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The $D_1$-Triangulation of $\mathbb{R}^n$ for Simplicial Algorithms for Computing Solutions of Nonlinear Equations

by

Chuangyin Dang


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THE D₁-TRIANGULATION OF Rⁿ FOR SIMPLICIAL ALGORITHMS FOR COMPUTING SOLUTIONS OF NONLINEAR EQUATIONS*

CHUANGYIN DANG

We present a new triangulation of Rⁿ, which is called the D₁-triangulation, for computing zero points or fixed points of nonlinear mappings. The D₁-triangulation subdivides the unit cube and is based on very elementary pivot rules. We compare the D₁-triangulation to several well-known triangulations of Rⁿ which triangulate the unit cube. According to several measures of efficiency the new triangulation is superior, such as the number of simplices in the unit cube, the diameter of a triangulation, the average directional density, and the surface density.

1. Introduction. There are now a number of simplicial algorithms for computing zero points or fixed points using triangulations of Rⁿ, for example, Merrill's homotopy restart method [5] and van der Laan and Talman's variable dimension simplicial restart algorithms without an extra dimension [4]. The other variable dimension algorithms have been introduced by Wright [9] and by Kojima and Yamamoto [3]. Allgower and Georg's paper [1] is an excellent survey of this field.

It has been accepted by now that the efficiency of the various simplicial homotopy and restart algorithms for solving equations is influenced in a critical manner by the triangulation employed. To evaluate and design triangulations for these algorithms, Todd, and Saigal, Solow and Wolsey established several measures in [6] and [7], such as the number of simplices in the unit cube, the diameter of a triangulation, the average directional density, and the surface density. Eaves and Yorke [2] showed that the average directional density and the surface density are equivalent.

To improve the efficiency of simplicial fixed point algorithms, we construct a new triangulation of Rⁿ and show that according to these measures it is the best of the well-known triangulations of Rⁿ, which subdivide the unit cube.

In §2 the D₁-triangulation is introduced. We describe the pivot rules of the D₁-triangulation in §3. The number of simplices in the unit cube, the diameter, and the surface density are calculated in §§4, 5, and 6, respectively.

2. The D₁-triangulation of Rⁿ. Let y⁰, y¹, ..., yᵏ be a set of vectors in Rⁿ. If they are affinely independent, then we call their convex hull, σ, a k-simplex and write

$$\sigma = \text{conv} \{y^0, y^1, ..., y^k\}.$$  

A simplex τ is called a face of a simplex σ if all vertices of τ are vertices of σ. If \( \dim \tau = \dim \sigma - 1 \), we call τ a facet of σ. In addition, if y is the vertex of σ which is not a vertex of τ, τ is called the facet of σ opposite y.

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Let $C$ be a convex subset of $\mathbb{R}^n$ and let $\dim C = m$. We call $G$ a triangulation of $C$ if

1. $G$ is a collection of $m$-simplices,
2. $C = \bigcup_{\sigma \in G} \sigma$,
3. for any $\sigma^1, \sigma^2 \in G$, $\sigma^1 \cap \sigma^2$ is either empty or a common face of both $\sigma^1$ and $\sigma^2$,
4. each $x \in C$ has a neighborhood meeting only a finite number of simplices of $G$.

We denote the collection of $j$-simplices that are faces of simplices of $G$ by $G_j$, for $j = 0, 1, \ldots, m$.

For ease of notation, let $N = \{1, 2, \ldots, n\}$, let $D^0_{1e} = \{y \in \mathbb{R}^n|\text{ all components of } y \text{ are even}\}$, and for $i = 1, 2, \ldots, n$, let $u^i$ be the $i$th unit vector in $\mathbb{R}^n$.

As follows, we construct the simplices of a new triangulation of $\mathbb{R}^n$. We assume $n \geq 2$.

**Definition 2.1.** Let $s$ denote a sign vector in $\mathbb{R}^n$ such that $s_i \in \{-1, +1\}$ for all $i \in N$. Let $0 \leq p \leq n - 1$ be an integer. Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the $n$ elements of $N$ such that $\pi(p) < \cdots < \pi(n)$ if $p > 1$ and $\pi(1) < \cdots < \pi(n)$ if $p = 0$.

Let $y \in D^0_{1e}$. If $p = 0$, let $y^0 = y$ and

$$ y^k = y + s_{\pi(k)} u^{\pi(k)}, \quad k = 1, 2, \ldots, n. $$

If $p > 1$, let

$$ y^0 = y + s, $$

$$ y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, \quad k = 1, 2, \ldots, p - 1, \quad \text{and} $$

$$ y^k = y + s_{\pi(k)} u^{\pi(k)}, \quad k = p, \ldots, n. $$

**Lemma 2.1.** Let $y^0, y^1, \ldots, y^n$ be obtained from Definition 2.1. Then $y^0, y^1, \ldots, y^n$ are affinely independent.

