) < \varepsilon.$$  \hspace{1cm} (10)

By (10), there exists a sequence \((\alpha_n)\) in \([0, \infty)\) converging to zero, such that

$$\forall n, p \in \mathbb{N}: \ \rho(x_n, x_{n+p}) \leq \alpha_n.$$  \hspace{1cm} (11)

But then \((x_n)\) is a Cauchy sequence by part (a) of Lemma 2.1. Since \((X, d)\) is complete, \((x_n)\) has a limit \(x \in X\). We show that \(Ax = x\). Since \(\rho(x_n, \cdot)\) is lower semicontinuous and \(x_m \to x\), it follows from (11) that

$$\forall n \in \mathbb{N}: \ \rho(x_n, x) \leq \liminf_{m \to \infty} \rho(x_n, x_m) \leq \alpha_n,$$  \hspace{1cm} (12)

and, using (2) and (12), that

$$\forall n \in \mathbb{N}: \ \rho(x_n, Ax) = \rho(Ax_{n-1}, Ax) \leq \rho(x_{n-1}, x) \leq \alpha_{n-1}.$$  \hspace{1cm} (13)

From (12), (13), and part (b) of Lemma 2.1, it follows that \(Ax = x\), i.e., that \(x\) is a fixed point of \(A\). To see that \(\rho(x, x) = 0\), suppose — to the contrary — that \(\rho(x, x) > 0\). Then \(0 \leq \alpha(x, x) < 1\) by property (b) of Definition 2.2; by (2) and the fact that \(x\) is a fixed point, it follows that:

$$\rho(x, x) = \rho(Ax, Ax) \leq \alpha(x, x) \rho(x, x) < \rho(x, x),$$

a contradiction. Finally, to prove that \(x\) in the unique fixed point of \(A\), suppose that \(y \in X\) satisfies \(Ay = y\). Analogous to the proof that \(\rho(x, x) = 0\), it follows that \(\rho(x, y) = 0\), so part (b) of Lemma 2.1 implies that \(x = y\). \hfill \Box

4 Concluding Remarks

Some remarks concerning the generalizations embodied in Theorem 3.1:

- If we take \(\alpha\) to be a constant function in \([0, 1)\), we obtain Theorem 2 of Suzuki and Takahashi (1996);
• If we take $\rho = d$, we obtain the contraction theorem of Rakotch (1962, p. 463);

• If we take $\rho = d$ and take $\alpha$ to be a constant function in $[0, 1)$, we obtain the original Banach Contraction Principle.

If $A$ itself does not satisfy (2), but some power $A^n (n \in \mathbb{N})$ of $A$ does, the conclusion of Theorem 3.1 still holds: according to Theorem 3.1, $A^n$ has a unique fixed point $x$, but

$$Ax = A(A^n x) = A^n(Ax)$$

indicates that $Ax$ is also a fixed point of $A^n$. Hence $Ax = x$, i.e., $x$ is a fixed point of $A$. The fact that $x$ is the unique fixed point of $A$ and $\rho(x, x) = 0$ follows in the same way as before.

References


