The Pseudo-Geometric Graphs for Generalised Quadrangles of Order (3,t)

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Abstract

The values $t = 1, 3, 5, 6, 9$ satisfy the standard necessary conditions for existence of a generalised quadrangle of order $(3, t)$. This gives the following possible parameter sets for strongly regular graphs that are pseudo-geometric for such a generalised quadrangle: $(v, k, \lambda, \mu) = (16, 6, 2, 2)$, $(40, 12, 2, 4)$, $(64, 18, 2, 6)$, $(76, 31, 2, 7)$ and $(112, 30, 2, 10)$. It is well-known that there are two graphs with the first parameter set and that there is just one graph with the last set of parameters. Recently, the second author has shown that there are precisely 28 strongly regular graphs with the second parameter set. Non-existence of a strongly regular graph with the fourth set of parameters was proved by the first author. Here we complete the classification by announcing that there are exactly 167 non-isomorphic strongly regular graphs with parameters $(64, 18, 2, 6)$.

Key words and phrases: Strongly regular graph, generalised quadrangle, classification

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1 Introduction

A (finite) *Generalised Quadrangle* $\mathrm{GQ}(s,t)$ is an incidence structure of points and lines with the following properties:

- every line has $s+1$ points and every point is on $t+1$ lines;
- any two distinct points are incident with at most one line;
- given a line L and a point p not on L, there is a unique point on L collinear with p (two points are said to be collinear if there is a line incident with both).

By a *strongly regular graph* we mean a non-complete, non-void regular graph Γ on v vertices and of degree k, with the property that each pair of adjacent vertices has λ common neighbours and each pair of non-adjacent vertices has μ common neighbours. Such a graph is commonly referred to as a (v,k,λ,μ) strongly regular graph. The adjacency matrix of a strongly regular graph has an eigenvalue k, corresponding to the all-one eigenvector, and just two further eigenvalues which are the roots of the quadratic equation $x^2+(\mu-\lambda)x+\mu-k=0$. It is a simple exercise to see that the collinearity graph of a $\mathrm{GQ}(s,t)$ is strongly regular. This is the graph whose vertex set is the point set of the generalised quadrangle, where two vertices are adjacent if they are collinear as points. Such a graph is said to be *geometric*. Its parameter set is $((st+1)(s+1), s(t+1), s-1, t+1)$. However, there may be graphs with such a parameter set that do not arise in this way. In general, a strongly regular graph with the above parameters (for some s and t) is called *pseudo-geometric* for a $\mathrm{GQ}(s,t)$ (even if such a generalised quadrangle does not exist). It is not difficult to see that a pseudo-geometric graph for a $\mathrm{GQ}(s,t)$ is geometric if and only if every edge is contained in a clique of size $s+1$. Necessary conditions for existence of a pseudo geometric graph (and hence for existence of a $\mathrm{GQ}(s,t)$) are $1 \le t \le s^2$ if $s>1$, and $s+t$ divides $st(s+1)(t+1)$. These conditions are known as the Krein condition and the rationallity condition, respectively. The reader may care to consult Brouwer [1] for these and further references on strongly regular graphs.

Assume $s=3$. Then $t \in \{1,3,5,6,9\}$. There are precisely five generalised quadrangles with these orders, namely one $\mathrm{GQ}(3,1)$, two $\mathrm{GQ}(3,3)$'s, one $\mathrm{GQ}(3,5)$, no $\mathrm{GQ}(3,6)$ and one $\mathrm{GQ}(3,9)$. For details of this and further facts on generalised quadrangles the reader is referred to Payne and Thas [7]. The five parameter sets for the pseudo-geometric graphs are $(16,6,2,2)$, $(40,12,2,4)$, $(64,18,2,6)$, $(76,21,2,7)$ and $(112,30,2,10)$. It is known that there are two non-isomorphic graphs with the first parameter set (see [9]), 28 graphs with the second parameter set [10], no graph with the fourth parameter set (see [4]) and a unique (geometric) graph with the last set of parameters [2]. It is the purpose of this paper to determine all $(64,18,2,6)$ graphs by computer and so complete the classification of all pseudo-geometric graphs for a $\mathrm{GQ}(3,t)$. The computer search became feasible because of strong information on the structure (Theorem 1) that could be obtained by an argument similar to

the one that led to the non-existence of a $(76, 21, 2, 7)$ strongly regular graph. As a consequence we will reprove also the mentioned non-existence and uniqueness result concerning the last two parameter sets, by showing that such a graph is necessarely geometric.
Remark. If $s < 3$ the only possible values for s and t are $(s, t) = (2, 1)$, $(2, 2)$, $(2, 4)$ and $(1, t)$ with $t \geq 1$. For each of these orders there exists a unique generalised quadrangle and a unique pseudo-geometric graph.