**Proof.** If $p = 0$, then let

$$ z^1 = y^1 - y^0 = s_{\pi(1)} u^{\pi(1)}, $$

$$ z^2 = y^2 - y^1 = s_{\pi(2)} u^{\pi(2)} - s_{\pi(1)} u^{\pi(1)}, $$

$$ \ldots $$

$$ z^n = y^n - y^{n-1} = s_{\pi(n)} u^{\pi(n)} - s_{\pi(n-1)} u^{\pi(n-1)}. $$

Obviously, $z^1, \ldots, z^n$ are linearly independent.

If $p > 1$, then let

$$ z^k = y^k - y^{k-1} = -s_{\pi(k)} u^{\pi(k)}, \quad k = 1, 2, \ldots, p - 1, $$

$$ z^p = y^p - y^{p-1} = -\sum_{k=p+1}^{n} s_{\pi(k)} u^{\pi(k)}, \quad \text{and} $$

$$ z^k = y^k - y^{k-1} = s_{\pi(k)} u^{\pi(k)} - s_{\pi(k-1)} u^{\pi(k-1)}, \quad k = p + 1, \ldots, n. $$
Suppose that $z_1, \ldots, z^n$ are linearly dependent. Then there exists a $q = (q_1, \ldots, q_n)^T \neq 0$ such that $q_1 z_1 + \cdots + q_n z^n = 0$. If $p = n - 1$, it is necessary that $q_1 = \cdots = q_{n-2} = 0$, $-q_{n-1} + q_n = 0$ and $q_n = 0$. We conclude that $q_1 = \cdots = q_n = 0$. If $p < n - 1$, we must have that $q_1 = \cdots = q_{p-1} = 0$, $q_{p+1} = 0$, $q_k - q_{k+1} - q_p = 0$ for $k = p + 1, \ldots, n - 1$, and $q_n - q_p = 0$. Therefore, $q_p = q_n$, $q_{n-1} = 2q_n$, $q_{n-2} = 3q_n$, $\ldots$, $q_{p+2} = (n - (p + 1))q_n$, and $q_{p+2} + q_p = 0$. Hence, $(n - (p + 1))q_n = 0$. Since $p < n - 1$, we have $q_1 = q_2 = \cdots = q_n = 0$. Thus the hypothesis is incorrect, i.e., $z_1, \ldots, z^n$ are linearly independent. Therefore, $y^0, y^1, \ldots, y^n$ are affinely independent. The proof is completed.

Let $y^0, y^1, \ldots, y^n$ be obtained from Definition 2.1. Then their convex hull is an $n$-simplex by Lemma 2.1, which is denoted by $D_1(y, \pi, s, p)$. Let $D_1$ be the collection of all such simplices $D_1(y, \pi, s, p)$.

**Lemma 2.2.** \( \cup_{\sigma \in D_1} \sigma = \mathbb{R}^n \).  

**Proof.** Let $x$ be an arbitrary point of $\mathbb{R}^n$. For each $i \in N$, let

\[
  y_i = \begin{cases} 
    [x_i] & \text{if } [x_i] \text{ is even,} \\
    [x_i] + 1 & \text{otherwise,}
  \end{cases}
\]

\[
  s_i = \begin{cases} 
    +1 & \text{if } [x_i] \text{ is even,} \\
    -1 & \text{otherwise.}
  \end{cases}
\]

We have $0 \leq \text{diag}(s_1, \ldots, s_n)(x - y) \leq u$, where $u = (1, \ldots, 1)^T$. Let $\pi'$ be a permutation of $N$ such that

\[
0 \leq s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}) \leq \cdots \leq s_{\pi'(n)}(x_{\pi'(n)} - y_{\pi'(n)}) \leq 1.
\]

If $1_{i-1} s_i(x_i - y_i) \leq 1$, let

\[
q'_i = s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}), \ldots, q'_n = s_{\pi'(n)}(x_{\pi'(n)} - y_{\pi'(n)}),
\]

and $q'_0 = 1 - \sum_{j=1}^{n} q'_j$. Obviously, $q'_j \geq 0$ for all $j$ and $\sum_{j=0}^{n} q'_j = 1$. Let $\pi = (1, 2, \ldots, n)$, $p = 0$, $y^0 = y$, and $y^k = y + s_k u^k$ for $k = 1, 2, \ldots, n$. It is easily seen that $x = \sum_{j=0}^{n} q_j y^j$, where $q_0 = q'_0$ and, for $j = 1, \ldots, n$, $q_j = q'_h$ with $h$ the index for which $\pi'(h) = j$. Thus $x \in D_1(y, \pi, s, p)$.

