The Maximum Edge Biclique Problem is NP-Complete
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Abstract

We prove that the maximum edge biclique problem in bipartite graphs is NP-complete.

A biclique in a bipartite graph is a vertex induced subgraph which is complete. The problem of finding a biclique with a maximum number of vertices is known to be solvable in polynomial time but the complexity of finding a biclique with a maximum number of edges was still undecided.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. A pair of two disjoint subsets $A$ and $B$ of $V$ is called a biclique if $\{a, b\} \in E$ for all $a \in A$ and $b \in B$. Thus the edges $\{a, b\}$ form a complete bipartite subgraph of $G$ (which is not necessarily an induced subgraph if $G$ is not bipartite). A biclique $\{A, B\}$ clearly has $|A| + |B|$ vertices and $|A| \cdot |B|$ edges. In this note we restrict ourselves to case that $G$ is bipartite. The two colour classes of $G$ will be denoted by $V_1$ and $V_2$, so $V = V_1 \cup V_2$.

Already in the book of Garey and Johnson [2] (problem GT24) the complexity of deciding whether or not a bipartite graph contains a biclique of a certain size is discussed. If the requirement is that $|A| = |B| = k$ for some integer $k$ (this is called the balanced complete bipartite subgraph problem or balanced biclique problem), then the problem is NP-complete. If however the
requirement is that \(|A| + |B| \geq k\) (the \textit{maximum vertex biclique problem}), the problem can be solved in polynomial time via the matching algorithm. The complexity of the maximum vertex biclique problem for general graphs depends on the precise definition of a biclique in this case. With the above definition the problem is solvable in polynomial time since there is a one to one correspondence between bicliques in the bipartite double of the graph and bicliques in the graph itself. If one defines a biclique as an induced complete bipartite subgraph (so \(A\) and \(B\) are independent sets in \(G\)), then the maximum vertex biclique problem for general graphs is NP-complete (see \([10]\)). A natural third variant is the so-called \textit{maximum edge biclique problem (MBP)} where the requirement is that \(|A| \ast |B| \geq k\). Up to now the complexity of this problem was still undecided.

In various papers the complexity of MBP is mentioned and guessed to be NP-complete (\([1, 4, 3, 9]\). In \([1]\) some applications of MBP are discussed and it is shown that the weighted version of MBP is NP-complete. Furthermore the authors show that four variants of MBP are NP-complete. Using different techniques Hochbaum \([4]\), Haemers \([3]\) and Pasechnik \([9]\) derive upper bounds for the maximum number of edges in a biclique of a bipartite graph. Hochbaum \([4]\) presents a 2-approximation algorithm for the minimum number of edges needed to be removed so that the remainder is a biclique based on an LP-relaxation. Inspired by the work of Lovász on the Shannon capacity of a graph (\([6]\)), Haemers \([3]\) and Pasechnik \([9]\) derive similar inequalities for the maximum biclique problem. Pasechnik uses semidefinite programming techniques whereas Haemers uses eigenvalue techniques.

In the next section we prove that indeed MBP is NP-complete. The reduction used is inspired by the reduction that is used to prove the NP-completeness of the balanced biclique problem (see \([5]\)). As a consequence MBP is also NP-complete for general graphs.

## 2 The reduction

We define MBP as follows:

\textbf{Maximum edge biclique problem (MBP)}: Given a bipartite graph \(G = (V_1 \cup V_2, E)\) and a positive integer \(K\), does \(G\) contain a biclique with at least \(K\) edges?
Theorem 1 MBP is NP-complete.

Proof: We reduce 3SAT to MBP in two steps. Given an instance $\phi$ of 3SAT, we first construct a graph $G = (V, E)$ that has a clique of size $\frac{1}{2}|V|$ if and only if $\phi$ is satisfiable. This reduction is a modification of a well known and rather straightforward reduction from 3SAT to CLIQUE/INDEPENDENT SET ([7, 8]). Secondly we construct a bipartite graph $H = (V_1 \cup V_2, E')$ such that $H$ has a biclique containing a certain number of edges if and only if $G$ has a clique of size $\frac{1}{2}|V|$. This second step is a modification of the reduction from CLIQUE to BALANCED COMPLETE BIPARTITE SUBGRAPH referred to in [2] (problem GT24) and published in [5].

