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The maximum edge biclique problem is
NP-complete

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Abstract

We prove that the maximum edge biclique problem in bipartite graphs is NP-complete.

A biclique in a bipartite graph is a vertex induced subgraph which is complete. The problem of finding a biclique with a maximum number of vertices is known to be solvable in polynomial time but the complexity of finding a biclique with a maximum number of edges was still undecided.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. A pair of two disjoint subsets $A$ and $B$ of $V$ is called a biclique if $\{a, b\} \in E$ for all $a \in A$ and $b \in B$. Thus the edges $\{a, b\}$ form a complete bipartite subgraph of $G$ (which is not necessarily an induced subgraph if $G$ is not bipartite). A biclique $\{A, B\}$ clearly has $|A| + |B|$ vertices and $|A| \cdot |B|$ edges. In this note we restrict ourselves to case that $G$ is bipartite. The two colour classes of $G$ will be denoted by $V_1$ and $V_2$, so $V = V_1 \cup V_2$.

Already in the book of Garey and Johnson [2] (problem GT24) the complexity of deciding whether or not a bipartite graph contains a biclique of a certain size is discussed. If the requirement is that $|A| = |B| = k$ for some integer $k$ (this is called the balanced complete bipartite subgraph problem or balanced biclique problem), then the problem is NP-complete. If however the
requirement is that \(|A| + |B| \geq k\) (the maximum vertex biclique problem), the problem can be solved in polynomial time via the matching algorithm. The complexity of the maximum vertex biclique problem for general graphs depends on the precise definition of a biclique in this case. With the above definition the problem is solvable in polynomial time since there is a one to one correspondence between bicliques in the bipartite double of the graph and bicliques in the graph itself. If one defines a biclique as an induced complete bipartite subgraph (so \(A\) and \(B\) are independent sets in \(G\)), then the maximum vertex biclique problem for general graphs is NP-complete (see [10]). A natural third variant is the so-called maximum edge biclique problem (MBP) where the requirement is that \(|A| \ast |B| \geq k\). Up to now the complexity of this problem was still undecided.

In various papers the complexity of MBP is mentioned and guessed to be NP-complete ([1, 4, 3, 9]. In [1] some applications of MBP are discussed and it is shown that the weighted version of MBP is NP-complete. Furthermore the authors show that four variants of MBP are NP-complete. Using different techniques Hochbaum [4], Haemers [3] and Pasechnik [9] derive upper bounds for the maximum number of edges in a biclique of a bipartite graph. Hochbaum [4] presents a 2-approximation algorithm for the minimum number of edges needed to be removed so that the remainder is a biclique based on an LP-relaxation. Inspired by the work of Lovász on the Shannon capacity of a graph ([6]), Haemers [3] and Pasechnik [9] derive similar inequalities for the maximum biclique problem. Pasechnik uses semidefinite programming techniques whereas Haemers uses eigenvalue techniques.

In the next section we prove that indeed MBP is NP-complete. The reduction used is inspired by the reduction that is used to prove the NP-completeness of the balanced biclique problem (see [5]). As a consequence MBP is also NP-complete for general graphs.

2 The reduction

We define MBP as follows:

Maximum edge biclique problem (MBP): Given a bipartite graph \(G = (V_1 \cup V_2, E)\) and a positive integer \(K\), does \(G\) contain a biclique with at least \(K\) edges?
Theorem 1 MBP is NP-complete.

Proof: We reduce 3SAT to MBP in two steps. Given an instance \( \phi \) of 3SAT, we first construct a graph \( G = (V, E) \) that has a clique of size \( \frac{1}{2}|V| \) if and only if \( \phi \) is satisfiable. This reduction is a modification of a well known and rather straightforward reduction from 3SAT to CLIQUE/INDEPENDENT SET ([7, 8]). Secondly we construct a bipartite graph \( H = (V_1 \cup V_2, E') \) such that \( H \) has a biclique containing a certain number of edges if and only if \( G \) has a clique of size \( \frac{1}{2}|V| \). This second step is a modification of the reduction from CLIQUE to BALANCED COMPLETE BIPARTITE SUBGRAPH referred to in [2] (problem GT24) and published in [5].

We are given an instance \( \phi \) of 3SAT with \( m \) clauses \( C_1, \ldots, C_m \), with each clause being \( C_i = (\alpha_{i1} \lor \alpha_{i2} \lor \alpha_{i3}) \), with the \( \alpha_{ij} \)'s being either Boolean variables or negations thereof. Now construct the graph \( G = (V, E) \) as follows:

\[
V = \{v_{ij} : i = 1, \ldots, m; j = 1, 2, 3\} \cup \{v_i : i = 1, \ldots, m\}
\]
\[
E = \{\{v_{ij}, v_{kl}\} : i \neq k; \alpha_{ij} \neq \neg \alpha_{kl}\}
\]
\[
\cup \{\{v_{ij}, v_k\} : i = 1, \ldots, m; j = 1, 2, 3; k = 1, \ldots, m\} \cup \{\{v_i, v_j\} : i \neq j\}
\]

Clearly a maximal clique in \( G \) contains all vertices \( v_i \) and at most one vertex out of each triple \( \{v_{i1}, v_{i2}, v_{i3}\} \). It is easy to check that \( G \) has a (maximal) clique of size \( 2m \) (\( = \frac{1}{2}|V| \)) if and only if \( \phi \) is satisfiable.