If $\sum_{j=1}^{n} s_j(x_j - y_j) \geq 1$, then we show that there exists an integer $1 \leq p \leq n - 1$ such that the following system has a nonnegative solution:

\[
\sum_{j=0}^{1} q'_j = s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}), \quad j = 1, \ldots, p - 1,
\]

\[
\sum_{i=0}^{p-1} q'_i + q'_k = s_{\pi'(k)}(x_{\pi'(k)} - y_{\pi'(k)}), \quad k = p, \ldots, n,
\]

\[
q'_0 + q'_1 + \cdots + q'_n = 1.
\]

\[
q'_0 + q'_1 + \cdots + q'_n = 1.
\]
In fact, rewriting the system, we obtain

\[ q_0' = s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}) \]
\[ q_{j-1}' = s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) \]
\[ -s_{\pi'(j-1)}(x_{\pi'(j-1)} - y_{\pi'(j-1)}), \quad j = 2, \ldots, p - 1, \]
\[ q_{p-1}' = -s_{\pi'(p-1)}(x_{\pi'(p-1)} - y_{\pi'(p-1)}) \]
\[ + \left( \sum_{j=p}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1 \right) / (n - p), \]
\[ q_k' = s_{\pi'(k)}(x_{\pi'(k)} - y_{\pi'(k)}) \]
\[ + \left( 1 - \sum_{j=p}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) \right) / (n - p), \quad k = p, \ldots, n. \]

Let \( N_0 = \{0, 1, \ldots, n\} \). If \( q_{n-2}' \geq 0 \) for \( p = n - 1 \), it is clear that \( q_j' \geq 0 \) for all \( j \in N_0 \), otherwise, there exists a \( p_0, 1 \leq p_0 \leq n - 2 \), such that

\[ -s_{\pi'(p_0-1)}(x_{\pi'(p_0-1)} - y_{\pi'(p_0-1)}) + \left( \sum_{j=p_0}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1 \right) / (n - p_0) \geq 0 \]
and

\[ -s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + \left( \sum_{j=p_0+1}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1 \right) / (n - p_0 - 1) < 0. \]

Hence,

\[ s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + \left( 1 - \sum_{j=p_0}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) \right) / (n - p_0) \]
\[ \geq s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + (1 - s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) \]
\[ - (n - p_0 - 1)s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) - 1) / (n - p_0 - 1) = 0. \]

Therefore, by taking \( p \) equal to \( p_0 \), \( q_j' \geq 0 \) for all \( j \in N_0 \).

Let \( 1 \leq p \leq n - 1 \) be such that the system above has a nonnegative solution and let \( \pi \) be such that \( \pi(k) = \pi'(k), k = 1, 2, \ldots, p - 1 \), and \( \pi(p) < \cdots < \pi(n) \).

Let

\[ y^0 = y + s, \]
\[ y^k = y^{k-1} - s_{\pi'(k)}u_{\pi'(k)}, \quad k = 1, \ldots, p - 1, \]
\[ y^k = y + s_{\pi'(k)}u_{\pi'(k)}, \quad k = p, \ldots, n. \]
Let \( q'_j \) be obtained from the system, for \( j = 0, 1, \ldots, n \). Then it is easily seen that 
\( x = \sum_{j=0}^n q'_j y^j \), where \( q_0 = q'_0 \) and, for \( j = 1, \ldots, n \), \( q_j = q'_j \) with \( h \) the index for which \( \pi(h) = \pi(j) \). Thus \( x \in D_1(y, \pi, s, p) \).

From these results, the lemma follows immediately. \( \square \)

**Lemma 2.3.** For any \( \sigma^1 \) and \( \sigma^2 \in D_1 \), \( \sigma^1 \cap \sigma^2 \) is either empty or a common face of both \( \sigma^1 \) and \( \sigma^2 \).

**Proof.** Let \( x \in \mathbb{R}^n \) be arbitrary. By Lemma 2.2, we may assume that \( x \in \sigma \) for some

\[ \sigma = [y^0, y^1, \ldots, y^n] = D_1(y, \pi, s, p), \]

i.e., \( x = \sum_{i=0}^n q_i y^i \), with \( q_i \geq 0 \) for all \( i \) and \( \sum_{i=0}^n q_i = 1 \). Then \( x \) lies in a face of \( \sigma \) whose vertices are \( y^j \) for \( j \in J := \{ j \in N_0 | q_j > 0 \} \). We show below how each \( y^j \), \( j \in J \), can be generated from \( x \) independent of \( y \), \( \pi \), \( s \), and \( p \). Thus these vertices are found for any simplex of \( D_1 \) containing \( x \).

For each \( i \in N \), let

\[ r_i = \begin{cases} 
\lfloor x_i \rfloor & \text{if } \lfloor x_i \rfloor \text{ is even,} \\
\lfloor x_i \rfloor + 1 & \text{if } \lfloor x_i \rfloor \text{ is odd,}
\end{cases} \]

and

\[ t_i = \begin{cases} 
1 & \text{if } x_i - r_i > 0, \\
0 & \text{if } x_i - r_i = 0, \\
-1 & \text{if } x_i - r_i < 0.
\end{cases} \]

Let \( w = \sum_{i=1}^n t_i (x_i - r_i) \). Further, let

\[ y_i(t_j) = \begin{cases} 
[r_i + t_j] & \text{if } i = j, \\
r_i & \text{otherwise,}
\end{cases} \]

for \( i = 1, \ldots, n \), and let \( y(t_j) = (y_1(t_j), \ldots, y_n(t_j))^T \). Then

\[ \{y(t_1), \ldots, y(t_n), r\} = \{y_j | j \in J\} \text{ if } w < 1, \text{ and} \]

\[ \{y(t_1), \ldots, y(t_n)\} \setminus \{r\} = \{y_j | j \in J\} \text{ if } w = 1. \]

