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Abstract

In this paper we consider information collecting (IC) situations where an action taker in an uncertain situation can improve his action choices by gathering information from some players who are more informed about the situation. Then the problem of sharing the gains when cooperating with informants is tackled by constructing an appropriate game, the IC–game corresponding to the IC–situation. It turns out that the cone of IC–games, given a fixed set of players, coincides with the cone of 0–normalized monotonic games with a veto player. Also special classes of convex IC–games and big boss IC–games are considered, for which more is known about the solution concepts.

Keywords: information, cooperative games, solutions, monotonic games, veto player.

JEL-codes: C71, D89.

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1 Introduction

This paper concentrates on cooperative decision-making under uncertainty. It deals with collecting information to improve decision-making. To be more detailed, an action taker has incomplete information about relevant facts to act optimally and he can obtain more information from other agents who are more informed about the situation.

One can think for example of collecting information about the intensity of a happening such that a provider of goods for the visitors of the happening can act better. Another example is about a hidden object that can be detected more adequately if one collects some information. Interesting questions that arise are: from whom to obtain information, what to pay for it? We construct a model of information collecting situations (IC–situation) and assign to such a situation a cooperative game (the IC–game) to handle the question of the transfers of payoffs to the informants. Such an IC–situation is a hybrid of a one-person decision problem under uncertainty and an Aumann structure, which describes the knowledge of the informants. The corresponding IC–game offers the possibility to consider compensations for informed agents that correspond to the various solution concepts developed in this field of cooperative game theory.

In this context we like to mention the paper of Slikker et al. (1999) on information sharing games, which is in the same spirit as this paper. In the Slikker et al. paper, however, all players are action takers. The corresponding information sharing games exhaust the cone of games with a population monotonic allocation scheme.

The outline of this paper is as follows. In section 2 we formally introduce IC–situations and IC–games. We give three examples dealing with information collecting situations in uncertain environments. The first example deals with a search problem, the second with the problem of participating in a gamble or not, and the third with the problem of preparing the right amount of goods for a sales outlet. In section 3 we show that the IC–games (with a fixed action taker and a fixed set of informants) form a convex cone in the game space that coincides with the cone of 0–normalized monotonic games with a fixed veto player. In the last section, section 4, we discuss some solutions for IC–games and also for subclasses of IC–games that are convex games or big boss games. Some special results are derived for 3-person IC–games.

2 Information collecting situations and IC–games

The information collecting situation (IC–situation) to be introduced is a hybrid of a one-person decision situation under uncertainty and an Aumann structure.

Here we mean with a one-person decision situation under uncertainty a tuple

$$< \{0\}, (\Omega, \mathcal{F}, \mu), A, \{r_a : \Omega \longrightarrow \mathbb{R} \mid a \in A\} >,$$

where agent 0 is the decision-maker (action taker) who has to choose an action a from the action set A. The set of possible states (relevant to the decision situation) is denoted by Ω and the probability measure μ, which is defined on the σ-algebra $\mathcal{F}$ of subsets of Ω, describes the prior belief of agent 0 over all the states ω in Ω. Elements of the σ-algebra $\mathcal{F}$ will be called events. Agent 0 receives the reward $r_a(\omega)$ if he chooses action a and when ω turns out to be the true state. In the following the reward function $r_a : \Omega \longrightarrow \mathbb{R}$ is supposed to be a bounded $\mathcal{F}$–measurable function and we also suppose that agent 0 is risk-neutral.

An Aumann structure (cf. R.J. Aumann (1976), R. Fagin et al. (1999) p. 332) is a tuple

$$< \{1, 2, ..., n\}, (\Omega, \mathcal{F}, \mu), (\mathcal{I}_i)_{i \in \underline{N}} >$$

where $(\Omega, \mathcal{F}, \mu)$ is a probability space as above, $\underline{N} = \{1, 2, ..., n\}$ is a set of partially informed agents about the state $\omega \in \Omega$ at hand, and the information of each $i \in \underline{N}$ is described by a finite $\mathcal{F}$–measurable partition $\mathcal{I}_i$ of Ω. If $\omega \in \Omega$ is the true state, then agent i knows that the event $I_i(\omega)$ happens, where $I_i(\omega)$ is that element (atom) of the partition $\mathcal{I}_i$ of Ω that contains ω.

An IC–*situation* is now defined as a tuple

$$< N, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i \mid i \in \underline{N}\}, A, \{r_a : \Omega \longrightarrow \mathbb{R} \mid a \in A\} >$$

where $N = \{0\} \cup \underline{N} = \{0, 1, 2, ..., n\}$, and where the other components are as above.

An IC–situation models an interactive situation, where the action taker 0 has to choose an action a, and where his resulting reward $r_a(\omega)$ depends on the state $\omega \in \Omega$. Without cooperating with players in $\underline{N}$ he only knows the probability measure μ, describing the probability of events $E \in \mathcal{F}$. But the optimal expected reward can be improved upon by using the partial knowledge of agents in $\underline{N}$, because these agents have information on the state, described by their information partitions, which is available before player 0 has to choose an action.