2 Subgraphs

Let Γ be a $(12t + 4, 3t + 3, 2, t + 1)$ strongly regular graph. So Γ is pseudo geometric for a $\mathrm{GQ}(3, t)$ and the adjacency matrix of Γ has eigenvalues $12t + 4$, 2 and $-t - 1$. Given any vertex x of Γ, the subgraph Γ_x induced by the neighbours of x is regular of degree $\lambda = 2$, and so is a disjoint union of cycles. However, more can be said.

Lemma 1 *Γ_x consists of a union of disjoint cycles whose lengths are multiples of* 3.

This lemma can be proved in a short way by eigenvalue interlacing (see [3, Lemma 6.2.4], or [5, Proposition 7.4]). Also the following result admits a proof by eigenvalue techniques ([3, Theorem 2.1.4], or [5, Theorem 3.5]).

Lemma 2 *Let Γ' be an induced subgraph of Γ and suppose that Γ' has v' vertices and average degree k'. Then $v' \geq 4(k' - 2)$ and equality implies that Γ' is regular (of degree k').*

Assume that Γ is not geometric. Then there exists an edge that is not contained in a 4-clique. Because every edge has two common neighbours, Γ contains an induced subgraph $K_{1,1,2}$ (that is, a K_4 minus an edge). Let S be the vertex set of such a $K_{1,1,2}$ and let S_i denote those vertices of Γ that are adjacent to i vertices of S, $(i = 0, 1, \ldots, 4)$. Clearly $S_4 = \emptyset$, since $\lambda = 2$. Also $S_3 = \emptyset$ on account of the fact that a neighbour graph cannot contain a cycle of length 4. Counting the paths of length 2 from S to itself, via the vertices of S_i, $(i = 0, 1, 2)$ we find that

$$\begin{aligned} |S_0| + |S_1| + |S_2| &= 12t, \\ |S_1| + 2|S_2| &= 12t + 2, \\ |S_2| &= t + 3. \end{aligned}$$

Thus $|S_0| = t + 1$, $|S_1| = 10t - 4$ and $|S_2| = t + 3$.

Now let s_i denote the number of edges from S_0 to S_i, $(i = 0, 1, 2)$ so that $2s_0 + s_1 + s_2 = |S_0| \times (3t + 3) = 3(t + 1)^2$. Also, counting the number of paths of length 2 between S and S_0 gives $s_1 + 2s_2 = |S_0| \times |S| \times (t + 1) = 4(t + 1)^2$. Hence

$$s_2 = 2s_0 + (t + 1)^2.$$

Let x and y be the vertices of S that correspond to the missing edge. Then x and y both have $t+1$ neighbours in S_2. Put $S_0' = S_0 \cup \{x, y\}$ and let Γ' be the subgraph induced by $S_0' \cup S_2$. Then there are $2(t+1)+s_2 \geq (t+3)(t+1)$ edges between S_0' and S_2, so the everage degree of Γ' equals $k' \geq t+1$. Now Lemma 2 gives $2(t+3) \geq 4(t-1)$ hence $t \leq 5$. Moreover if $t = 5$ there are no edges inside S_0' and S_2, so Γ' is bipartite and (again by Lemma 2) regular of degree 6. By considering the graph induced by $S \cup S_0 \cup S_2$ we can conclude:

Theorem 1

- *Any* $(12t+4, 3t+3, 2, t+1)$ *strongly regular graph with* $t > 5$ *is geometric.*
- *Any* $(64, 18, 2, 6)$ *strongly regular graph that is not geometric must contain a subgraph on* 18 *vertices whose adjacency matrix* B *say, is of the following form*