Suppose that \( w > 1 \). Let \( T_1, \ldots, T_g \) be subsets of \( N \) such that \( \bigcup_{k=1}^g T_k = N \) and for each \( 1 \leq k \leq g \), \( t_i(x_i - r_i) = t_j(x_j - r_j) \) if \( i \in T_k \) and \( j \in T_k \) and for any \( 1 \leq e < f \leq g \), \( t_i(x_i - r_i) < t_j(x_j - r_j) \) if \( i \in T_e \) and \( j \in T_f \). Let \( T_0 = \emptyset \). Let \( i(k) \in T_k \) for \( k = 0, \ldots, g \). Since \( w > 1 \), there exist unique \( 0 \leq u < g \) and \( q \geq 0 \) such that

\[ t_{i(v)}(x_{i(v)} - r_{i(v)}) + (1 - |T_{v+1}| - \cdots - |T_g|)q \]

\[ + |T_{v+1}|(t_{i(v+1)}(x_{i(v+1)} - r_{i(v+1)}) - t_{i(v)}(x_{i(v)} - r_{i(v)})) \]

\[ + \cdots + |T_g|(t_{i(g)}(x_{i(g)} - r_{i(g)}) - t_{i(v)}(x_{i(v)} - r_{i(v)})) = 1, \]
and
\[ t_{i(j)}(x_{i(j)} - r_{i(j)}) - t_{i(v)}(x_{i(v)} - r_{i(v)}) = q > 0, \quad j = v + 1, \ldots, g. \]

For \( 0 \leq k \leq v \), let for \( i = 1, \ldots, n \),
\[ y_i(T_k) = \begin{cases} r_i + t_i & \text{if } i \not\in T_0 \cup T_1 \cup \cdots \cup T_k, \\ r_i & \text{otherwise,} \end{cases} \]
and let \( y(T_k) = (y_1(T_k), \ldots, y_n(T_k))^T \). For \( v + 1 \leq k \leq g \), let for each \( j \in T_k \),
\[ y_i(j) = \begin{cases} r_i + t_j & \text{if } i = j, \\ r_i & \text{otherwise}, \end{cases} \]
and for all \( i \), let \( \tilde{y}(j) = (\tilde{y}_1(j), \ldots, \tilde{y}_n(j))^T \). Let
\[ \tilde{g} = \begin{cases} g - 1 & \text{if } t_{i(j)}(x_{i(j)} - r_{i(j)}) - t_{i(v)}(x_{i(v)} - r_{i(v)}) = q = 0 \quad \text{for } j = g, \\ g & \text{otherwise}. \end{cases} \]

If \( q = 0 \), then
\[ \{ y(T_k) | 0 \leq k < v \} \cup \left( \bigcup_{k=v+1}^{\tilde{g}} \{ \tilde{y}(j) | j \in T_k \} \right) = \{ y^j | j \in J \}, \]
and if \( q > 0 \), then
\[ \{ y(T_k) | 0 \leq k < v \} \cup \left( \bigcup_{k=v+1}^{g} \{ \tilde{y}(j) | j \in T_k \} \right) = \{ y^j | j \in J \}. \]

From these results, we obtain the proof of the lemma. \( \square \)

**Theorem 2.4.** \( D_1 \) is a triangulation of \( \mathbb{R}^n \).

**Proof.** Let \( x \in \mathbb{R}^n \) be arbitrary. It is clear that \( x \) is only contained in a finite number of simplices of \( D_1 \). Using Lemma 2.1, Lemma 2.2, and Lemma 2.3, we complete the proof of the theorem. \( \square \)

The \( D_1 \)-triangulation of \( \mathbb{R}^3 \) is illustrated in Figure 1.

3. The pivot rules of the \( D_1 \)-triangulation. Let \( \sigma = [y^0, y^1, \ldots, y^n] = D_1(y, \pi, s, p) \) be given. We wish to obtain the unique \( n \)-simplex
\[ \tilde{\sigma} = [\tilde{y}^0, \tilde{y}^1, \ldots, \tilde{y}^n] = D_1(\tilde{y}, \tilde{\pi}, \tilde{s}, \tilde{p}), \]
containing all vertices of \( \sigma \) except \( y^i \). Table 1 shows how \( \tilde{y}, \tilde{\pi}, \tilde{s}, \) and \( \tilde{p} \) depend on \( y, \pi, s, p, \) and \( i \). From this table it is easy to obtain each vertex \( \tilde{y}^k, k = 0, 1, \ldots, n, \) of \( \tilde{\sigma} \), and in particular its new vertex.

4. Comparison of the numbers of simplices in the unit cube. Let \( I^n = \{ x \in \mathbb{R}^n | 0 \leq x \leq u \} \) be the unit cube in \( \mathbb{R}^n \).

