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Abstract

In this paper we model the formation of directed communication networks. A directed communication network is represented by a directed graph. Firstly, we study an allocation rule satisfying two appealing properties, component efficiency and directed fairness. We show that such an allocation rule exists if and only if we restrict ourselves to a class of directed graphs that naturally comes to the fore in the setting of hierarchical structures. Subsequently, we discuss several possibilities to model the formation of directed communication networks and provide some preliminary results.

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1 Introduction

Networks play an important role in relationships and interaction between economic agents. A relationship between individuals allows for a variety of interpretations. It can, for example, represent communication possibilities, information transmission, or a trade relation. In the organization of a firm, the relationships that jointly form a network might represent the formal flow of information between individuals or it can represent the hierarchical structure of a firm. The position of an individual in a network will usually influence his productivity. Consequently, the architecture of the internal relationships within a group of economic agents may affect the joint productivity of these agents.

The game-theoretical analysis of networks was initiated by Myerson (1977). He assumes that the joint profits that can be obtained by a group of players depends on its connected components only and not on the specific architectures of these components. The possibilities to cooperate with each other are modeled by means of an undirected graph, whereas the economic possibilities of the players, apart from the restricted possibilities to cooperate, are captured in a cooperative game with transferable utilities. Myerson (1977) focuses on the issue how joint profits should be divided and proposes that an allocation rule should be used that satisfies two appealing properties. Firstly, it should be efficient in the sense that profits that can be ascribed to a connected group of agents should be divided between the members of this group. Secondly, the division should be fair in the sense that an additional relation between two players should have the same cardinal effects on the payoffs that are attributed to the players that are involved in this relationship. The main result of Myerson (1977) states that the division of joint profits in a manner that satisfies these two properties is unique determined.

The framework of Myerson (1977), i.e., a cooperative game in conjunction with an undirected graph, is usually called a communication situation. These situations have received a considerable amount of attention since. Borm et al. (1992) and Hamiache (1999) focus on alternative allocation rules. Aumann and Myerson (1988) were the first to deal with the formation of networks in this setting. They introduced a model in which relationships between agents are formed sequentially. Their model has subsequently been studied by Feinberg (1998) and Slikker and Norde (2000). Both papers focus on incomplete stable structures. Qin (1996) and Dutta et al. (1998) on the other hand study a model, introduced by Myerson (1991), in which bilateral relationships are formed simultaneously rather than sequentially. For a survey of the literature on communication situations we refer to van den Nouweland (1993) and Slikker (2000).

Recently, a growing interest in the analysis of social and economic networks can be observed. Jackson and Wolinsky (1996) introduced a model that is close to the communication situations of Myerson (1977). Their model additionally allows for influence of the architecture of a connected group of players on their joint profitability. In their analysis, Jackson and Wolinsky (1996) focus on the conflict between efficiency and stability, showing that a stable network that is efficient does not always exist. The conflict between efficiency and stability is further investigated by Dutta and Mutuswami (1997). Watts (1997) considers a dynamic model of network formation, thereby restricting the analysis to a stylized model that was introduced by Jackson and Wolinsky (1996), the connections model. Johnson and Gilles (1999) also study the connections model, supplemented with a spatial cost topology. They investigate the pairwise stable networks and the subgame perfect implementation of pairwise stable and efficient networks.

In the models described so far, relations between agents are assumed to be symmetric. Some situations, for example the hierarchical structure of a firm, demand for modeling for a distinction between the two agents that are involved in a relation. Situations with asymmetric bilateral relations are sometimes modeled by means of directed graphs. Cooperative situations with asymmetric relations between players have been studied by Gilles et al. (1992), Derks and Gilles (1995), van den Brink and Gilles (1996), and Gilles and Owen (1999). In these papers, a cooperation structure is interpreted as a permission structure, where a player can fully participate in a cooperative situation if he has permission by one or all of his superiors. Van den Brink and Gilles (1994) and van den Brink and Borm (1995) study these cooperation structures from a different point of view. Rather than an analysis of the division of cooperative gains, they provide two approaches to measure social power in hierarchically structured populations.

The formation of asymmetric relations between players is the subject of study in Bala and Goyal (2000). In their approach the costs of establishing a cooperation relation are incurred by the person who initiates this link. Two interpretations of such a cooperation relation are considered. Firstly, a cooperation relation represents one-way communication, meaning that the person who establishes a directed communication relation has access to the information of the player with whom he forms a directed communication relation, but not vice-versa. Hence, in this model a relation between two players is asymmetric. Secondly, Bala and Goyal (2000) consider two-way communication, in which both players involved in a directed communication relation have access to each other's information. Bala and Goyal (2000) characterize the equilibrium networks in these models and subsequently take a dynamic approach to the formation of networks. They derive that equilibrium networks emerge rapidly. These models appear to have simple architectures and are often socially efficient. Jackson (1999) also studies the formation of asymmetric communication networks. He focuses on the relation between stability and efficiency.

In this paper we follow Bala and Goyal (2000) in assuming that relations between two players are asymmetric. We assume that the economic possibilities of the players are captured in a directed reward function that assigns a value to every directed graph on a fixed set of agents. We first focus on allocation rules for situations where the economic possibilities are described by such a reward function and a specific network has formed. Here we want to follow Myerson (1977) as close as possible, but we want to take into account the asymmetry of a relation between two players. We show that there exists an allocation rule that satisfies an efficiency property and a fairness property, which takes into account these asymmetries in a relation, if and only if the players can be partitioned in hierarchical classes. Subsequently, we introduce and discuss several models that represent the formation of hierarchical structures.

In considering asymmetric communication relations between the agents we distinguish between the initiator of a relation and the receiver. The allocation rule described above treats an initiator and a receiver asymmetrically. This asymmetry is captured in a parameter $\alpha \geq 1$, where benchmark parameter $\alpha = 1$ stands for symmetric treatment of initiator and receiver in a directed communication relation. With respect to the model of Bala and Goyal (2000), $\alpha = 1$ corresponds to two-way communication. One-way communication as described in Bala and Goyal (2000) would require analysis of α 'equal' to infinity, representing that benefits of a relation accrue solely to the initiator of a link. We will mainly concentrate on the intermediate cases, i.e., $1 < \alpha < \infty$.

Several major theories provide different explanations of why firms exist and employees submit themselves to an incomplete contract in their relation with an employer. The employer can be seen as the initiator and the employee as the receiver of a relation between them. Coase (1937) considers an employment relation to be a relationship in which the employer has full control of the employee and can order him around at will. In the agency literature this is replaced by a more subtle relation. Here, the employer is called principal and the employee his agent. The employer has an information deficit due to adverse selection and/or moral hazard problems. He does not have complete control of the agent. Subsequently, the literature turns to the design of an optimal employment contract between employer and employee. Such a contract usually contains (minimal) incentives for the agent to do his work properly. In the asset approach this viewpoint is weakened further. The employer can essentially only deny the employee access to the productive assets of the firm. All residual rights result from the fundamental right to 'veto' the employee. In any of these interpretations it does not become clear why a firm should have a hierarchical organization structure. The main theorem of this paper deals with this issue. This theorem states that if the initiator of a directed communication relation (possibly a hierarchical relation) expects a higher marginal influence from an additional relation on his payoff, then this can only be implemented within networks in which players are divided in hierarchical classes. Hence, the asymmetry in these marginal influences of directed communication relations demands for a hierarchical communication network.

The setup of the remainder of this paper is as follows. In section 2 we provide preliminaries with respect to directed graphs and directed reward situations. Allocation rules are the subject of study in section 3. Finally, in section 4 we give the first impulse to modeling the formation of asymmetric communication networks in which the consent of both players in a directed communication relation is required for the formation of this relation.

2 Preliminaries

A *directed graph* is a pair (N, A) where $N = \{1, \ldots, n\}$ is a set of vertices and $A \subseteq \big\{(i,j) \mid i,j \in N; i \neq j\big\}$ a set of (directed) arcs.[1] The vertices can be interpreted as players and the arcs as communication relations: an arc (i,j) states that player i is the *initiator* of this directed communication relation and that player j is the *receiver*. The set of all directed graphs with player set N is denoted by DG^N. In case there is no ambiguity about the set of vertices we will sometimes identify a directed graph with its set of directed arcs. To stress the interpretation of a directed graph we will from now on refer to $(N, A) \in \mathrm{DG}^N$ as a *directed communication network*.

A collection $\mathcal{A}$ of directed communication networks is called *closed* (under taking inclusions) if for all $A \in \mathcal{A}$ and all $A' \subseteq A$ it holds that $A' \in \mathcal{A}$. In this paper we will restrict our attention to collections of directed communication networks that are closed. Note that the set of all directed communication networks is closed as is, for example, the set of all bipartite directed graphs, which are structures that contain only directed arcs from a subset of the player set, the set of initiators, to its complement, the set of receivers. We remark that in a bipartite graph it is possible that a player is not involved in any directed communication relation.

