The Maximum Edge Biclique Problem is NP-Complete
Peeters, M.J.P.

Publication date:
2000

Citation for published version (APA):
The maximum edge biclique problem is NP-complete

René Peeters
Tilburg University
May 9, 2000

Abstract

We prove that the maximum edge biclique problem in bipartite graphs is NP-complete.

A biclique in a bipartite graph is a vertex induced subgraph which is complete. The problem of finding a biclique with a maximum number of vertices is known to be solvable in polynomial time but the complexity of finding a biclique with a maximum number of edges was still undecided.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. A pair of two disjoint subsets $A$ and $B$ of $V$ is called a biclique if \{a, b\} $\in E$ for all \(a \in A\) and \(b \in B\). Thus the edges \{a, b\} form a complete bipartite subgraph of $G$ (which is not necessarily an induced subgraph if $G$ is not bipartite). A biclique \{A, B\} clearly has \(|A| + |B|\) vertices and \(|A| * |B|\) edges. In this note we restrict ourselves to case that $G$ is bipartite. The two colour classes of $G$ will be denoted by $V_1$ and $V_2$, so $V = V_1 \cup V_2$.

Already in the book of Garey and Johnson [2] (problem GT24) the complexity of deciding whether or not a bipartite graph contains a biclique of a certain size is discussed. If the requirement is that \(|A| = |B| = k\) for some integer $k$ (this is called the balanced complete bipartite subgraph problem or balanced biclique problem), then the problem is NP-complete. If however the
requirement is that \(|A| + |B| \geq k\) (the \textit{maximum vertex biclique problem}), the problem can be solved in polynomial time via the matching algorithm. The complexity of the maximum vertex biclique problem for general graphs depends on the precise definition of a biclique in this case. With the above definition the problem is solvable in polynomial time since there is a one to one correspondence between bicliques in the bipartite double of the graph and bicliques in the graph itself. If one defines a biclique as an induced complete bipartite subgraph (so \(A\) and \(B\) are independent sets in \(G\)), then the maximum vertex biclique problem for general graphs is NP-complete (see [10]). A natural third variant is the so-called \textit{maximum edge biclique problem (MBP)} where the requirement is that \(|A| \ast |B| \geq k\). Up to now the complexity of this problem was still undecided.

In various papers the complexity of MBP is mentioned and guessed to be NP-complete ([1, 4, 3, 9]. In [1] some applications of MBP are discussed and it is shown that the weighted version of MBP is NP-complete. Furthermore the authors show that four variants of MBP are NP-complete. Using different techniques Hochbaum [4], Haemers [3] and Pasechnik [9] derive upper bounds for the maximum number of edges in a biclique of a bipartite graph. Hochbaum [4] presents a 2-approximation algorithm for the minimum number of edges needed to be removed so that the remainder is a biclique based on an LP-relaxation. Inspired by the work of Lovász on the Shannon capacity of a graph ([6]), Haemers [3] and Pasechnik [9] derive similar inequalities for the maximum biclique problem. Pasechnik uses semidefinite programming techniques whereas Haemers uses eigenvalue techniques.

In the next section we prove that indeed MBP is NP-complete. The reduction used is inspired by the reduction that is used to prove the NP-completeness of the balanced biclique problem (see [5]). As a consequence MBP is also NP-complete for general graphs.

\section{The reduction}

We define MBP as follows:

\textbf{Maximum edge biclique problem (MBP):} Given a bipartite graph \(G = (V_1 \cup V_2, E)\) and a positive integer \(K\), does \(G\) contain a biclique with at least \(K\) edges?
Theorem 1 MBP is NP-complete.

Proof: We reduce 3SAT to MBP in two steps. Given an instance $\phi$ of 3SAT, we first construct a graph $G = (V, E)$ that has a clique of size $\frac{1}{2}|V|$ if and only if $\phi$ is satisfiable. This reduction is a modification of a well known and rather straightforward reduction from 3SAT to CLIQUE/INDEPENDENT SET ([7, 8]). Secondly we construct a bipartite graph $H = (V_1 \cup V_2, E')$ such that $H$ has a biclique containing a certain number of edges if and only if $G$ has a clique of size $\frac{1}{2}|V|$. This second step is a modification of the reduction from CLIQUE to BALANCED COMPLETE BIPARTITE SUBGRAPH referred to in [2] (problem GT24) and published in [5].