**Theorem 4.1.** The number of simplices of the \( D_1 \)-triangulation in the unit cube is equal to
\[ d_n = n + n(n - 1) + \cdots + n(n - 1) \cdots 4 \cdot 3 + 2. \]
**TABLE 1**

The Pivot Rules of the $D_1$-Triangulation

<table>
<thead>
<tr>
<th>$p$</th>
<th>$i$</th>
<th>$y$</th>
<th>$s$</th>
<th>$\pi$</th>
<th>$\bar{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$y$</td>
<td>$s$</td>
<td>$\pi$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>0</td>
<td>$i &gt; 1$</td>
<td>$y$</td>
<td>$s - 2s_{\pi(i)} \mu_{\pi(i)}$</td>
<td>$\pi$</td>
<td>$p$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$y$</td>
<td>$s$</td>
<td>$\pi$</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>$2 \leq p \leq n - 1$</td>
<td>0</td>
<td>$y$</td>
<td>$s - 2s_{\pi(1)} \mu_{\pi(1)}$</td>
<td>$\pi$</td>
<td>$p$</td>
</tr>
<tr>
<td>$2 \leq p \leq n - 1$</td>
<td>$1 &lt; i &lt; p - 1$</td>
<td>$y$</td>
<td>$s$</td>
<td>$(\pi(1), \ldots, \pi(i + 1), \ldots, \pi(n))$</td>
<td>$p$</td>
</tr>
<tr>
<td>$2 \leq p \leq n - 1$</td>
<td>$p - 1$</td>
<td>$y$</td>
<td>$s$</td>
<td>$(\pi(1), \ldots, \pi(p - 2), \pi(p), \ldots, \pi(j), \ldots, \pi(n))$</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>$1 &lt; p &lt; n - 1$</td>
<td>$i &gt; p - 1$</td>
<td>$y$</td>
<td>$s$</td>
<td>$(\pi(1), \ldots, \pi(p - 1), \pi(p), \ldots, \pi(i + 1), \ldots, \pi(n))$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>$n - 1$</td>
<td>$n - 1$</td>
<td>$y + 2s_{\omega(n)} \mu_{\omega(n)}$</td>
<td>$s - 2s_{\omega(n)} \mu_{\omega(n)}$</td>
<td>$\pi$</td>
<td>$p$</td>
</tr>
<tr>
<td>$n - 1$</td>
<td>$n$</td>
<td>$y + 2s_{\omega(n-1)} \mu_{\omega(n-1)}$</td>
<td>$s - 2s_{\omega(n-1)} \mu_{\omega(n-1)}$</td>
<td>$\pi$</td>
<td>$p$</td>
</tr>
</tbody>
</table>

*Where $j$ is such that $\pi(j) < \pi(p - 1) < \pi(j + 1)$. 

**FIGURE 1.** $D_1$-Triangulation of the Unit Cube in $\mathbb{R}^3$. 
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Proof. Let $Q = \{D(x, \pi, s, p) | y = 0, s = (1, 1, \ldots, 1)^T\}$. From Definition 2.1, in $Q$, there is only one simplex for which $p = 0$, one simplex for which $p = 1$, and $n!/(n - q + 1)!$ simplices for which $p = q, 2 \leq q \leq n - 1$. Thus

$$|Q| = 1 + 1 + n!/(n - 1)! + n!/(n - 2)! + \cdots + n!/2!$$

$$= 2 + n + n(n - 1) + \cdots + n(n - 1) \cdots 4 \cdot 3.$$

Since $\bigcup_{\sigma \in Q} \sigma = l^n$, the proof of the theorem follows immediately. $\square$

For the definitions of the $K_1$-, $J_1$- and $H_1$-triangulations, we refer to [8].

Theorem 4.2. The number of simplices in $l^n$ of Freudenthal's $K_1$-triangulation, that of Tucker's $J_1$-triangulation, and that of Saigal's $H_1$-triangulation is $n!$.

Theorem 4.3. If $n \geq 3$, then $d_n < n!$. As $n$ goes to infinity, $d_n/n!$ converges to $e - 2$.

Proof. For $n = 3$, we have $d_3 < 3!$, since $d_3 = 5$ and $3! = 6$. Suppose $d_{n-1} < (n - 1)!$. Thus $nd_{n-1} < n!$. From

$$nd_{n-1} = n(n - 1) + n(n - 1)(n - 2) + \cdots + n(n - 1) \cdots 4 \cdot 3 + 2n$$

$$= d_n + (n - 2),$$

we obtain $d_n < n!$, since $n \geq 3$. By the induction principle, the conclusion $d_n < n!$ for $n \geq 3$ follows directly. Furthermore,

$$d_n/n! = 1/(n - 1)! + 1/(n - 2)! + \cdots + 1/2! + 1/n!,$$

so $d_n/n!$ converges to $e - 2$ as $n$ goes to infinity. $\square$

From these results, we obtain that the number of simplices of the $D_1$-triangulation is the smallest for these triangulations.