The communication relations between players have another interpretation as well: since the only relations between players are directed communication relations, cooper-

[1] $S \subseteq T$ denotes that S is a subset of T. $S \subset T$ denotes that S is a strict subset of T.

ation between players is via these relations only. This restriction in cooperation between the players results for a fixed directed communication network A in a partition of the player set. Two players i and j are *connected* if $i = j$ or the players are connected, either directly, i.e., $(i, j) \in A$ or $(j, i) \in A$, or indirectly, i.e., there exists a *path* $(x_1, e_1, x_2, \ldots, x_{k-1}, e_{k-1}, x_k)$, $k \geq 3$, with $x_1 = i$, $x_k = j$, and for all $l \in \{1, \ldots, k-1\}$ it holds that $e_l \in \{(x_l, x_{l+1}), (x_{l+1}, x_l)\}$ and $e_l \in A$. Two players are in the same partition element if and only if they are connected. The resulting partition is denoted by N/A. For any $R \subseteq N$, the partition into components in $(R, A(R))$ is denoted by R/A, where $A(R) = \{(i, j) \in A \mid \{i, j\} \subseteq R\}$. Though an arc $(i, j) \in A$ is directed, it represents a fully developed communication link. Hence, connectedness, as defined above, is based on arbitrary communication paths, without paying attention to the direction of the links in a path. We remark that $A(R)$ sometimes refers to directed communication network $(N, A(R))$ and sometimes to directed communication network $(R, A(R))$.

For an arbitrary path $P = (x_1, e_1, x_2, \ldots, e_{l-1}, x_l)$ we denote

$$t(P) = \left|\left\{r \in \{1, \ldots, l-1\} \,|\, e_r = (x_{r+1}, x_r)\right\}\right| - \left|\left\{r \in \{1, \ldots, l-1\} \,|\, e_r = (x_r, x_{r+1})\right\}\right|.$$

Finally, a *cycle* is a path $P = (x_1, e_1, x_2, \ldots, e_{l-1}, x_l)$ with $x_1 = x_l$ and $x_1, \ldots, x_{l-1}$ all distinct points.

Let $\mathcal{A} \subseteq \mathrm{DG}^N$ be a closed set of directed communication networks. A *directed reward function* r on $\mathcal{A}$ is a function $r : \mathcal{A} \to \mathbb{R}$ that assigns a real number to every directed communication network $A \in \mathcal{A}$. The value $r(A)$ represents the profit that can be obtained by all players together if they cooperate according to this directed communication network. Throughout this paper we assume that $r(\emptyset) = 0$, which states that if there are no relations between the players then no profit can be made. Furthermore, we assume that the directed reward function is *component additive*, i.e.,

$$r(A) = \sum_{C \in N/A} r(A(C)) \text{ for all } A \in \mathcal{A}.$$

A triple (N, r, A) with r a directed reward function on $\mathcal{A}$ and $A \in \mathcal{A}$ will be called a *directed reward situation* on $\mathcal{A}$. Denote the set of all directed reward situations with player set N on $\mathcal{A}$ by $\mathrm{DRS}^{N,\mathcal{A}}$.

An *allocation rule* γ for directed reward situations assigns a vector $\gamma(N, r, A) \in \mathbb{R}^N$ to every directed reward situation (N, r, A) in a class of directed reward situations. If there is no ambiguity about the underlying player set and the directed reward function we sometimes refer to $\gamma(A)$ instead of $\gamma(N, r, A)$.

A cooperative game is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ the characteristic function, with $v(\emptyset) = 0$. Every game (N, v) can be written in a unique way as a linear combination of unanimity games $(N, u_R)_{R\subseteq N}$, where $u_R(T) = 1$ if $R \subseteq T$ and $u_R(T) = 0$ otherwise, i.e., $v = \sum_{R\subseteq N} \lambda_R(v) u_R$. In case there is no ambiguity about the underlying game we simply write λ_R instead of $\lambda_R(v)$. The *Shapley value* Φ of a game (cf. Shapley (1953)) is now easily described by

$$\Phi_i(N, v) = \sum_{R\subseteq N,\ i\in R} \frac{\lambda_R}{|R|} \text{ for all } i \in N.$$

Weighted Shapley values can be described similarly. Let $w \in \mathbb{R}^N_{++}$ and for all $R \subseteq N$ define $w_R = \sum_{i\in R} w_i$. The *weighted Shapley value with weights* w of a game (cf. Kalai and Samet (1988)) is then described by

$$\Phi^w_i(N, v) = \sum_{R\subseteq N,\ i\in R} \frac{w_i}{w_R} \lambda_R \text{ for all } i \in N.$$

We will sometimes refer to the weighted Shapley value with weights w as the w-Shapley value.

A game in strategic form will be denoted by $\Gamma = (N; (X_i)_{i\in N}; (\pi_i)_{i\in N})$, where $N = \{1, \ldots, n\}$ denotes the player set, X_i the strategy space of player $i \in N$, and $\pi = (\pi_i)_{i\in N}$ the payoff function which assigns to every strategy-tuple $x = (x_i)_{i\in N} \in \prod_{i\in N} X_i = X$ a vector in $\mathbb{R}^N$. For notational convenience we write $x_{-i} = (x_l)_{l\in N\setminus\{i\}}$, $x_{-ij} = (x_l)_{l\in N\setminus\{i,j\}}$, and $x_R = (x_l)_{l\in R}$.

Monderer and Shapley (1996) formally defined the class of weighted potential games. For any $w \in \mathbb{R}^N_{++}$ the function $P^w : \prod_{i\in N} X_i \to \mathbb{R}$ is called a *w-potential* for Γ if for every $i \in N$, every $x \in X$, and every $t_i \in X_i$ it holds that

$$\pi_i(x_i, x_{-i}) - \pi_i(t_i, x_{-i}) = w_i \left(P^w(x_i, x_{-i}) - P^w(t_i, x_{-i}) \right). \tag{1}$$

The game Γ is called a w-potential game if it admits a w-potential. The game Γ is called a weighted potential game if it is a w-potential game for some $w \in \mathbb{R}^N_{++}$.

The following set of collections of cooperative games forms the basis for a representation theorem of weighted potential games.

$$\mathcal{G}_{N,X} := \left\{ \{(N, v_x)\}_{x\in X} \in (TU^N)^X \mid v_x(R) = v_t(R) \text{ if } x_R = t_R \text{ for all } x, t \in X,\ R \subseteq N \right\}. \tag{2}$$

The representation theorem of Slikker et al. (2000) describes a relation between weighted potential games and weighted Shapley values of cooperative games.[2]

[2]Ui (2000) provides a similar theorem for (unweighted) potential games in terms of Shapley values.

Theorem 2.1 Let $\Gamma = (N; (X_i)_{i \in N}; (\pi_i)_{i \in N})$ be a game in strategic form. Γ is a w-potential game if and only if there exists $\{(N, v_x)\}_{x \in X} \in \mathcal{G}_{N,X}$ such that

$$\pi_i(x) = \Phi_i^w(v_x) \text{ for all } i \in N \text{ and all } x \in X. \tag{3}$$

Proof: See Slikker et al. (2000). □

3 Allocation rules

In this section we study allocation rules for directed reward situations. Besides an efficiency requirement we demand that an allocation rule treats directed communication relations symmetric. However, we would like the allocation rule to distinguish between the two players forming a directed communication relation. These two features are captured in the property *directed fairness*. We show that in general there is no allocation rule that satisfies the efficiency requirement and directed fairness. We then proceed in two different directions. Firstly, we study some alternative fairness principle that is more widely applicable but less appealing in the setting of this paper. Secondly, we study the necessary and sufficient conditions on a set of directed communication networks to ensure existence of an allocation rule satisfying the efficiency requirement and directed fairness.

Let $\mathcal{A} \subseteq \mathrm{DG}^N$ be a closed collection of directed graphs. Consider the following property for an allocation rule γ on $\mathrm{DRS}^{N,\mathcal{A}}$:

Component efficiency (CE): For all directed reward situations $(N, r, A) \in \mathrm{DRS}^{N,\mathcal{A}}$ and all $C \in N/A$ it holds that

$$\sum_{i \in C} \gamma_i(N, r, A) = r(A(C)). \tag{4}$$

Furthermore, for any $\alpha \geq 1$ consider the following property for an allocation rule γ on $\mathrm{DRS}^{N,\mathcal{A}}$:

α-Directed fairness (α-DF): For all directed reward situations $(N, r, A) \in \mathrm{DRS}^{N,\mathcal{A}}$ and all $i, j \in N$ with $(i, j) \in A$ it holds that

$$\gamma_i(N, r, A) - \gamma_i(N, r, A \backslash \{(i, j)\}) = \alpha \Big[\gamma_j(N, r, A) - \gamma_j(N, r, A \backslash \{(i, j)\}) \Big]. \tag{5}$$

Component efficiency states that the profits of a component should be divided among the players in this component. The property α-directed fairness states that, though the change in payoff for the initiator (i) and the receiver (j) might be different, it holds that if one of the differences is non-zero then the ratio between the two differences is the same for all directed communication relations. This constant ratio is denoted by α and represents the difference between an initiator and a receiver in a directed communication relation. If $\alpha = 1$ then the initiator and the receiver experience the same influence of an additional link. We will mainly restrict ourselves to $\alpha > 1$. This states that an initiator experiences a greater influence of an additional link than a receiver. We say that an allocation rule satisfies *directed fairness* if this allocation rule satisfies α-directed fairness, for some $\alpha > 1$. We note that the mathematical analysis that follows would not change if we consider $\alpha \in (0, 1)$, but such a weight is in contrast with the natural assumption that an initiator experiences a greater influence of an additional link than a receiver. Furthermore, we remark that the distinction between an initiator and a receiver is not determined a priori. In fact, we do not even exclude that both (i, j) and (j, i) belong to a directed communication network.