$$B = \left[\begin{array}{cccc|cccccc|cccc|cccc}
0&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&1&1&0&0&0&0&0&0&0&1&1&1&1&0&0&1&1\\
\hline
0&0&0&0&0&0&0&0&0&0&&&&&&&&\\
0&0&0&0&0&0&0&0&0&0&&&&&&&&\\
0&0&0&0&0&0&0&0&0&0&\multicolumn{4}{c|}{X}&\multicolumn{4}{c}{Y}\\
0&0&0&0&0&0&0&0&0&0&&&&&&&&\\
0&0&0&0&0&0&0&0&0&0&&&&&&&&\\
0&0&0&0&0&0&0&0&0&0&&&&&&&&\\
\hline
1&0&1&0&&&&&&&0&0&0&0&0&0&0&0\\
1&0&1&0&\multicolumn{6}{c|}{X^\top}&0&0&0&0&0&0&0&0\\
1&0&1&0&&&&&&&0&0&0&0&0&0&0&0\\
1&0&1&0&&&&&&&0&0&0&0&0&0&0&0\\
\hline
1&1&0&0&&&&&&&0&0&0&0&0&0&0&0\\
1&0&1&0&\multicolumn{6}{c|}{Y^\top}&0&0&0&0&0&0&0&0\\
0&1&0&1&&&&&&&0&0&0&0&0&0&0&0\\
0&0&1&1&&&&&&&0&0&0&0&0&0&0&0
\end{array}\right],$$

where each column of X *has four* 1*'s, each column of* Y *has five* 1*'s and* $[X|Y]$ *has row sums* 6.

Notice that when $t = 5$ the bipartite complement of Γ' has degree 2, so it is a disjoint union of cycles of even length. This gives seven possibilities for Γ'. Using this information it is an easy task to identify all above matrices. There are in total 24.

3 Computer search

Fix $t = 5$. So Γ has parameters $(64, 18, 2, 6)$ but is not the collinearity graph of a GQ$(3, 5)$.

It is well-known that the eigenvalues of any subgraph of Γ interlace those of Γ and more generally, that the eigenvalues of any princpal submatrix of a Hermitian matrix H interlace those of H. For details of this the reader is referred to [3]. Now Γ has eigenvalues 18, 2 and -6, so that if A is the adjacency matrix of Γ the matrix $J-4(A-2I)$ is positive semi-definite with eigenvalues 0 and 32. Here, as usual J and I denote the all-one matrix and identity matrix, respectively. Similarly, $8(A+6I)-3J$ is also positive semi-definite with eigenvalues 0 and 64. The following result is an immediate consequence of [3, Theorem 1.2.2].

Proposition 1 *If A is the adjacency matrix of any subgraph of a $(64, 18, 2, 6)$ strongly regular graph then $J-4(A-2I)$ has eigenvalues between 0 and 32 and $8(A+6I)-3J$ has eigenvalues that lie between 0 and 64.*

Every one of the 24 subgraphs found passed these tests.

The question is: how can these subgraphs be embedded in a Γ, if at all? A preliminary attempt to do it directly indicated that such an approach would take an inordinate amount of computer time and was rejected in favour of an intermediate step whereby each subgraph was augmented by the 10 vertices which need to be adjacent to the first vertex of S. Thus an attempt was made to embed B in a matrix of the form

$$\begin{bmatrix} B & N \\ N^\top & C \end{bmatrix},$$

where N has 10 columns and 18 rows, the first of which is the all-one vector. We call such a matrix a *feasible* extension of B if it satisfies the conditions of Proposition 1.

A brief explanation is now given as to how this extending of B was done. For each column $\mathbf{x}$ of N the matrix

$$\begin{bmatrix} B & \mathbf{x} \\ \mathbf{x}^\top & 0 \end{bmatrix}$$

must satisfy Proposition 1. The first task was to generate all columns $\mathbf{x}$ that satisfied this condition and were compatible with the above matrix being a submatrix of the adjacency matrix of the strongly regular graph. Then the rows of N were generated, entry by entry, checking at each stage that each column of N obtained thus far was a sub-column of one of the permissible columns. This test was easy and swift to implement because the permissible columns could be stored as binary integers. Once a candidate N had been found attention was turned to C. Here adjacencies were assigned in such a way that whenever a subgraph on $20, 21, \ldots, 28$ vertices was determined it was tested to ensure that it passed the conditions of Proposition 1. When this was done it was found that only six of the 24 subgraphs on 18 vertices could be extended to subgraphs on 28 vertices. These six subgraphs are given in the appendix, together with the number of extensions they yielded.

The same technique that was used to extend the subgraphs of Theorem 1 was now used to extend each of the extensions still further to the full strongly regular graph. This last stage was relatively quick and clearly benefited from the application of the test outlined in Proposition 1. In all 35 of the 89 extensions obtained gave rise to 166 strongly regular graphs.