We are given an instance $\phi$ of 3SAT with $m$ clauses $C_1, \ldots, C_m$, with each clause being $C_i = (\alpha_{i1} \lor \alpha_{i2} \lor \alpha_{i3})$, with the $\alpha_{ij}$'s being either Boolean variables or negations thereof. Now construct the graph $G = (V, E)$ as follows:

\[
\begin{align*}
V &= \{v_{ij} : i = 1, \ldots, m; j = 1, 2, 3\} \cup \{v_i : i = 1, \ldots, m\} \\
E &= \{\{v_{ij}, v_{kl}\} : i \neq k; \alpha_{ij} \neq -\alpha_{kl}\} \\
&\quad \cup \{\{v_{ij}, v_k\} : i = 1, \ldots, m; j = 1, 2, 3; k = 1, \ldots, m\} \cup \{\{v_i, v_j\} : i \neq j\}
\end{align*}
\]

Clearly a maximal clique in $G$ contains all vertices $v_i$ and at most one vertex out of each triple $\{v_1, v_2, v_3\}$. It is easy to check that $G$ has a (maximal) clique of size $2m$ ($= \frac{1}{2}|V|$) if and only if $\phi$ is satisfiable.

Let $k = \frac{1}{2}|V|$. Now construct an instance $H = (V_1 \cup V_2, E')$, $K$ of MBP as follows: Let

\[
\begin{align*}
V_1 &= V \\
V_2 &= E \cup \{e_1, \ldots, e_k, k^{2-k}\} \\
E' &= \{\{v, e\} : v \in V; e \in E; v \not\in e\} \cup \{\{v, e_i\} : v \in V; i = 1, \ldots, \frac{1}{2}k^2 - k\} \\
K &= k^3 - \frac{3}{2}k^2
\end{align*}
\]

This construction can clearly be performed in polynomial time. Suppose $G$ has a clique $C$ of size $k$. Take $A := V - C$ and $B := \{e_1, \ldots, e_{\frac{1}{2}k^2 - k}\} \cup \{\{c, d\} : c, d \in C; c \neq d\}$. Then $\{A, B\}$ is a biclique with $|A| \ast |B| = \frac{1}{2}|V|$. 

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$k \cdot (\frac{1}{2}k^2 - k + \frac{1}{2}k(k-1)) = k^3 - \frac{3}{2}k^2$. So if $G$ has a clique of size $k$ then $H$ has a biclique with $k^3 - \frac{3}{2}k^2$ edges. On the other hand, if $H$ has a biclique with at least $k^3 - \frac{3}{2}k^2$ edges, then $G$ must have a clique of size $k$. We complete the proof by showing this.

Let $\{A, B\}$ be a biclique of $H$ with $A \subseteq V_1$ and $B \subseteq V_2$. We shall prove that $|A| \cdot |B| \leq k^3 - \frac{3}{2}k^2$ and that equality implies that $G$ has a clique of size $k$. Without loss of generality, $e_i \in B$ for $i = 1, \ldots, \frac{1}{2}k^2 - k$. Let $a := |A|$ and $b := |B| - (\frac{1}{2}k^2 - k)$.

The $b$ elements of $B \cap E$ correspond with edges in $G$ whose endpoints are not in $A$. There are $2k - a$ vertices of $G$ that are not in $A$ so $b \leq \frac{1}{2}(2k - a)(2k - a - 1)$, with equality if and only if $V - A$ is a clique with edge set $B \cap E$.

We consider two cases:

1. Suppose $a \geq k$, so $|V - A| = k - c$ with $c := a - k$ (So $0 \leq c \leq k$). Then $b \leq \frac{1}{2}(k - c)(k - c - 1)$, so

$$|A| \cdot |B| \leq [k + c] \cdot \left[\frac{1}{2}k^2 - k + \frac{1}{2}(k - c)(k - c - 1)\right]$$

This reduces to

$$|A| \cdot |B| - (k^3 - \frac{3}{2}k^2) \leq \frac{1}{2}c^2 - (k - 1)c - 2k$$

Now $c^2 - (k - 1)c - 2k$ is negative for $0 \leq c \leq k$, so $|A| \cdot |B| \leq k^3 - \frac{3}{2}k^2$ with equality if and only if $|A| = k$ and $V - A$ is a clique of size $k$ in $G$.

2. Suppose $a \leq k$, so $|V - A| = k + c$ with $c := k - a$ (So $0 \leq c \leq k$). Since $G$ has no cliques with more than $k$ vertices, the number of edges $b$ in the subgraph of $G$ induced by $V - A$ is at most $\frac{1}{2}(k + c)(k + c - 1) - c$. This leads to

$$|A| \cdot |B| \leq [k - c] \cdot \left[\frac{1}{2}k^2 - k + \frac{1}{2}(k + c)(k + c - 1) - c\right]$$

This reduces to

$$|A| \cdot |B| - (k^3 - \frac{3}{2}k^2) \leq \frac{1}{2}c^2(-c + 3 - k)$$
Since we may assume that $k \geq 4$, again the right hand side is negative for $1 \leq c \leq k$ and zero for $c = 0$. So $|A| \ast |B| \leq k^3 - \frac{3}{2}k^2$, with equality if and only if $|A| = k$ and $V - A$ is a clique of size $k$ in $G$.

References