The following example shows that in general there is no allocation rule that satisfies component efficiency and α-directed fairness with $\alpha > 1$.

Example 3.1 Let $N = \{1, 2, 3\}$, $\mathcal{A}$ the set of all subsets of $A^* = \{(1, 2), (2, 3), (3, 1)\}$, and r the directed reward function on $\mathcal{A}$ with $r(\{(1, 2), (2, 3), (3, 1)\}) = 1$ and $r(A) = 0$ otherwise. Suppose γ is an allocation rule that satisfies CE and α-DF. We will show that it must hold that $\alpha = 1$.

Firstly, by CE it follows that $\gamma_i(N, r, \emptyset) = 0$ for all $i \in N$. Subsequently, consider $A = \{(j, k)\}$. Then the remaining player i receives 0 by CE. Furthermore, $\gamma_j(N, r, \{(j, k)\}) = \alpha\Big[\gamma_k(N, r, \{(j, k)\})\Big]$ by α-DF. Using CE it then follows for the payoffs of players j and k that $\gamma_j(N, r, \{(j, k)\}) = \gamma_k(N, r, \{(j, k)\}) = 0$. With similar arguments we find that for all $A \subset A^*$ and all $i \in \{1, 2, 3\}$ it holds that $\gamma_i(N, r, A) = 0$.

Subsequently, consider A^*. By α-DF it follows that

$$\gamma_1(N, r, A^*) - \gamma_1(N, r, A^*\backslash\{(1, 2)\}) = \alpha\Big[\gamma_2(N, r, A^*) - \gamma_2(N, r, A^*\backslash\{(1, 2)\})\Big];$$
$$\gamma_2(N, r, A^*) - \gamma_2(N, r, A^*\backslash\{(2, 3)\}) = \alpha\Big[\gamma_3(N, r, A^*) - \gamma_3(N, r, A^*\backslash\{(2, 3)\})\Big];$$
$$\gamma_3(N, r, A^*) - \gamma_3(N, r, A^*\backslash\{(3, 1)\}) = \alpha\Big[\gamma_1(N, r, A^*) - \gamma_1(N, r, A\backslash\{(3, 1)\})\Big].$$

Using that $\gamma_i(N, r, A) = 0$ for all $A \subset A^*$ and all $i \in N$ we find

$$\begin{aligned}\gamma_1(N, r, A^*) &= \alpha\gamma_2(N, r, A^*)\\ &= \alpha^2\gamma_3(N, r, A^*)\\ &= \alpha^3\gamma_1(N, r, A^*)\end{aligned}$$

Hence, it follows that $\alpha = 1$ or $\gamma_1(N, r, A^*) = 0$. The last possibility would imply that $\sum_{i \in N} \gamma_i(N, r, A^*) = 0$, since $\gamma_2(N, r, A^*) = \alpha^2\gamma_1(N, r, A^*)$ and $\gamma_3(N, r, A^*) = \alpha\gamma_1(N, r, A^*)$, contradicting CE. We conclude that $\alpha = 1$ should hold, otherwise there is no allocation rule that satisfies α-directed fairness and component efficiency. $\diamond$

So, in general there exists no allocation rule that satisfies CE and α-DF for some $\alpha > 1$. However, for several classes of directed graphs that are interesting in analyzing directed communication networks there exists an allocation rule satisfying these properties. These classes are studied in section 3.2. Before that, in section 3.1 we study an alternative fairness-criterion.

3.1 Allocation rules with fixed weights

In this section we introduce and study weighted allocation rules, called weighted directed communication values. We show that these values can be characterized by two properties, component efficiency and w-fairness.

Before we introduce these allocation rules we need some additional notation. Let $\mathcal{A}$ be a closed collection of directed communication networks. For every directed reward situation $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$ define an associated cooperative game $(N, v^{r,A})$ as follows:

$$v^{r,A}(T) = r(A(T)) \text{ for all } T \subseteq N.$$

Then the *directed communication value* δ of directed reward situation $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$ is defined by

$$\delta(N, r, A) = \Phi(N, v^{r,A}),$$

where Φ denotes the Shapley value.

Similarly, the *weighted directed communication value* δ^w of directed reward situation $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$ with weights $w \in \mathbb{R}^N_{++}$ is defined by

$$\delta^w(N, r, A) = \Phi^w(N, v^{r,A}),$$

where Φ^w denotes the w-Shapley value. Note that $\delta = \delta^w$ with $w = (1, 1, \ldots, 1)$. We will sometimes refer to the weighted directed communication value with weights w as the w-directed communication value.

We will characterize weighted directed communication values by two properties, component efficiency and an alternative fairness criterion. Therefore, for all $w \in \mathbb{R}^N_{++}$ consider the following property for an allocation rule on $\text{DRS}^{N,\mathcal{A}}$:

w-Fairness (w-F): For all directed reward situations $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$, and all $(i, j) \in A$ it holds that

$$\frac{\gamma_i(N, r, A) - \gamma_i(N, r, A\backslash\{(i, j)\})}{w_i} = \frac{\gamma_j(N, r, A) - \gamma_j(N, r, A\backslash\{(i, j)\})}{w_j}.$$

So, w-fairness states that the change in payoff for two players forming an additional directed communication relation is proportional to their weights. Note that although at first sight this property may seem similar to α-directed fairness, there is a big difference: the weights in w-fairness are determined by the nature of the players, whereas in α-directed fairness the weights are determined by the nature of the relation between the two players. We remark that w-fairness with $w = (1, \ldots, 1)$ coincides with α-directed fairness with $\alpha = 1$.

The following lemma shows that the w-directed communication value satisfies the two properties component efficiency and w-fairness. In the proof we use some results of Kalai and Samet (1988). In this paper, it is shown that the w-Shapley value satisfies the *dummy property*, *additivity*, and *partnership consistency*. The *dummy property* states that $\Phi^w_i(N, v) = v(\{i\})$ for all (N, v) with $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$. *Additivity* states that $\Phi^w(N, v + z) = \Phi^w(N, v) + \Phi^w(N, z)$ for all cooperative games (N, v) and (N, z). To describe *partnership consistency* we need the notion of *partnership*. A coalition $S \subseteq N$ is a *partnership* in (N, v) if for all $T \subset S$ and all $R \subseteq N\backslash S$, $v(R \cup T) = v(R)$. Partnership consistency of Φ^w states that for every partnership S in (N, v) it holds that

$$\Phi^w_i(v) = \Phi^w_i\left(\Phi^w_S(v)\; u_S\right), \text{ for every } i \in S,$$

where $\Phi^w_S(v) = \sum_{j \in S} \Phi^w_j(v)$.

In the following lemma we show that the w-directed communication value satisfies component efficiency and w-fairness.

Lemma 3.1 Let $w \in \mathbb{R}^N_{++}$. The w-directed communication value, δ^w, satisfies component efficiency and w-fairness on $\text{DRS}^{N,\mathcal{A}}$.

Proof: First we will show that δ^w satisfies component efficiency. Let $(N,r,A) \in \mathrm{DRS}^{N,\mathcal{A}}$ and C a component of (N,A). We define two cooperative games (N,v^C) and $(N,v^{N\backslash C})$. For all $T \subseteq N$ let

$$\begin{aligned} v^C(T) &:= r(A(T \cap C)), \\ v^{N\backslash C}(T) &:= r(A(T\backslash C)). \end{aligned}$$

Since C is a component of (N,A) and r is component additive it holds that $v^{r,A} = v^C + v^{N\backslash C}$. Since all $i \in C$ are dummy players in the game $(N,v^{N\backslash C})$, we conclude, from the dummy player property of the w-Shapley value, that $\Phi_i^w(v^{N\backslash C}) = 0$ for all $i \in C$. In the same way we find for all $i \in N\backslash C$ that $\Phi_i^w(v^C) = 0$. Using this and the additivity of the w-Shapley values we find

$$\begin{aligned} \sum_{i\in C} \Phi_i^w(v^A) &= \sum_{i\in C} \Phi_i^w(v^C) + \sum_{i\in C} \Phi_i^w(v^{N\backslash C}) \\ &= \sum_{i\in C} \Phi_i^w(v^C) = \sum_{i\in N} \Phi_i^w(v^C) = v^C(N) = r(A(C)), \end{aligned}$$

where the fourth equality follows from the efficiency of the w-Shapley value.