With the computation completed we can now announce

Theorem 2 *There exist precisely* 167 *non-isomorphic* $(64, 18, 2, 6)$ *strongly regular graphs, one being the collinearity graph of the generalised quadrangle of order* $(3, 5)$.

It has been known for some time that there exist at least twelve strongly regular graphs with the parameter set $(64, 18, 2, 6)$. Eleven of these can be constructed from systems of linked 2-$(16, 6, 2)$ designs, classified by Mathon [6] and a further one was obtained by Peeters [8]. Among those of Mathon is the collinearity graph of the GQ$(3, 5)$.

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A Appendix

As mentioned in the text, the six choices for the matrix B from Theorem 1 given in Table 1 can be extended still further and can be embedded in what were termed feasible submatrices of order 28. There are 89 of these in total, but it turned out that only those that arose from numbers I, VI, V and VI could be extended all the way to a strongly regular graph. In Table 1 the notation (x, y) is used to indicate that the corresponding submatrix gives rise to x extensions, which in turn yield y strongly regular graphs.

There are eleven graphs on eighteen vertices each of which is the union of disjoint cycles of lengths a multiple of 3, but of these there are three that do not occur as a neighbour graph in any of the 167 strongly regular graphs. They are the 18-cycle (C_{18}), the union of two 9-cycles ($2C_9$), and the union of a triangle and a 15-cycle ($K_3 + C_{15}$). All others, apart from $K_3 + C_6 + C_9$ occur as neighbour graphs in a transitive strongly regular graph, of which there are eight. Also, every graph except one possesses an involution. This exceptional graph has in fact a trivial automorphism group.

Table 2 contains a listing of the frequencies of the eight possible neighbour graphs in the form of an eight-tuple $(n_1, n_2, \ldots, n_8)$, where, n_i $(1 \leq i \leq 8)$ denotes the frequency of the neighbour graph as given below.

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$6K_3$	$4K_3 + C_6$	$3K_3 + C_9$	$2K_3 + C_6$	$2K_3 + C_{12}$	$K_3 + C_6 + C_9$	$3C_6$	$C_6 + C_{12}$

Due to lack of space it is impossible to include any of the graphs here. The interested reader is referred to the home page of the second author **http://www.maths.gla.ac.uk/˜es** where a full listing may be obtained.

11111100	11111010	11111100	11111001	11111100	11101110
11110011	11001111	11101011	11100111	11110011	11011101
11001111	11110101	11100111	10011111	11001111	11001111
11001111	11001111	10011111	11101110	10101111	10111011
00111111	00111111	01011111	01011111	01011111	01110111
00111111	00111111	00111111	00111111	00111111	00111111
I(12,161)	II(38,0)	III(14,0)	IV(10,101)	V(5,131)	VI(10,142)

Table 1: The choices for $[X|Y]$ that give extendible submatrices B.