We are given an instance $\phi$ of 3SAT with $m$ clauses $C_1, \ldots, C_m$, with each clause being $C_i = (\alpha_{i1} \lor \alpha_{i2} \lor \alpha_{i3})$, with the $\alpha_{ij}$’s being either Boolean variables or negations thereof. Now construct the graph $G = (V, E)$ as follows:

$$V = \{v_{ij} : i = 1, \ldots, m; j = 1, 2, 3\} \cup \{v_i : i = 1, \ldots, m\}$$
$$E = \{\{v_{ij}, v_{kl}\} : i \neq k; \alpha_{ij} \neq -\alpha_{kl}\}$$
$$\cup \{\{v_{ij}, v_k\} : i = 1, \ldots, m; j = 1, 2, 3; k = 1, \ldots, m\} \cup \{\{v_i, v_j\} : i \neq j\}$$

Clearly a maximal clique in $G$ contains all vertices $v_i$ and at most one vertex out of each triple $\{v_{i1}, v_{i2}, v_{i3}\}$. It is easy to check that $G$ has a (maximal) clique of size $2m (=\frac{1}{2}|V|)$ if and only if $\phi$ is satisfiable.

Let $k = \frac{1}{2}|V|$. Now construct an instance $H = (V_1 \cup V_2, E')$, $K$ of MBP as follows: Let

$$V_1 = V$$
$$V_2 = E \cup \{e_1, \ldots, e_{\frac{1}{2}k^2 - k}\}$$
$$E' = \{\{v, e\} : v \in V; e \in E; v \notin e\} \cup \{\{v, e_i\} : v \in V; i = 1, \ldots, \frac{1}{2}k^2 - k\}$$
$$K = k^3 - \frac{3}{2}k^2$$

This construction can clearly be performed in polynomial time. Suppose $G$ has a clique $C$ of size $k$. Take $A := V - C$ and $B := \{e_1, \ldots, e_{\frac{1}{2}k^2 - k}\} \cup \{\{c, d\} : c, d \in C; c \neq d\}$. Then $\{A, B\}$ is a biclique with $|A| * |B| = 3$.
k \ast (\frac{1}{2}k^2 - k + \frac{1}{2}k(k-1)) = k^3 - \frac{3}{2}k^2. \text{ So if } G \text{ has a clique of size } k \text{ then } H \text{ has a biclique with } k^3 - \frac{3}{2}k^2 \text{ edges. On the other hand, if } H \text{ has a biclique with at least } k^3 - \frac{3}{2}k^2 \text{ edges, then } G \text{ must have a clique of size } k. \text{ We complete the proof by showing this.}

Let \( \{A, B\} \) be a biclique of \( H \) with \( A \subseteq V_1 \) and \( B \subseteq V_2 \). We shall prove that \(|A| \ast |B| \leq k^3 - \frac{3}{2}k^2\) and that equality implies that \( G \) has a clique of size \( k \). Without loss of generality, \( e_i \in B \) for \( i = 1, \ldots, \frac{1}{2}k^2 - k \). Let \( a := |A| \) and \( b := |B| - (\frac{1}{2}k^2 - k) \).

The \( b \) elements of \( B \cap E \) correspond with edges in \( G \) whose endpoints are not in \( A \). There are \( 2k - a \) vertices of \( G \) that are not in \( A \) so \( b \leq \frac{1}{2}(2k - a)(2k - a - 1) \), with equality if and only if \( V - A \) is a clique with edge set \( B \cap E \).

We consider two cases:

1. Suppose \( a \geq k \), so \( |V - A| = k - c \) with \( c := a - k \) (So \( 0 \leq c \leq k \)). Then \( b \leq \frac{1}{2}(k - c)(k - c - 1) \), so

\[
|A| \ast |B| \leq [k + c] \ast [\frac{1}{2}k^2 - k + \frac{1}{2}(k - c)(k - c - 1)]
\]

This reduces to

\[
|A| \ast |B| - (k^3 - \frac{3}{2}k^2) \leq \frac{1}{2}c(c^2 - (k - 1)c - 2k)
\]

Now \( c^2 - (k - 1)c - 2k \) is negative for \( 0 \leq c \leq k \), so \( |A| \ast |B| \leq k^3 - \frac{3}{2}k^2 \) with equality if and only if \( |A| = k \) and \( V - A \) is a clique of size \( k \) in \( G \).

2. Suppose \( a \leq k \), so \( |V - A| = k + c \) with \( c := k - a \) (So \( 0 \leq c \leq k \)). Since \( G \) has no cliques with more than \( k \) vertices, the number of edges \( b \) in the subgraph of \( G \) induced by \( V - A \) is at most \( \frac{1}{2}(k + c)(k + c - 1) - c \). This leads to

\[
|A| \ast |B| \leq [k - c] \ast [\frac{1}{2}k^2 - k + \frac{1}{2}(k + c)(k + c - 1) - c]
\]

This reduces to

\[
|A| \ast |B| - (k^3 - \frac{3}{2}k^2) \leq \frac{1}{2}c^2(-c + 3 - k)
\]
Since we may assume that $k \geq 4$, again the right hand side is negative for $1 \leq c \leq k$ and zero for $c = 0$. So $|A| \ast |B| \leq k^3 - \frac{3}{2}k^2$, with equality if and only if $|A| = k$ and $V - A$ is a clique of size $k$ in $G$.

\[\square\]

References