Secondly, we will show that the w-directed communication value satisfies w-fairness. Let $(N,r,A) \in \mathrm{DRS}^{N,\mathcal{A}}$ and $(i,j) \in A$. Define $A' := A\backslash\{(i,j)\}$ and $v' := v^{r,A} - v^{r,A'}$. For all $T \subseteq N$ with $\{i,j\} \not\subseteq T$ we then have

$$v'(T) = r(A(T)) - r(A'(T)) = 0$$

since $A(T) = A'(T)$. This means that $\{i,j\}$ is a partnership in v'. From partnership consistency of Φ^w, it follows that

$$\Phi_i^w(v') = \Phi_i^w\left(\left(\Phi_i^w(v') + \Phi_j^w(v')\right)u_{\{i,j\}}\right) = \frac{w_i}{w_i + w_j}\left(\Phi_i^w(v') + \Phi_j^w(v')\right).$$

Similarly,

$$\Phi_j^w(v') = \frac{w_j}{w_i + w_j}\left(\Phi_i^w(v') + \Phi_j^w(v')\right).$$

So,

$$\frac{\Phi_i^w(v')}{w_i} = \frac{\Phi_j^w(v')}{w_j}.$$

From this we find

$$\frac{\delta_i^w(N,r,A) - \delta_i^w(N,r,A')}{w_i} = \frac{\Phi_i^w(v')}{w_i} = \frac{\Phi_j^w(v')}{w_j} = \frac{\delta_j^w(N,r,A) - \delta_j^w(N,r,A')}{w_j},$$

where the first and third equalities follow from the definition of the game (N, v') and the additivity of the w-Shapley values. Hence, δ^w satisfies w-fairness. □

Subsequently, we show that there exists at most one allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and w-fairness.

Lemma 3.2 Let $w \in \mathbb{R}^N_{++}$. There is at most one allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and w-fairness.

Proof: Suppose there are two rules γ^1 and γ^2 that satisfy component efficiency and w-fairness. We will show that γ^1 coincides with γ^2 on $\text{DRS}^{N,\mathcal{A}}$ by induction to the number of arcs.

Firstly, let $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$ be a directed reward situation with $|A| = 0$. Then it follows directly by component efficiency that $\gamma^1(N, r, A) = \gamma^2(N, r, A)$.

Secondly, let $p \geq 1$ and suppose that $\gamma^1(N, r, A) = \gamma^2(N, r, A)$ for all directed reward situations in $\text{DRS}^{N,\mathcal{A}}$ with $|A| \leq p - 1$. Let $(N, r, A) \in \text{DRS}^{N,\mathcal{A}}$ be a directed reward situation with $|A| = p$. Let $(i, j) \in A$. From w-fairness of γ^1 we then find

$$\frac{1}{w_i}\left(\gamma^1_i(A) - \gamma^1_i(A\backslash\{(i,j)\})\right) = \frac{1}{w_j}\left(\gamma^1_j(A) - \gamma^1_j(A\backslash\{(i,j)\})\right).$$

Using this, the induction hypothesis, and the w-fairness of γ^2 respectively, we find

$$\begin{aligned} w_j\gamma^1_i(A) - w_i\gamma^1_j(A) &= w_j\gamma^1_i(A\backslash\{(i,j)\}) - w_i\gamma^1_j(A\backslash\{(i,j)\}) \\ &= w_j\gamma^2_i(A\backslash\{(i,j)\}) - w_i\gamma^2_j(A\backslash\{(i,j)\}) \\ &= w_j\gamma^2_i(A) - w_i\gamma^2_j(A). \end{aligned}$$

So

$$\frac{\gamma^1_i(A) - \gamma^2_i(A)}{w_i} = \frac{\gamma^1_j(A) - \gamma^2_j(A)}{w_j}.$$

This expression is valid for all pairs $\{i, j\}$ with $(i, j) \in A$ or $(j, i) \in A$. Hence, it is also valid for all pairs $\{s, t\}$ that are in the same component.

Let $i \in N$ and let $C \in N/A$ with $i \in C$. For all $j \in C$ we now have

$$\frac{1}{w_j}\left(\gamma^1_j(A) - \gamma^2_j(A)\right) = \frac{1}{w_i}\left(\gamma^1_i(A) - \gamma^2_i(A)\right).$$

Let $d := \frac{1}{w_i}(\gamma_i^1(A) - \gamma_i^2(A))$. Then for all $j \in C$: $\gamma_j^1(A) - \gamma_j^2(A) = w_j d$. Component efficiency of γ^1 and γ^2 gives us

$$\sum_{j \in C} \gamma_j^1(A) = \sum_{j \in C} \gamma_j^2(A) = r(A(C)).$$

Thus,

$$0 = \sum_{j \in C} \left(\gamma_j^1(A) - \gamma_j^2(A)\right) = \sum_{j \in C} w_j d.$$

Since $w \in \mathbb{R}^N_{++}$ it follows that $d = 0$. Since i was chosen arbitrarily, we conclude that $\gamma^1(A) = \gamma^2(A)$.

This completes the proof. □

Combining the two lemmas above, the following theorem follows directly.

Theorem 3.1 Let $w \in \mathbb{R}^N_{++}$. The w-directed communication value is the unique allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and w-fairness.

Proof: Follows directly from lemmas 3.1 and 3.2. □

3.2 Directed fairness on subclasses of networks

In this section we study what conditions on the class of directed graphs are necessary and sufficient for the existence of an allocation rule that satisfies component efficiency and directed fairness.

Before we consider these conditions we prove the following lemma.

Lemma 3.3 Let $\alpha > 1$ and let $\mathcal{A}$ be a closed collection of directed communication networks. There is at most one allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and α-directed fairness.

Proof: Suppose there are two rules γ^1 and γ^2 that satisfy component efficiency and α-directed fairness. We will show that γ^1 coincides with γ^2 on $\text{DRS}^{N,\mathcal{A}}$ by induction to the number of arcs.

Firstly, let $(N,r,A) \in \text{DRS}^{N,\mathcal{A}}$ be a directed reward situation with $|A| = 0$. Then it follows directly by component efficiency that $\gamma^1(N,r,A) = \gamma^2(N,r,A)$.

Secondly, let $p \geq 1$ and suppose that $\gamma^1(N,r,A) = \gamma^2(N,r,A)$ for all directed reward situations in $\text{DRS}^{N,\mathcal{A}}$ with $|A| \leq p-1$. Let $(N,r,A) \in \text{DRS}^{N,\mathcal{A}}$ be a directed reward situation with $|A| = p$. Let $(i,j) \in A$. From α-directed fairness of γ^1 we then find

$$\gamma_i^1(A) - \gamma_i^1(A \backslash \{(i,j)\}) = \alpha \left(\gamma_j^1(A) - \gamma_j^1(A \backslash \{(i,j)\}) \right).$$

Using this, the induction hypothesis, and the α-directed fairness of γ^2 respectively, we find

$$\begin{aligned}
\gamma_i^1(A) - \alpha\gamma_j^1(A) &= \gamma_i^1(A \backslash \{(i,j)\}) - \alpha\gamma_j^1(A \backslash \{(i,j)\}) \\
&= \gamma_i^2(A \backslash \{(i,j)\}) - \alpha\gamma_j^2(A \backslash \{(i,j)\}) \\
&= \gamma_i^2(A) - \alpha\gamma_j^2(A).
\end{aligned}$$

So,

$$\gamma_i^1(A) - \gamma_i^2(A) = \alpha\Big(\gamma_j^1(A) - \gamma_j^2(A)\Big).$$

This expression is valid for all pairs $\{i,j\}$ with $(i,j) \in A$.

Let $i \in N$ and let $C \in N/A$ be such that $i \in C$. For all $j \in C \backslash \{i\}$ there exists at least one path from i to j. Denote an arbitrary but fixed path from i to j by $P_{ij} = (x_1, e_1, x_2, \ldots, x_{l-1}, e_{l-1}, x_l)$. For all $r \in \{1, \ldots, l-1\}$ it holds that

$$\gamma_{x_r}^1(A) - \gamma_{x_r}^2(A) = \begin{cases} \alpha \left(\gamma_{x_{r+1}}^1(A) - \gamma_{x_{r+1}}^2(A) \right) & \text{if } e_r = (x_r, x_{r+1}); \\ \frac{1}{\alpha} \left(\gamma_{x_{r+1}}^1(A) - \gamma_{x_{r+1}}^2(A) \right) & \text{otherwise.} \end{cases}$$

Now, define $w_j^C = \alpha^{t(P_{ij})}$ for all $j \in C \backslash \{i\}$. Furthermore, define $w_i^C = 1$. For all $j \in C$ we have

$$\left(\gamma_j^1(A) - \gamma_j^2(A) \right) = w_j^C \left(\gamma_i^1(A) - \gamma_i^2(A) \right).$$

Let $d = (\gamma_i^1(A) - \gamma_i^2(A))$. Then for all $j \in C : \gamma_j^1(A) - \gamma_j^2(A) = w_j^C d$. Component efficiency of γ^1 and γ^2 gives us

$$\sum_{j \in C} \gamma_j^1(A) = \sum_{j \in C} \gamma_j^2(A) = r(A(C)).$$

Thus,

$$0 = \sum_{j \in C} \left(\gamma_j^1(A) - \gamma_j^2(A) \right) = \sum_{j \in C} w_j^C d.$$

Since $w_j^C > 0$ for all $j \in C$ it follows that $d = 0$. Since i was chosen arbitrarily, we conclude that $\gamma^1(A) = \gamma^2(A)$. A contradiction.

This completes the proof. □

Consider the following properties for a directed communication network (N, A):

Hierarchical-classes property (HCP): There exists an (ordered) partition $\mathcal{B} = (B_1, \ldots, B_m)$ of N such that for all $(i, j) \in A$ there exists $k \in \{1, \ldots, m-1\}$ with $i \in B_{k+1}$ and $j \in B_k$.

Cycle property (CP): For every cycle $P = (x_1, e_1, x_2, \ldots, x_l, e_l, x_{l+1})$ with $x_{l+1} = x_1$ it holds that $t(P) = 0$.