5. The diameter of the $D_1$-triangulation. Let $G$ be a triangulation of $\mathbb{R}^n$ such that its restriction to $l^n$, $G|l^n = \{\sigma \subset l^n | \sigma \in G\}$, triangulates $l^n$ and all vertices of $G|l^n$ are vertices of $l^n$. Let $\tau$ and $\tau'$ be two facets of $G$ in the boundary of $l^n, \partial l^n$. Let $\sigma_0, \sigma_1, \ldots, \sigma_m$ be a sequence of simplices of $G$ such that $\sigma_i$ and $\sigma_{i-1}$ are adjacent, for $i = 1, 2, \ldots, m$. If $\tau$ is a facet of $\sigma_0$ and $\tau'$ a facet of $\sigma_m$, then we say that the sequence of $\sigma_0, \sigma_1, \ldots, \sigma_m$ is a path of length $m + 1$ from $\tau$ to $\tau'$. We define the distance between $\tau$ and $\tau'$ to be the minimum length of a path between $\tau$ and $\tau'$. The diameter of $G$ is the maximal distance between any two facets in the boundary. It is denoted by $\text{diam}(G)$.

Theorem 5.1.

$$\text{diam}(K_1) = 1 + n(n - 1)/2 = O(n^2),$$

$$\text{diam}(J_1) = \text{diam}(K_1),$$

$$\text{diam}(H_1) \geq (n^3 - n + 6)/6 = O(n^3), \text{ and}$$

$$\text{diam}(D_1) = 2n - 3 = O(n).$$
PROOF. Let \( \sigma = [y^0, y^1, \ldots, y^n] = K_i(0, \pi) \) and \( \tau = [y^0, \ldots, y^{n-1}] \), where \( \pi = (1, 2, \ldots, n) \). Let

\[
\tilde{\sigma} = [\tilde{y}^0, \tilde{y}^1, \ldots, \tilde{y}^n] = K_i(0, \tilde{\pi}) \quad \text{and} \quad \tilde{\tau} = [\tilde{y}^0, \ldots, \tilde{y}^{n-1}],
\]

where \( \tilde{\pi} = (n, n-1, \ldots, 1) \). Let \( \sigma_1, \ldots, \sigma_{m-1} \) in \( G | l^n \) be such that \( \sigma_{i-1} \) and \( \sigma_i \) are adjacent for \( i = 2, \ldots, m-1 \), \( \sigma \) and \( \sigma_1 \) are adjacent, and also \( \sigma_{m-1} \) and \( \tilde{\sigma} \). It is easily seen that the smallest \( m \) is equal to \( n(n-1)/2 \). The distance between \( \tau \) and \( \tilde{\tau} \) is obviously the greatest of all distances between two facets in \( \partial l^n \). Therefore,

\[
diam(K_i) \geq n(n-1)/2 + 1.
\]

Since \( J_i | l^n \) is the same as \( K_i | l^n \), \( \text{diam}(J_i) = \text{diam}(K_i) \).

Let \( \sigma = [y^0, y^1, \ldots, y^n] = H_i(y^2x, \pi) \) and \( \tau = [y^1, \ldots, y^n] \), where \( y^2x = (1, 0, \ldots, 0)^T \) and \( \pi = (1, 2, \ldots, n) \). Let

\[
\tilde{\sigma} = [\tilde{y}^0, \tilde{y}^1, \ldots, \tilde{y}^n] = H_i(\tilde{y}^2x, \tilde{\pi}) \quad \text{and} \quad \tilde{\tau} = [\tilde{y}^0, \ldots, \tilde{y}^{n-1}],
\]

where \( \tilde{y}^2x = (1, \ldots, 1)^T \) and \( \pi = (n, n-1, \ldots, 1) \).

Let \( \sigma_1, \ldots, \sigma_{m-1} \) be a sequence such that \( \sigma_{i-1} \) and \( \sigma_i \) are adjacent for \( i = 2, \ldots, m-1 \), \( \sigma \) and \( \sigma_1 \) are adjacent, and also \( \sigma_{m-1} \) and \( \tilde{\sigma} \). Then the smallest \( m \) is equal to \( (n^3 - n + 6)/6 - 1 \). Thus the distance between \( \tau \) and \( \tilde{\tau} \) is \( (n^3 - n + 6)/6 \). This means \( \text{diam}(H_i) \geq 0(n^3) \).

Finally, let \( \sigma = [y^0, y^1, \ldots, y^n] = D_1(y, \pi, s, p) \) and \( \tau = [y^1, y^2, \ldots, y^n] \), where \( y = 0, s = (1, 1, \ldots, 1)^T \), \( p = n-1 \), and \( \pi = (1, 2, \ldots, n) \). Let

\[
\tilde{\sigma} = [\tilde{y}^0, \tilde{y}^1, \ldots, \tilde{y}^n] = D_1(\tilde{y}, \tilde{\pi}, \tilde{s}, \tilde{p}) \quad \text{and} \quad \tilde{\tau} = [\tilde{y}^0, \ldots, \tilde{y}^{n-1}],
\]