(64, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 64, 0)	(0, 64, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 64, 0, 0, 0, 0)
(0, 0, 0, 0, 64, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 64, 0)	(0, 0, 0, 0, 0, 0, 0, 64)	(24, 36, 0, 0, 0, 0, 4, 0)
(4, 0, 0, 36, 0, 0, 24, 0)	(0, 0, 64, 0, 0, 0, 0, 0)	(0, 32, 0, 0, 32, 0, 0, 0)	(4, 0, 48, 12, 0, 0, 0, 0)
(16, 32, 0, 16, 0, 0, 0, 0)	(0, 16, 0, 32, 0, 0, 16, 0)	(0, 0, 0, 32, 32, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 32, 32)
(8, 0, 32, 24, 0, 0, 0, 0)	(12, 24, 0, 28, 0, 0, 0, 0)	(0, 28, 0, 24, 0, 0, 12, 0)	(8, 24, 0, 24, 0, 0, 8, 0)
(0, 32, 0, 0, 32, 0, 0, 0)	(0, 16, 0, 32, 0, 0, 0, 16)	(0, 0, 0, 0, 0, 0, 16, 48)	(0, 0, 0, 24, 24, 0, 0, 16)
(0, 0, 0, 24, 24, 0, 16, 0)	(0, 0, 0, 16, 0, 0, 24, 24)	(0, 0, 0, 0, 16, 0, 24, 24)	(0, 16, 32, 0, 16, 0, 0, 0)
(0, 0, 0, 32, 0, 0, 16, 16)	(0, 8, 0, 0, 0, 32, 8, 16)	(0, 0, 0, 16, 0, 32, 16, 0)	(0, 0, 0, 32, 0, 32, 0, 0)
(0, 16, 32, 0, 16, 0, 0, 0)	(0, 0, 0, 16, 16, 0, 16, 16)	(0, 0, 0, 16, 16, 0, 16, 16)	(0, 0, 0, 0, 0, 32, 0, 32)
(0, 0, 24, 4, 24, 0, 0, 12)	(0, 24, 24, 12, 0, 0, 0, 4)	(8, 8, 0, 24, 8, 16, 0, 0)	(4, 16, 16, 12, 16, 0, 0, 0)
(0, 8, 0, 16, 8, 16, 0, 16)	(0, 24, 16, 8, 0, 0, 8, 8)	(0, 24, 16, 8, 0, 0, 8, 8)	(0, 16, 16, 16, 0, 16, 0, 0)
(0, 0, 16, 8, 16, 0, 16, 8)	(0, 16, 0, 32, 0, 16, 0, 0)	(0, 0, 0, 16, 0, 32, 0, 16)	(0, 0, 16, 0, 16, 16, 0, 16)
(0, 0, 16, 8, 16, 0, 16, 8)	(0, 0, 0, 0, 0, 16, 16, 32)	(4, 0, 0, 16, 8, 24, 0, 12)	(0, 16, 24, 0, 12, 0, 12, 0)
(0, 8, 0, 20, 0, 24, 0, 12)	(0, 1, 0, 0, 0, 12, 15, 36)	(0, 3, 0, 12, 0, 12, 13, 24)	(0, 3, 0, 12, 0, 12, 21, 16)
(0, 21, 0, 24, 0, 12, 3, 4)	(0, 4, 0, 36, 0, 12, 0, 12)	(0, 0, 0, 0, 0, 36, 12, 16)	(6, 4, 0, 22, 8, 16, 4, 4)
(8, 16, 8, 20, 8, 0, 0, 4)	(0, 8, 8, 20, 16, 0, 0, 12)	(4, 26, 16, 4, 4, 0, 10, 0)	(0, 6, 16, 8, 20, 0, 14, 0)
(4, 8, 0, 20, 8, 16, 0, 8)	(2, 0, 0, 10, 8, 16, 8, 20)	(4, 4, 8, 24, 0, 16, 4, 4)	(0, 8, 16, 8, 8, 16, 0, 8)
(8, 28, 8, 12, 0, 0, 4, 4)	(0, 12, 16, 8, 8, 0, 12, 8)	(0, 20, 0, 16, 4, 8, 8, 8)	(0, 0, 8, 12, 16, 0, 16, 12)
(0, 10, 0, 16, 0, 24, 6, 8)	(0, 12, 16, 8, 8, 0, 12, 8)	(0, 4, 0, 16, 0, 16, 12, 16)	(0, 4, 0, 8, 12, 8, 16, 16)
(0, 0, 8, 12, 0, 16, 8, 20)	(0, 0, 0, 16, 0, 16, 16, 16)	(0, 0, 0, 16, 0, 32, 0, 16)	(0, 2, 0, 0, 0, 24, 14, 24)
(0, 10, 0, 18, 6, 24, 0, 6)	(0, 0, 0, 10, 0, 12, 24, 18)	(0, 6, 0, 6, 10, 24, 0, 18)	(0, 6, 0, 30, 0, 12, 10, 6)