The hierarchical-classes property states that the players can be partitioned in hierarchical classes such that there exist only directed communication relations from a hierarchical class to the hierarchical class directly below this class. Note that the class of graphs that satisfy the hierarchical-classes property are sometimes used to represent a hierarchical structure. Here, one can think of top-down relations between individuals in a firm. The cycle property states that every cycle in a directed communication network contains as many arcs directed one way as it contains directed the other way.

We will say that a collection of graphs satisfies the hierarchical-classes property (cycle property) if every directed communication network in this collection satisfies the hierarchical-classes property (cycle property). Note that if a directed communication network satisfies the hierarchical-classes property (cycle property) then all subgraphs of this directed communication network satisfy the hierarchical-classes property (cycle property) as well.

The following lemma shows that the hierarchical-classes property and the cycle property are equivalent.

Lemma 3.4 Let $(N, A) \in \mathrm{DG}^N$ be a directed communication network. Then (N, A) satisfies the hierarchical-classes property if and only if (N, A) satisfies the cycle property.

Proof: Firstly, we show the if-part. Suppose (N, A) satisfies the cycle property. We will assign a number to each vertex. Let $C \in N/A$ be a component in the directed communication network (N, A). Let $i \in C$ be fixed. Set $p_i^C = 0$. For all $j \in C \backslash \{i\}$ consider an arbitrary but fixed path from i to j, $P_{ij} = (x_1, e_1, x_2, \ldots, e_{l-1}, x_l)$ with $x_1 = i$ and $x_l = j$. Define

$$p_j^C = t(P_{ij}).$$

By CP it follows that p_j^C is independent of the choice of a specific path between i and j. Furthermore, note that this construction implies that for all $a, b \in C$ it holds that if $(a, b) \in A$ then $p_a^C = p_b^C + 1$. In this way p_j^C is constructed for all $C \in N/A$ and all $j \in C$. Subsequently, define

$$B_k^C = \{j \in C \mid p_j^C - \min_{r \in C} p_r^C + 1 = k\}$$

for all $C \in N/A$ and all $k \in \{1, \ldots, m^*\}$, where

$$m^* = \max_{C \in N/A} [\max_{r \in C} p_r^C - \min_{r \in C} p_r^C + 1]\}.$$

Then, for all $a, b \in N$ it holds that if $(a, b) \in A$ and $a \in B_{k+1}^C$ for some $C \in N/A$ and $k \in \{1, \ldots, m^* - 1\}$ then $b \in B_k^C$ since $p_a^C = p_b^C + 1$. Now, $(B_1, \ldots, B_{m^*})$ with $B_k = \cup_{C \in N/A} B_k^C$ for all $k \in \{1, \ldots, m^*\}$ is an ordered partition of N such that for all $(r, t) \in A$ there exists $k \in \{1, \ldots, m^* - 1\}$ with $r \in B_{k+1}$ and $t \in B_k$. We conclude that (N, A) satisfies the hierarchical-classes property.

It remains to show the only-if-part. Suppose (N, A) satisfies the hierarchical-classes property. Then there exists an ordered partition $(B_1, \ldots, B_m)$ of N such that for all $(i, j) \in A$ there exists $k \in \{1, \ldots, m - 1\}$ with $i \in B_{k+1}$ and $j \in B_k$. Consider a cycle $(x_1, e_1, x_2, \ldots, x_l, e_l, x_{l+1})$, with $x_1 = x_{l+1}$. For every $r \in \{1, \ldots, l\}$ with $x_r \in B_k$ it holds that if $e_r = (x_r, x_{r+1})$ then $x_{r+1} \in B_{k-1}$ and if $e_r = (x_{r+1}, x_r)$ then $x_{r+1} \in B_{k+1}$. Since x_1 and x_{l+1} coincide, they belong to the same partition element implying that

$$|\{r \in \{1, \ldots, l\} \mid e_r = (x_r, x_{r+1})\}| = |\{r \in \{1, \ldots, l\} \mid e_r = (x_{r+1}, x_r)\}|.$$

This completes the proof. □

In the following lemma we will show that there exists an allocation rule that satisfies component efficiency and α-directed fairness with $\alpha > 1$ on any closed collection of graphs satisfying the hierarchical-classes property. Therefore, we need some additional notation. For every (N, A) that satisfies the hierarchical-classes property denote by $\mathcal{B}(A) = (B_1^A, \ldots, B_m^A)$ an arbitrary but fixed partition of the player set such that for all $(i, j) \in A$ there exists $k \in \{1, \ldots, m - 1\}$ with $i \in B_{k+1}^A$ and $j \in B_k^A$. We remark that $\mathcal{B}(A)$ is unique if $N/A = 1$. Furthermore, define the weight vector $w_\alpha^A \in \mathbb{R}_{++}^N$ by $(w_\alpha^A)_i = \alpha^k$ for all $i \in N$, where k is such that $i \in B_k^A$.

Lemma 3.5 Let $\alpha > 1$, let (N, A) be a directed communication network that satisfies the hierarchical-classes property, and let $\mathcal{A} = \{A' \mid A' \subseteq A\}$. Then $\delta^{w_\alpha^A}$ satisfies component efficiency and α-directed fairness on $\mathrm{DRS}^{N,\mathcal{A}}$.

Proof: It follows from theorem 3.1 that $\delta^{w^A_\alpha}$ satisfies component efficiency. It remains to show that $\delta^{w^A_\alpha}$ satisfies α-directed fairness. From theorem 3.1 it follows that $\delta^{w^A_\alpha}$ satisfies w^A_α-fairness. Hence, for all $A_1 \subseteq A$ and all $(i,j) \in A_1$ it holds that

$$\frac{\delta_i^{w^A_\alpha}(A_1) - \delta_i^{w^A_\alpha}(A_1 \backslash \{(i,j)\})}{(w^A_\alpha)_i} = \frac{\delta_j^{w^A_\alpha}(A_1) - \delta_j^{w^A_\alpha}(A_1 \backslash \{(i,j)\})}{(w^A_\alpha)_j}.$$

Since $(w^A_\alpha)_i = \alpha(w^A_\alpha)_j$ it follows that

$$\delta_i^{w^A_\alpha}(A_1) - \delta_i^{w^A_\alpha}(A_1 \backslash \{(i,j)\}) = \alpha\Big(\delta_j^{w^A_\alpha}(A_1) - \delta_j^{w^A_\alpha}(A_1 \backslash \{(i,j)\})\Big).$$

Hence, $\delta^{w^A_\alpha}$ satisfies α-directed fairness. □

We remark that although w^A_α is defined by means of an arbitrary partition for which the hierarchical-classes property is satisfied, it follows by lemmas 3.3 and 3.5 that the allocation rule $\delta^{w^A_\alpha}$ is independent of the choice of $\mathcal{B}(A)$.

The following lemma states that if two allocation rules, each defined on a class of directed reward situations based on a closed collection of directed communication networks, both satisfy component efficiency and α-directed fairness then they coincide on the intersection of these classes.

Lemma 3.6 Let $\alpha > 1$ and let $\mathcal{A}_1, \mathcal{A}_2$ be two closed collections of graphs that satisfy the hierarchical-classes property. If γ_1 and γ_2 satisfy component efficiency and α-directed fairness on $\text{DRS}^{N,\mathcal{A}_1}$ and $\text{DRS}^{N,\mathcal{A}_2}$, respectively, then γ_1 and γ_2 coincide on $\text{DRS}^{N,\mathcal{A}_1 \cap \mathcal{A}_2}$

Proof: Both γ_1 and γ_2 satisfy CE and α-DF on $\text{DRS}^{N,\mathcal{A}_1 \cap \mathcal{A}_2}$. By lemma 3.3 it follows that γ_1 and γ_2 coincide on $\text{DRS}^{N,\mathcal{A}_1 \cap \mathcal{A}_2}$. □

The following theorem deals with arbitrary closed classes of graphs that satisfy the hierarchical-classes property. For any collection of directed reward situations based on such a class there is a unique allocation rule that satisfies component efficiency and α-directed fairness.

Theorem 3.2 Let $\alpha > 1$ and let $\mathcal{A}$ be a closed collection of graphs that satisfies the hierarchical-classes property. Then there exists a unique allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and α-directed fairness.

Proof: Let

$$\mathcal{A}^{\max} = \left\{ A \in \mathcal{A} \,\middle|\, \text{there is no } A' \in \mathcal{A} \text{ with } A \subset A' \right\}.$$

Let r be a directed reward function on $\mathcal{A}$. Let $A \in \mathcal{A}$. Choose $A' \in \mathcal{A}^{\max}$ such that $A \subseteq A'$. Define

$$\gamma(N, r, A) = \delta^{w_\alpha^{A'}}(N, r, A).$$

By lemma 3.6 it follows that $\gamma(N, r, A)$ is independent of the choice of $A' \in \mathcal{A}^{\max}$. Hence, γ is well-defined and γ coincides with $\delta^{w_\alpha^{A'}}$ on $\mathcal{A}' = \{A \mid A \subseteq A'\}$ for all $A' \in \mathcal{A}^{\max}$. By lemma 3.5 it follows that γ satisfies CE and α-DF on $\text{DRS}^{N,\mathcal{A}'}$ for all $A' \in \mathcal{A}^{\max}$. We conclude that γ satisfies CE and α-DF on $\text{DRS}^{N,\mathcal{A}}$.