where \( \tilde{y} = 0, \tilde{s} = (1, 1, \ldots, 1)^T, \tilde{p} = n-1 \), and \( \tilde{\pi} = (n, n-1, \ldots, 3, 1, 2) \). Let \( \sigma_1, \ldots, \sigma_{m-1} \) be a sequence such that \( \sigma \) and \( \sigma_1 \), \( \sigma_{m-1} \) and \( \tilde{\sigma} \), for \( i = 2, \ldots, m-1 \), and \( \tilde{\sigma} \) and \( \sigma_{m-1} \) are adjacent. Then the smallest \( m \) is equal to \( 2n - 4 \). The distance between \( \tau \) and \( \tilde{\tau} \) is obviously the greatest of all distances between two facets in \( \partial l^n \). Therefore,

\[
\text{diam}(D_1) = 2n - 3.
\]

From these results, the theorem follows immediately. \( \square \)

6. The average directional density and surface density. From Eaves and Yorke [2], we know that for a triangulation the average directional density and surface density are equivalent. We calculate below the surface density and obtain the average directional density from the surface density.

First we calculate the surface density of the \( D_1 \)-triangulation. Let

\[
\sigma^0 = [0, u^1, \ldots, u^n], \quad \sigma^1 = [u, u^1, \ldots, u^n],
\]

\[
\sigma^2 = [u, u-u^1, u^2, \ldots, u^n], \ldots, \sigma^{n-1} = [u, u-u^1, \ldots, u-u^1-u^2-\cdots-u^{n-2}, u^{n-1}, u^n].
\]

The volume of a simplex \( \sigma \) is denoted by \( V(\sigma) \). The surface area of a simplex \( \sigma \) is denoted by \( \text{SA}(\sigma) \). Let

\[
\tau^0_0 = [u^1, u^2, \ldots, u^n], \quad \tau^0_1 = [0, u^2, \ldots, u^n], \ldots, \tau^0_{n-1} = [0, u^1, \ldots, u^{n-2}, u^n],
\]

\[
\tau^0_n = [0, u^1, \ldots, u^{n-1}].
\]
be the facets of $\sigma^0$. Then
\[
SA(\sigma^0) = \sum_{i=0}^{n} V(\tau_{n}^{0}) = nV(\tau_{n}^{0}) + V(\tau_{0}^{0}).
\]

Clearly,
\[
V(\tau_{n}^{0}) = \frac{1}{(n-1)!} |\det[u^1, u^2, \ldots, u^n]| = \frac{1}{(n-1)!},
\]
and
\[
V(\tau_{0}^{0}) = \frac{1}{(n-1)!} |\det[u/\sqrt{n}, u^2 - u^1, \ldots, u^n - u^1]| = \sqrt{n}/(n-1)!,
\]
so $SA(\sigma^0) = (n + \sqrt{n})/(n-1)!$. Since $V(\sigma^0) = 1/n!$, we obtain that
\[
SA(\sigma^0)/V(\sigma^0) = n(n + \sqrt{n}).
\]

For $k = 2, \ldots, n-1$, let
\[
\tau_{j}^{k} = [u, u - u^1, \ldots, u - u^1 - \cdots - u^{k-1}, u^k, \ldots, u^{k-1}, u_k, \ldots, u^n],
\]
\[j = k, \ldots, n,
\]
\[
\tau_{0}^{k} = [u - u^1, \ldots, u - u^1 - \cdots - u^{k-1}, u^k, \ldots, u^n], \text{ and}
\]
\[
\tau_{j}^{k} = [u, u - u^1, \ldots, u - u^1 - \cdots - u^{j-1}, u - u^1 - \cdots - u^{j-1}, u^j + 1, \ldots, u^n],
\]
\[j = 1, 2, \ldots, k - 1,
\]
denote the facets of $\sigma^k$. Then
\[
SA(\sigma^k) = V(\tau_{0}^{k}) + \sum_{j=1}^{k-1} V(\tau_{j}^{k}) + (n - k + 1)V(\tau_{n}^{k}).
\]

Let
\[
a_1 = a_2 = \cdots = a_{n-k} = \left((n-k+1)^2 - 3(n-k+1) + 3\right)^{-1/2} \text{ and}
\]
\[
a_{n-k+1} = -\left((n-k+1)^2 - 3(n-k+1) + 3\right)^{-1/2}(n-k-1).
\]

Then
\[
V(\tau_{n}^{k}) = \frac{1}{(n-1)!} |\det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & q_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & q_{n-k} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & q_{n-k+1}
\end{bmatrix}|
\]
\[
= \left((n-k+1)^2 - 3(n-k+1) + 3\right)^{1/2}/(n-1)!.\]
Further

\[ V(\tau_0^k) = \frac{1}{(n-1)!} \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \end{vmatrix} \]

\[ = \frac{(n-k)}{(n-1)!}. \]

Now suppose \( 1 \leq j \leq k - 1 \). If \( j < k - 1 \), let \( q_j = 1/\sqrt{2} \), \( q_{j+1} = -1/\sqrt{2} \), and \( q_{j+2} = \cdots = q_n = 0 \). If \( j = k - 1 \), let

\[ q_{k-1} = -(n-k)((n-k+1)^2 - (n-k+1) + 1)^{-1/2} \]

and

\[ q_k = \cdots = q_n = ((n-k+1)^2 - (n-k+1) + 1)^{-1/2}. \]