(0, 0, 0, 28, 0, 24, 0, 12)	(0, 0, 0, 12, 0, 24, 0, 28)	(0, 8, 16, 8, 8, 16, 0, 8)	(0, 10, 16, 8, 8, 8, 6, 8)
(2, 8, 8, 14, 8, 16, 0, 8)	(4, 16, 8, 16, 4, 8, 4, 4)	(4, 22, 4, 22, 0, 8, 2, 2)	(0, 12, 16, 12, 4, 16, 0, 4)
(2, 8, 8, 14, 8, 16, 0, 8)	(4, 12, 8, 12, 12, 16, 0, 0)	(2, 12, 16, 14, 12, 0, 0, 8)	(0, 4, 8, 12, 12, 8, 8, 12)
(0, 10, 0, 20, 0, 16, 6, 12)	(0, 8, 16, 8, 8, 16, 0, 8)	(0, 10, 16, 8, 8, 8, 6, 8)	(0, 6, 4, 14, 4, 8, 14, 14)
(4, 8, 8, 16, 8, 16, 0, 4)	(0, 10, 0, 12, 8, 16, 6, 12)	(0, 12, 24, 0, 12, 16, 0, 0)	(2, 2, 8, 18, 0, 16, 6, 12)
(0, 10, 0, 12, 8, 16, 6, 12)	(2, 12, 16, 14, 12, 0, 0, 8)	(0, 12, 24, 8, 12, 0, 0, 8)	(0, 4, 4, 18, 4, 8, 16, 10)
(0, 4, 16, 12, 0, 8, 12, 12)	(0, 10, 8, 16, 8, 0, 14, 8)	(0, 0, 4, 6, 8, 8, 16, 22)	(0, 2, 0, 16, 0, 24, 14, 8)
(0, 4, 0, 16, 0, 16, 4, 24)	(0, 0, 0, 24, 0, 16, 8, 16)	(0, 8, 8, 12, 8, 16, 0, 12)	(0, 0, 0, 20, 0, 16, 8, 20)
(0, 4, 0, 16, 0, 16, 12, 16)	(0, 0, 0, 8, 0, 16, 16, 24)	(0, 8, 0, 24, 0, 16, 8, 8)	(0, 4, 16, 4, 12, 16, 0, 12)
(0, 12, 0, 24, 0, 24, 4, 0)	(0, 8, 0, 28, 0, 16, 0, 12)	(2, 12, 16, 6, 12, 16, 0, 0)	(0, 0, 0, 8, 0, 24, 16, 16)
(2, 16, 6, 18, 1, 12, 3, 6)	(1, 0, 12, 15, 0, 24, 0, 12)	(0, 4, 0, 15, 0, 30, 6, 9)	(0, 1, 0, 12, 0, 36, 3, 12)
(0, 0, 0, 13, 0, 18, 6, 27)	(4, 15, 8, 18, 4, 8, 5, 2)	(0, 6, 8, 18, 2, 24, 0, 6)	(2, 10, 4, 22, 2, 16, 4, 4)
(2, 10, 8, 14, 8, 8, 6, 8)	(0, 5, 8, 10, 12, 8, 7, 14)	(0, 2, 4, 8, 6, 16, 8, 20)	(0, 2, 8, 6, 6, 24, 0, 18)
(2, 13, 4, 20, 2, 8, 9, 6)	(0, 0, 0, 16, 0, 16, 8, 24)	(0, 3, 4, 10, 6, 8, 15, 18)	(0, 2, 0, 14, 0, 20, 10, 18)
(0, 2, 0, 18, 0, 24, 6, 14)	(0, 0, 0, 10, 0, 20, 12, 22)	(0, 4, 0, 20, 0, 24, 4, 12)	(0, 2, 0, 8, 0, 24, 14, 16)
(0, 0, 0, 12, 0, 24, 8, 20)	(2, 8, 8, 14, 8, 16, 0, 8)	(1, 6, 6, 16, 4, 16, 6, 9)	(0, 4, 4, 12, 4, 20, 4, 16)
(1, 10, 6, 18, 2, 16, 4, 7)	(2, 6, 4, 18, 4, 20, 2, 8)	(1, 8, 12, 11, 8, 16, 0, 8)	(0, 6, 6, 11, 4, 16, 6, 15)
(2, 13, 12, 12, 6, 8, 5, 6)	(0, 11, 8, 14, 2, 12, 7, 10)	(0, 5, 8, 12, 4, 20, 3, 12)	(0, 7, 12, 10, 10, 8, 7, 10)
(0, 2, 6, 9, 6, 16, 8, 17)	(1, 6, 4, 15, 4, 16, 6, 12)	(0, 4, 6, 12, 5, 12, 11, 14)	(0, 3, 8, 10, 6, 12, 11, 14)
(0, 5, 0, 20, 0, 20, 7, 12)	(0, 2, 0, 11, 0, 26, 8, 17)	(0, 0, 0, 11, 0, 18, 10, 25)	(0, 3, 0, 17, 0, 26, 7, 11)
(0, 5, 0, 25, 0, 18, 5, 11)	(0, 2, 0, 16, 0, 24, 6, 16)	(1, 5, 4, 15, 4, 20, 3, 12)	

Table 2: Frequencies of neighbour graphs