Lemma 3.3 implies that γ is the unique allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies CE and α-DF. □

The following theorem shows that if a collection of directed communication networks contains a directed communication network that does not satisfy the hierarchical-classes property then there is no allocation rule that satisfies component efficiency and directed fairness.

Theorem 3.3 Let (N, A^*) be a directed communication network that does not satisfy the hierarchical-classes property. Let $\mathcal{A}$ be a closed collection of graphs with $A^* \in \mathcal{A}$. Then there is no allocation rule γ that satisfies component efficiency and α-directed fairness with $\alpha > 1$.

Proof: By lemma 3.4 it follows that (N, A^*) does not satisfy CP. Consider a cycle $P = (x_1, e_1, x_2, \ldots, x_l, e_l, x_{l+1})$ with $x_1 = x_{l+1}$ and $t(P) \neq 0$. Denote $A' = \{e_1, \ldots, e_l\}$. Consider the directed reward function r on $\mathcal{A}$ defined by

$$r(A) = \begin{cases} 1 & \text{if } A = A'; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose γ is an allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies CE and α-DF with $\alpha > 1$. Then, with arguments similar to those in example 3.1, it follows that

$$\gamma_i(N, r, A) = 0 \text{ for all } A \subset A' \text{ and all } i \in N.$$

Then α-DF implies that for all $r \in \{1, \ldots, l\}$ it holds that

$$\gamma_{x_{r+1}}(N, r, A') = \begin{cases} \alpha\gamma_{x_r}(N, r, A') & \text{if } (x_{r+1}, x_r) \in A'; \\ \frac{1}{\alpha}\gamma_{x_r}(N, r, A') & \text{otherwise.} \end{cases} \tag{6}$$

Hence,

$$\gamma_{x_{l+1}}(N, r, A') = \alpha^{t(P)} \gamma_{x_1}(N, r, A').$$

Since $t(P) \neq 0$ and $x_1 = x_{l+1}$ it follows that $\alpha = 1$ or $\gamma_{x_1}(N, r, A') = 0$. Since $\alpha > 1$ it follows that $\gamma_{x_1}(N, r, A') = 0$. But then it follows by equation (6) that $\gamma_{x_r}(N, r, A') = 0$ for all $r \in \{1, \dots, l\}$. This contradicts CE since $\{x_1, \dots, x_l\}$ is a component in (N, A'). □

Combining theorems 3.2 and 3.3 we get the following theorem

Theorem 3.4 Let $\alpha > 1$ and let $\mathcal{A}$ be a closed collection of directed communication networks. Then $\mathcal{A}$ satisfies the hierarchical-classes property if and only if there exists an allocation rule on $\text{DRS}^{N,\mathcal{A}}$ that satisfies component efficiency and α-directed fairness. Moreover, if such an allocation rule exists then it is unique.

Proof: Follows directly from theorems 3.2 and 3.3. □

The value of theorem 3.4 will be called the α-hierarchical value and will be denoted by h^α. In case there is no ambiguity about α this value will sometimes be called the hierarchical value. We stress once more that a hierarchical value exists only if the players can be partitioned in hierarchical classes for any network in the class of directed communication networks, i.e., the collection of directed communication networks satisfies the hierarchical-classes property.

4 Hierarchy formation

In this section we will study several ways to model the formation of a directed communication network. We discuss several possibilities for the modeling of the strategies of the players and the formation of a directed communication network given the strategies of the players. We start with the description of a general model and discuss some of the problems that arise. Subsequently, we will refine our model in several directions. We want to apply the hierarchical value developed in the previous section, which is properly defined for classes of directed communication networks that satisfy the hierarchical-classes property. Therefore, in this section we will usually refer to a directed communication network as a hierarchical structure. Additionally, we will sometimes call an initiator of a hierarchical relation a boss and we will sometimes call a receiver a subordinate.

Firstly, we describe a general model of hierarchical structure formation. Let N be a set of players, $\mathcal{A} = \mathrm{DG}^N$ the collection of directed communication networks on player set N, and r a directed reward function on $\mathcal{A}$. Let γ be an allocation rule on $\mathrm{DRS}^{N,\mathcal{A}}$. Define the *hierarchy formation game* $\Gamma^{hf}(N, r, \gamma)$ determined by the tuple $(N; (S_i)_{i\in N}; (f_i^\gamma)_{i\in N})$ where for all $i \in N$

$$S_i = \Big\{(s_i^1, s_i^2) \mid s_i^1, s_i^2 \subseteq N\backslash\{i\}; s_i^1 \cap s_i^2 = \emptyset\Big\}$$

represents the strategy set of player i. A strategy of player i, $s_i = (s_i^1, s_i^2)$, denotes the set of players he wants to be a subordinate to (s_i^1) and the set of players he wants to be a boss of (s_i^2). A strategy profile $s = (s_1, \ldots, s_n) \in \prod_{i\in N} S_i$, induces a set of directed links $A(s)$ given by

$$A(s) = \big\{(i, j) \mid i \in s_j^1;\ j \in s_i^2\big\}.$$

The interpretation is that a (directed) arc is formed if and only if both players involved in the arc are willing to form it. The payoff function $f^\gamma = (f_i^\gamma)_{i\in N}$ is then defined as the allocation rule applied to the hierarchical structure formed, i.e.,

$$f^\gamma(s) = \gamma(N, r, A(s)) \text{ for all } s \in S.$$

In case there is no ambiguity about the underlying player set and the directed reward function we will simply write $\Gamma^{hf}(\gamma)$ instead of $\Gamma^{hf}(N, r, \gamma)$. In the remainder we will consider an arbitrary directed reward function r on $\mathrm{DRS}^{N,\mathcal{A}}$.

Consider the following example.

Example 4.1 Let $N = \{1, 2, 3\}$ and $\mathcal{A} = \mathrm{DG}^N$ the set of all directed graphs on player set N. Let r be a directed reward function on $\mathcal{A}$ and let γ be an allocation rule. Consider the hierarchy formation game $\Gamma^{hf}(N, r, \gamma)$ and suppose that the players play strategies

$$\begin{aligned} s_1 &= (\{2\}, \{3\}); \\ s_2 &= (\{3\}, \{1\}); \\ s_3 &= (\{1\}, \{2\}). \end{aligned}$$

According to the formation rule this results in the formation of the arcs $(2, 1)$, $(3, 2)$, and $(1, 3)$. Hence, the resulting structure does not satisfy the hierarchical-classes property. $\diamond$

Example 4.1 illustrates a problem that arises in the model $\Gamma^{hf}(N, r, \gamma)$. Our general model may result in (directed) cycles. According to theorem 3.1 there exists no allocation rule that satisfies the properties component efficiency and α-directed fairness on a class of directed communication networks that does not satisfy HCP.

There are several ways to deal with this problems. We mention a few. Firstly, one could adjust the formation rule in the sense that if a directed cycle results, all relations will break down or, more subtle, all relations in each component that contains a cycle will break down. Another possibility is to break down all those links that belong to at least one cycle only. These methods can be applied to ensure that the resulting directed communication network satisfies the hierarchical-classes property. Here, we will take two different approaches. In section 4.1 we analyze a model of hierarchy formation where players are a priori partitioned in hierarchical classes. In section 4.2 we adjust the general model of hierarchy formation by adding the choice of a hierarchical class to the strategies of the players.

4.1 Formation with fixed hierarchical classes

In this section we model the formation of directed communication networks in case the players are a priori partitioned in hierarchical classes. We will present a slight modification of our general model of hierarchy formation. Generally, we will apply the *hierarchical value*, discussed in section 3, as an allocation rule. We will show that in that case the game is a weighted potential game.

Firstly, we need some additional notation. Let N be a set of players and $\mathcal{B} = (B_1, \ldots, B_m)$ an ordered partition of the set of players. For notational convenience let $B_0 = B_{m+1} = \emptyset$. Let $\alpha > 1$ be the weight that represents the relative strength of an initiator in a directed communication relation. For all $k \in \{1, \ldots, m\}$ and all $i \in B_k$ define

$$(w_\alpha^{\mathcal{B}})_i = \alpha^k.$$

Furthermore, with $(S_i)_{i \in N}$ the strategy sets as in the hierarchy formation game, we define for all $Y = \prod_{i \in N} Y_i$ with $Y_i \subseteq S_i$ for all $i \in N$,

$$\mathcal{A}(Y) = \{A(y) \mid y \in Y\}.$$

Not all classes of directed communication networks can be generated by our adjusted model of hierarchy formation. Therefore, we make some assumptions on the class of directed communication networks graphs we consider. Consider an ordered partition $\mathcal{B}$ of the player set. Let $\mathcal{A}$ be a set of directed communication networks that satisfies the following conditions:

1. $\mathcal{A}$ is closed under taking inclusions.

2. For all $A \in \mathcal{A}$ and all $(i,j) \in A$ it holds that there exists $k \in \{1,\ldots,m-1\}$ with $j \in B_k$ and $i \in B_{k+1}$.

3. There exist $Y_i \subseteq S_i = \{(s_i^1, s_i^2) \mid s_i^1, s_i^2 \subseteq N\backslash\{i\}; s_i^1 \cap s_i^2 = \emptyset\}$ for all $i \in N$ such that $\mathcal{A}(Y) = \mathcal{A}$.