Let \( k = \frac{1}{2}\sqrt{|V|} \). Now construct an instance \( H = (V_1 \cup V_2, E') \), \( K \) of MBP as follows: Let

\[
V_1 = V
\]
\[
V_2 = E \cup \{e_1, \ldots, e_{\frac{k^2}{2} - k}\}
\]
\[
E' = \{\{v, e\} : v \in V; e \in E; v \not\in e\} \cup \{\{v, e_i\} : v \in V; i = 1, \ldots, \frac{1}{2}k^2 - k\}
\]
\[
K = k^3 - \frac{3}{2}k^2
\]

This construction can clearly be performed in polynomial time. Suppose \( G \) has a clique \( C \) of size \( k \). Take \( A := V - C \) and \( B := \{e_1, \ldots, e_{\frac{k^2}{2} - k}\} \cup \{\{c, d\} : c, d \in C; c \neq d\} \). Then \( \{A, B\} \) is a biclique with \( |A| \ast |B| = \)

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\[ k \ast \left( \frac{1}{2} k^2 - k + \frac{1}{2} k(k-1) \right) = k^3 - \frac{3}{2} k^2. \] So if \( G \) has a clique of size \( k \) then \( H \) has a biclique with \( k^3 - \frac{3}{2} k^2 \) edges. On the other hand, if \( H \) has a biclique with at least \( k^3 - \frac{3}{2} k^2 \) edges, then \( G \) must have a clique of size \( k \). We complete the proof by showing this.

Let \( \{A, B\} \) be a biclique of \( H \) with \( A \subseteq V_1 \) and \( B \subseteq V_2 \). We shall prove that \( |A| \ast |B| \leq k^3 - \frac{3}{2} k^2 \) and that equality implies that \( G \) has a clique of size \( k \). Without loss of generality, \( e_i \in B \) for \( i = 1, \ldots, \frac{1}{2} k^2 - k \). Let \( a := |A| \) and \( b := |B| - \left( \frac{1}{2} k^2 - k \right) \).

The \( b \) elements of \( B \cap E \) correspond with edges in \( G \) whose endpoints are not in \( A \). There are \( 2k - a \) vertices of \( G \) that are not in \( A \) so \( b \leq \frac{1}{2} (2k - a)(2k - a - 1) \), with equality if and only if \( V - A \) is a clique with edge set \( B \cap E \).

We consider two cases:

1. Suppose \( a \geq k \), so \( |V - A| = k - c \) with \( c := a - k \) (So \( 0 \leq c \leq k \)). Then \( b \leq \frac{1}{2} (k - c)(k - c - 1) \), so

\[ |A| \ast |B| \leq [k + c] \ast \left[ \frac{1}{2} k^2 - k + \frac{1}{2} (k - c)(k - c - 1) \right] \]

This reduces to

\[ |A| \ast |B| - (k^3 - \frac{3}{2} k^2) \leq \frac{1}{2} c(c^2 - (k - 1)c - 2k) \]

Now \( c^2 - (k - 1)c - 2k \) is negative for \( 0 \leq c \leq k \), so \( |A| \ast |B| \leq k^3 - \frac{3}{2} k^2 \) with equality if and only if \( |A| = k \) and \( V - A \) is a clique of size \( k \) in \( G \).

2. Suppose \( a \leq k \), so \( |V - A| = k + c \) with \( c := a - k \) (So \( 0 \leq c \leq k \)). Since \( G \) has no cliques with more than \( k \) vertices, the number of edges \( b \) in the subgraph of \( G \) induced by \( V - A \) is at most \( \frac{1}{2} (k + c)(k + c - 1) - c \). This leads to

\[ |A| \ast |B| \leq [k - c] \ast \left[ \frac{1}{2} k^2 - k + \frac{1}{2} (k + c)(k + c - 1) - c \right] \]

This reduces to

\[ |A| \ast |B| - (k^3 - \frac{3}{2} k^2) \leq \frac{1}{2} c^2 (-c + 3 - k) \]
Since we may assume that $k \geq 4$, again the right hand side is negative for $1 \leq c \leq k$ and zero for $c = 0$. So $|A| \ast |B| \leq k^3 - \frac{3}{2}k^2$, with equality if and only if $|A| = k$ and $V - A$ is a clique of size $k$ in $G$.

\[\square\]

References