Then for all \( j \in \{1, \ldots, k-1\} \),

\[ V(\tau_j^k) = \frac{1}{(n-1)!} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & q_j \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & q_n \end{vmatrix} \]

\[ = \begin{cases} (n-k)\sqrt{2} / (n-1)! & \text{if } j \neq k - 1, \\ ((n-k+1)^2 - (n-k+1) + 1)^{1/2} / (n-1)! & \text{if } j = k - 1. \end{cases} \]

Thus,

\[ \text{SA}(\sigma^k) = \frac{(n-k)}{(n-1)!} + (n-k+1) \]

\[ \times ((n-k+1)^2 - 3(n-k+1) + 3)^{1/2} / (n-1)! \]

\[ + (k-2)(n-k)\sqrt{2} / (n-1)! \]

\[ + ((n-k+1)^2 - (n-k+1) + 1)^{1/2} / (n-1)!. \]
Moreover,

\[
V(\sigma^k) = \frac{1}{(n!)} \det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0
\end{bmatrix}
= (n-k)/n!.
\]

Hence,

\[
\frac{SA(\sigma^k)}{V(\sigma^k)} = n(n-k+(n-k+1)((n-k+1)^2-3(n-k+1)+3)^{1/2} + (k-2)(n-k)\sqrt{2} + ((n-k+1)^2
- (n-k+1) + 1)^{1/2})/(n-k).
\]

Let

\[
\tau_0^1 = [u^1, \ldots, u^n], \quad \tau_1^1 = [u, u^2, \ldots, u^n],
\tau_2^1 = [u, u^1, u^3, \ldots, u^n], \ldots, \tau_n^1 = [u, u^1, \ldots, u^{n-1}]
\]

be the facets of \(\sigma^1\). Then

\[
SA(\sigma^1) = nV(\tau_n^1) + V(\tau_0^1).
\]

Let

\[
q_1 = \cdots = q_{n-1} = (n^2 - 3n + 3)^{-1/2} \quad \text{and} \quad q_n = -(n-2)(n^2 - 3n + 3)^{-1/2}.
\]

Then

\[
V(\tau_n^1) = \frac{1}{(n-1)!} \det \begin{bmatrix}
0 & 1 & \cdots & 1 & q_1 \\
1 & 0 & \cdots & 1 & q_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & q_{n-1} \\
1 & 1 & \cdots & 1 & q_n
\end{bmatrix}
= (n^2 - 3n + 3)^{1/2}/(n-1)!,
\]

\[
V(\tau_0^1) = n^{1/2}/(n-1)!, \quad \text{and} \quad V(\sigma^1) = (n-1)/n!.
\]

Moreover,

\[
SA(\sigma^1) = \left( n(n^2 - 3n + 3)^{1/2} + n^{1/2} \right)/(n-1)!.
\]
TABLE 2

Comparison of the $K_1$, $J_1$, and $D_1$-Triangulations

<table>
<thead>
<tr>
<th>Triangulation</th>
<th>Number of Simplices in a Unit Cube</th>
<th>Diameter of a Triangulation</th>
<th>Average Directional Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1(J_1)$</td>
<td>$n!$</td>
<td>$O(n^2)$</td>
<td>$n(2 + (n - 1)/2)g_n$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>$n + n(n-1) + \cdots + n(n-1)$</td>
<td>$O(n)$</td>
<td>$SD(D_1)g_n$</td>
</tr>
<tr>
<td></td>
<td>$\cdots \cdot 3 + 2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hence,

$$SA(\sigma^1)/V(\sigma^1) = n(n(n^2 - 3n + 3)^{1/2} + n^{1/2})/(n - 1).$$

From the above results we obtain that the surface density of the $D_1$-triangulation equals

$$SD(D_1) = \max\{SA(\sigma^i)/V(\sigma^i) | i = 0, 1, \ldots, n - 1\}.$$ 

Let

$$g_n = \Gamma(n/2)/((n - 1)\Gamma(1/2)\Gamma((n - 1)/2)).$$

From [2] we know that the average directional density of a triangulation is $g_n$ times its surface density. Hence, the average directional density of the $D_1$-triangulation is equal to

$$\text{ADD}(D_1) = SD(D_1)g_n.$$

It is well known that both the average directional density of the $K_1$-triangulation and the one of the $J_1$-triangulation are equal to $n(2 + (n - 1)/2)g_n$. It is obvious that we have that \(\text{ADD}(D_1) < \text{ADD}(K_1) = \text{ADD}(J_1)\), and that \(\text{ADD}(D_1)/\text{ADD}(K_1)\) converges to 1 as $n$ goes to infinity. Thus, the average directional density of the $D_1$-triangulation is smaller than the one of the $K_1$- or the $J_1$-triangulation. Table 2 summarizes the results above.

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