The first condition is made throughout this paper. The second condition makes sure that the graphs respect the ordered partition of the players. The last condition is made to make sure that $\mathcal{A}$ is a set of graphs that can result in a hierarchy formation game. A set $\mathcal{A}$ that satisfies these conditions will be called a $\mathcal{B}$-constrained closed set.

Some examples of classes of graphs that satisfy the conditions above are the class of all graphs that respect the ordered player partition and the set of graphs in which every player has at most one initiator in the class directly above his own class.

Now, the *hierarchy formation game with fixed hierarchical classes* $\Gamma^{fc}(N,\mathcal{B},\mathcal{A},r,\gamma)$ with N, $\mathcal{B}$, and $\mathcal{A}$ as described above, r a reward function on $\mathcal{A}$, and γ an allocation rule on $\text{DRS}^{N,\mathcal{A}}$ is the tuple $(N;(X_i)_{i\in N};(f_i^\gamma)_{i\in N})$ where for all $i \in N$

$$X_i = \Big\{(x_i^1, x_i^2) \mid \exists A \in \mathcal{A} : x_i^1 = \{j \mid (j,i) \in A\} \text{ and } x_i^2 = \{j \mid (i,j) \in A\}\Big\}.$$

The payoff function is determined by

$$f^\gamma(x) = \gamma(N, r, A(x)) \text{ for all } x \in X.$$

We remark that some work needs to be done to show that $\mathcal{A}(X) = \mathcal{A}$. This is captured in the following lemma.

Lemma 4.1 Let $\Gamma^{fc}(N,\mathcal{B},\mathcal{A},r,\gamma)$ be a hierarchy formation game with fixed hierarchical classes determined by the tuple $(N;(X_i)_{i\in N};(f_i^\gamma)_{i\in N})$. Then $\mathcal{A}(X) = \mathcal{A}$.

Proof: By definition of X it follows directly that $\mathcal{A}(X) \supseteq \mathcal{A}$. It remains to show that $\mathcal{A}(X) \subseteq \mathcal{A}$. Therefore, let $Y = (Y_i)_{i\in N}$ be such that $\mathcal{A}(Y) = \mathcal{A}$. Such an Y exists by condition 3 in the description of $\mathcal{B}$-constrained closed sets. We will show that $\mathcal{A}(X) \subseteq \mathcal{A}(Y)$, which suffices in order to show that $\mathcal{A}(X) = \mathcal{A}$. Let $x \in X$. For every $i \in N$ we have $x_i = (x_i^1, x_i^2) \in X_i$. Then, by definition of X_i there exists $A \in \mathcal{A}$ such that

$$x_i^1 = \{j \mid (j,i) \in A\};$$
$$x_i^2 = \{j \mid (i,j) \in A\}.$$

Since $A \in \mathcal{A} = \mathcal{A}(Y)$ there exists $y_i = (y_i^1, y_i^2) \in Y_i$ with

$$y_i^1 \supseteq x_i^1 \tag{7}$$

and

$$y_i^2 \supseteq x_i^2. \tag{8}$$

Clearly, the profile $y = (y_i)_{i \in N}$ satisfies $A(y) \supseteq A(x)$. Since $\mathcal{A}(Y) = \mathcal{A}$ is closed, this implies that $\mathcal{A}(X) \subseteq \mathcal{A}(Y)$.

We conclude that $\mathcal{A}(X) = \mathcal{A}$. □

We remark that relations (7) and (8) above imply that the strategies in X_i can be seen as minimal strategies, not containing any announcements for forming a hierarchical relation with a player that cannot form according to the collection of directed communication networks under consideration.

We will show that if the $w_\alpha^\mathcal{B}$-directed communication value is applied, then this game is a weighted potential game. We will use that the $w_\alpha^\mathcal{B}$-directed communication value satisfies component efficiency and α-hierarchical fairness on $\text{DRS}^{N,\mathcal{A}}$ which is implied by the following lemma.

Lemma 4.2 Let $\mathcal{B}$ be an (ordered) partition of player set N, $\mathcal{A}$ a $\mathcal{B}$-constrained closed set of graphs, and $\alpha > 1$. Then the $w_\alpha^\mathcal{B}$-directed communication value coincides with the α-hierarchical value on $\text{DRS}^{N,\mathcal{A}}$.

Proof: Lemma 3.3 shows that there is at most one allocation rule that satisfies component efficiency and α-directed fairness. Theorem 3.1 implies that the $w_\alpha^\mathcal{B}$-directed communication value $\delta^{w_\alpha^\mathcal{B}}$ satisfies component efficiency and $w_\alpha^\mathcal{B}$-fairness. We will show that this implies that $\delta^{w_\alpha^\mathcal{B}}$ satisfies α-directed fairness.

Let $A \in \mathcal{A}$ and let $(i,j) \in A$. Then it follows from $w_\alpha^\mathcal{B}$-fairness that

$$\frac{\delta_i^{w_\alpha^\mathcal{B}}(A) - \delta_i^{w_\alpha^\mathcal{B}}(A \backslash \{(i,j)\})}{(w_\alpha^\mathcal{B})_i} = \frac{\delta_j^{w_\alpha^\mathcal{B}}(A) - \delta_j^{w_\alpha^\mathcal{B}}(A \backslash \{(i,j)\})}{(w_\alpha^\mathcal{B})_j}.$$

Since $(w_\alpha^\mathcal{B})_i = \alpha (w_\alpha^\mathcal{B})_j$ it follows that

$$\delta_i^{w_\alpha^\mathcal{B}}(N,r,A) - \delta_i^{w_\alpha^\mathcal{B}}(N,r,A \backslash \{(i,j)\}) = \alpha \Big[\delta_j^{w_\alpha^\mathcal{B}}(N,r,A) - \delta_j^{w_\alpha^\mathcal{B}}(N,r,A \backslash \{(i,j)\}) \Big].$$

Hence, $\delta^{w_\alpha^\mathcal{B}}$ satisfies α-directed fairness.

Theorem 3.4 completes the proof. □

The following theorem states that if the $w_\alpha^{\mathcal{B}}$-directed communication value is applied, then $\Gamma^{fc}(N, \mathcal{B}, \mathcal{A}, r, \gamma)$ game is a weighted potential game.

Theorem 4.1 The hierarchy formation game $\Gamma^{fc}(N, \mathcal{B}, \mathcal{A}, r, \delta^{w_\alpha^{\mathcal{B}}})$ is a weighted potential game.

Proof: Consider the following set of cooperative games, indexed by the set of strategy profiles of $\Gamma^{fc}(\delta^{w_\alpha^{\mathcal{B}}})$, $\{(N, v^{r,A(x)})\}_{x \in X}$. Let $R \subseteq N$ and $x = (x_R, x_{N\backslash R}) \in X$. Combining $v^{r,A(x)}(R) = \sum_{C \in R/A(x)} r((A(x))(C))$ with the fact that $R/A(x)$ and $(A(x))(C)$ for all $C \in R/A(x)$ do not depend on $x_{N\backslash R}$ we find that $v^{r,A(x)}(R)$ does not depend on $x_{N\backslash R}$. By (2) it follows that this implies that $\{(N, v^{A(x)})\}_{x \in X} \in \mathcal{G}_{N,X}$. Since $f_i^{\delta^{w_\alpha^{\mathcal{B}}}}(x) = \delta_i^{w^{\mathcal{B}}}(A(x)) = \Phi_i^{w^{\mathcal{B}}}(v^{r,A(x)})$ by definition, it follows by theorem 2.1 that $\Gamma^{fc}(\delta^{w_\alpha^{\mathcal{B}}})$ is a $w_\alpha^{\mathcal{B}}$-potential game. □

Note that in view of lemma 4.2 we could reformulate theorem 4.1 in terms of the α-hierarchical value rather than in terms of the α-directed communication value.

4.2 Formation with endogenous hierarchical classes

In the previous subsection we have described a game that models the formation of directed communication networks in case the players are a priori partitioned in hierarchical classes. In this section we will not make this assumption. Again we are mainly interested in models where we can apply the hierarchical value as developed in section 3.

The main difference with the models described so far, is that the players do not only specify the sets of players they want to be a boss of and be a subordinate to, but also a number that represents the hierarchical class they want to belong to.

Let N be a set of players, r a directed reward function on the set of all directed communication networks on N that satisfy HCP, and γ an allocation rule for directed reward situations. Then the *hierarchy formation game without fixed hierarchical classes* $\Gamma^{nfc}(N, r, \gamma, m)$ with $m \leq n$ is determined by the tuple $(N; (S_i)_{i \in N}; (f_i^\gamma)_{i \in N})$, where for all $i \in N$

$$S_i = \left\{ (s_i^1, s_i^2, s_i^3) \mid s_i^1, s_i^2 \subseteq N\backslash\{i\}; s_i^1 \cap s_i^2 = \emptyset; s_1^3 \in \{1, \ldots, m\} \right\}$$

represents the strategy set of player i. A strategy of player i, $s_i = (s_i^1, s_i^2, s_i^3)$, denotes the set of players he wants to be a subordinate to (s_i^1), the set of players he wants to be a boss of (s_i^2), and the hierarchical class player i wants to belong to (s_i^3). We assume that $m \leq n$, which represents a model without fixed hierarchical classes, but with a fixed *number* of hierarchical classes. A strategy profile $s = (s_1, \ldots, s_n) \in \prod_{i \in N} S_i$, induces a set of directed links $A^{nfc}(s)$ given by

$$A^{nfc}(s) = \big\{(i,j) \mid i \in s_j^1;\ j \in s_i^2; s_i^3 = s_j^3 + 1\big\}.$$

The interpretation is that a directed link is formed if and only if both players involved in the link are willing to form it and the player who wants to be receiver is in the class directly below the class of the player that wants to be an initiator in this relation. The payoff function $f^\gamma = (f_i^\gamma)_{i \in N}$ is then defined as the allocation rule applied to the hierarchy formed, i.e.,

$$f^\gamma(s) = \gamma(N, r, A^{nfc}(s)), \text{ for all } s \in S.$$

Consider the following 2-player example.

Example 4.2 Consider a situation with two players. These players can make joint profits if they cooperate. Cooperation only occurs by means of directed communication relations. Additionally, player 1 has more leading qualities than player 2. This results if player 1 is boss and player 2 subordinate in a profit equal to 1, and if player 2 is boss and player 1 subordinate in a profit equal to $a \in (0,1)$.

We model this situation by means of a directed reward function. Let $N = \{1,2\}$ and r the directed reward function defined by

$$r(A) = \begin{cases} 0 & \text{if } A = \emptyset; \\ 1 & \text{if } A = \{(1,2)\}; \\ a & \text{if } A = \{(2,1)\}, \end{cases}$$

where $a \in (0,1)$.

We want to study the formation of a directed communication network in this situation. Suppose the allocation rule that is agreed upon by the players is the α-hierarchical value. For illustrational purposes we set $\alpha = 2$, but a similar analysis can be given for any $\alpha > 1$.

Consider $\Gamma^{nfc}(N, r, h^\alpha, 2)$ with $\alpha = 2$. The strategy sets of the players are

$$S_1 = \{(\emptyset,\emptyset,1), (\emptyset,\{2\},1), (\{2\},\emptyset,1), (\emptyset,\emptyset,2), (\{2\},\emptyset,2), (\emptyset,\{2\},2)\};$$

$$S_2 = \{(\emptyset, \emptyset, 1), (\emptyset, \{1\}, 1), (\{1\}, \emptyset, 1), (\emptyset, \emptyset, 2), (\{1\}, \emptyset, 2), (\emptyset, \{1\}, 2)\}.$$

If player 1 chooses his sixth strategy and player 2 his third strategy, then player 1 will be boss and player 2 subordinate, and together they will receive 1. By α-directed fairness and component efficiency it is easily seen that player 1 will receive $\frac{2}{3}$ and player 2 will receive $\frac{1}{3}$. Similarly, if player 2 chooses his sixth strategy and player 1 his third strategy players 1 and 2 will receive $\frac{a}{3}$ and $\frac{2a}{3}$, respectively. All other strategy profiles result in no cooperation and zero payoffs for both players. Hence, both players have only 2 undominated strategies. The payoffs for the players restricted to undominated strategies for all players are denoted in figure 1.

	$s_2 = (\{1\}, \emptyset, 1)$	$t_2 = (\emptyset, \{1\}, 2)$
$s_1 = (\{2\}, \emptyset, 1)$	0, 0	$\frac{a}{3}, \frac{2a}{3}$
$t_1 = (\emptyset, \{2\}, 2)$	$\frac{2}{3}, \frac{1}{3}$	0, 0

Figure 1: Part of the payoff matrix of $\Gamma^{nfc}(N, v, h^\alpha, 2)$ with $\alpha = 2$.

By considering the complete payoff matrix it can be shown that the game in strategic form is not a (weighted) potential game. The game in strategic form described above has multiple equilibria. For example, a strategy profile representing that both players do not want to form anything is a Nash equilibrium since two players are needed to form a directed communication relation, so no player can form a link by unilaterally deviating. If we restrict ourselves to undominated strategies then (s_1, t_2) and (t_1, s_2) are the only pure Nash equilibria and $(\frac{1}{1+2a}s_1 + \frac{2a}{1+2a}t_1, \frac{a}{2+a}s_2 + \frac{2}{2+a}t_2)$, resulting in expected payoffs $(\frac{4a^2+2a}{6a^2+15a+6}, \frac{2a^2+4a}{6a^2+15a+6})$ is a Nash equilibrium in mixed strategies. Note that the equilibrium in mixed strategies is payoff dominated by (t_1, s_2).

Apparently, for $a \in (\frac{1}{2}, 1)$ selecting between the pure equilibria is not straightforward. However, if $a \leq \frac{1}{2}$, then (s_1, t_2) would be natural to be selected since it is the unique strong Nash equilibrium.

We conclude that in this example the formation of an efficient hierarchical structure, i.e., the hierarchical structure which highest total profit, seems to form with certainty only if the difference with the total profits that can be obtained in other directed communication networks is sufficiently large. $\diamond$

The model considered in this subsection provides an endogenous explanation for the division of players in different hierarchical classes. Although a player profits more from a directed communication relation if he is in a higher hierarchical class, not all players will choose the highest hierarchical class. Note that no profits for any player can be made if all players prefer the highest hierarchical class, since in that case no directed communication relations result.

The example shows that it can be expected that the efficient directed communication network results only if the difference between the gains in this directed communication network and other directed communication networks is sufficiently large. Whether this results can be extended to more general settings should be the topic of further research.

References

Aumann, R. and Myerson, R. (1988). Endogenous formation of links between players and coalitions: an application of the Shapley value. In Roth, A., editor, *The Shapley Value*, pages 175–191. Cambridge University Press, Cambridge, United Kingdom.

Bala, V. and Goyal, S. (2000). A non-cooperative model of network formation. *Econometrica* (to appear).

Borm, P., Owen, G., and Tijs, S. (1992). On the position value for communication situations. *SIAM Journal on Discrete Mathematics*, 5:305–320.

Coase, R. (1937). The nature of a firm. *Economica*, 4:386–405. Reprinted in *Readings in Price Theory*, eds. G. Stigler and K. Boulding (Homewood, Ill.:Irwin, 1952).

Derks, J. and Gilles, R. (1995). Hierarchical organization structures and constraints on coalition formation. *International Journal of Game Theory*, 24:147–163.

Dutta, B. and Mutuswami, S. (1997). Stable networks. *Journal of Economic Theory*, 76:322–344.

Dutta, B., van den Nouweland, A., and Tijs, S. (1998). Link formation in cooperative situations. *International Journal of Game Theory*, 27:245–256.

Feinberg, Y. (1998). An incomplete cooperation structure for a voting game can be strategically stable. *Games and Economic Behavior*, 24:2–9.

Gilles, R. and Owen, G. (1999). Cooperative games and disjunctive permission structures. CentER Discussion Paper 9920, Tilburg University, Tilburg, The Netherlands.

Gilles, R., Owen, G., and van den Brink, R. (1992). Games with permission structures: the conjunctive approach. *International Journal of Game Theory*, 20:277–293.

Hamiache, G. (1999). A value with incomplete information. *Games and Economic Behavior*, 26:59–78.

Jackson, M. (1999). The stability and efficiency of directed communication networks. Working paper.

Jackson, M. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71:44–74.

Johnson, C. and Gilles, R. (1999). Spatial social networks. CentER Discussion Paper 9987, Tilburg University, Tilburg, The Netherlands.

Kalai, E. and Samet, D. (1988). Weighted Shapley values. In Roth, A., editor, *The Shapley Value*, pages 83–99. Cambridge University Press, Cambridge, United Kingdom.

Monderer, D. and Shapley, L. (1996). Potential games. *Games and Economic Behavior*, 14:124–143.

Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229.

Myerson, R. (1991). *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts.

Qin, C. (1996). Endogenous formation of cooperation structures. *Journal of Economic Theory*, 69:218–226.

Shapley, L. (1953). A value for n-person games. In Tucker, A. and Kuhn, H., editors, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton.

Slikker, M. (2000). *Decision Making and Cooperation Restrictions*. PhD thesis, Tilburg University, Tilburg, The Netherlands.

Slikker, M., Dutta, B., van den Nouweland, A., and Tijs, S. (2000). Potential maximizers and network formation. *Mathematical Social Sciences*, 39:55–70.

Slikker, M. and Norde, H. (2000). Incomplete stable structures in symmetric convex games. Working paper.

Ui, T. (2000). A Shapley value representation of potential games. *Games and Economic Behavior*, 31:121–135.

van den Brink, R. and Borm, P. (1995). Digraph competitions and cooperative games. TI-Discussion Paper 95-142, Tinbergen Institute, Amsterdam, The Netherlands.

van den Brink, R. and Gilles, R. (1994). A social power index for hierarchically structured populations of economic agents. In Gilles, R. and Ruys, P., editors, *Imperfections and Behavior in Economic Organizations*, pages 195–227. Kluwer Academic Publishers, Boston.

van den Brink, R. and Gilles, R. (1996). Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behavior*, 12:113–126.

van den Nouweland, A. (1993). *Games and Graphs in Economic Situations.* PhD thesis, Tilburg University, Tilburg, The Netherlands.

Watts, A. (1997). A dynamic model of network formation. Working paper.