Let us be more precise about these expected rewards. Agent 0 can obtain on his own the expected payoff

$$w(0) = \sup_{a \in A} \int_\Omega r_a(\omega) d\mu(\omega).$$

Together with the coalition $S \subset \underline{N}$ the expected reward is given by

$$w(\{0\} \cup S) = \sum_{I \in \mathcal{I}_S} \sup_{a \in A} \int_I r_a(\omega) d\mu(\omega)$$

where $\mathcal{I}_S$ is the coursest partition of Ω that is a refinement of $\mathcal{I}_i$ for each $i \in S$. This is called *the join* of $\{\mathcal{I}_i \mid i \in S\}$. Then $I \subset \Omega$ is an atom of $\mathcal{I}_S$ if and only if I is a non-empty intersection of atoms $\{I_i \mid i \in S\}$, that is,

$$I = \cap_{i \in S} I_i \neq \emptyset, \text{ with } I_i \in \mathcal{I}_i.$$

Note that if the state ω happens, the event $I = \cap_{i \in S} I_i(\omega)$ is known by 0 before taking an action in case he collects information from all members of S.

If 0 works together with all players in $\underline{N}$ to improve his action choice, then the question arises how to compensate (pay) the informed players for their help. In this paper we contribute to this question by constructing a cooperative TU–game $<N, v>$ related to the IC–situation. This opens the possibility to consider compensation schemes which correspond to the various solution concepts developed in the field of cooperative game theory.

For the game $<N, v>$ the player set $N = \{0, 1, 2, ..., n\}$ and the characteristic function $v : 2^N \longrightarrow \mathbb{R}$ is defined as follows:

(a) $v(S) = 0$ for all $S \subset N$ with $0 \notin S$,

(b) $v(S) = w(S) - w(0)$ for all S with $0 \in S$.

Formula (a) expresses the fact that no reward improvements by cooperation can be made if the action taker 0 is not in the coalition. Formula (b) expresses that the worth $v(S)$ of coalition S is the improvement in expected reward made by not standing alone but collecting the information of all members in $S \setminus \{0\}$ before taking an (optimal) action.

We will call this game $<N, v>$ the IC–*game* corresponding to the underlying IC–situation. Properties of IC–games and solutions will be studied later. Now we will give three examples to illustrate the notions. The examples give indications to possible applications.

Example 1. (Catch the monster)

Consider the IC–situation

$$<N, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i \mid i \in \underline{N}\}, A, \{r_a : \Omega \to \mathbb{R} \mid a \in A\}>$$

where $N = \{0, 1, 2\}$, $\Omega = [0, 1]$, $\mathcal{F}$ is the family of Borel subsets of $[0, 1]$ and μ is the Lebesgue measure. Furthermore,

$$\mathcal{I}_1 = \{[0, 1/3), [1/3, 1]\},$$
$$\mathcal{I}_2 = \{[0, 2/3], (2/3, 1]\},$$

$A = [0, 1]$ and $r_a(x) = -36|x - a|$ for all $x, a \in [0, 1]$.

One can think of a situation where a (0–dimensional) monster is hidden in one of the points x of the closed interval $[0, 1]$ and player 0 has to catch the monster operating from a suitable basis $a \in [0, 1]$. The cost of catching the monster is a multiple (36) of the distance between the operating basis a of player 0 and the hiding place x of the monster. In choosing a good basis, player 0 can use the help of player 1 who knows whether the monster is to the left of $1/3$ or not and of player 2, who knows whether the monster is to the right of $2/3$ or not.

To calculate the corresponding IC–game $<N, v>$, note that

$$w(\{0\}) = \max_{a \in [0,1]} \int_0^1 -36|x - a| dx = \int_0^1 -36 \left| x - \frac{1}{2} \right| dx = -9$$

$$\begin{aligned} w(\{0, 1\}) &= \max_{a \in [0,1]} \int_0^{\frac{1}{3}} -36|x - a| dx + \max_{a \in [0,1]} \int_{\frac{1}{3}}^1 -36|x - a| dx \\ &= \int_0^{\frac{1}{3}} -36 \left| x - \frac{1}{6} \right| dx + \int_{\frac{1}{3}}^1 -36 \left| x - \frac{2}{3} \right| dx = -1 - 4 = -5 \end{aligned}$$

(Basis in $1/6$ if $x \in [0, 1/3)$ and basis in $2/3$ if $x \in [1/3, 1]$.) Similarly, $w(\{0, 2\}) = -5$. Furthermore, $\mathcal{I}_{\{1,2\}} = \{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$ and $w(\{0, 1, 2\}) = -3$. So, we have $v(\emptyset) = v(\{0\}) = v(\{1\}) = v(\{2\}) = v(\{1, 2\}) = 0$, $v(\{0, 1\}) = v(\{0, 2\}) = 4$ and $v(\{0, 1, 2\}) = 6$.

Example 2. (Gamble or not)

Consider the information collecting situation with $N = \{0, 1, 2\}$, $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mu(\omega) = 1/6$ for each $\omega \in \Omega$, $\mathcal{I}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, $\mathcal{I}_2 =$

$= \{\{1,3,5\},\{2,4,6\}\}$, $A = \{1,2,3,4,5,6,n\}$, $r_n(\omega) = 0$ for all $\omega \in \Omega$ and for $k \in \{1,2,...,6\}$ we have $r_k(\omega) = 60$ if $k = \omega$ and $r_k(\omega) = -18$ otherwise.

One can think of a situation where a fair die is thrown. Player 0 has the possibility to guess the outcome (actions $1,2,...,6$) or not to participate in the game (action n, payoff 0). A correct guess gives a reward of 60, a wrong guess a cost of 18. Without extra information this is not an attractive gamble for player 0. There are however two possible informants, the players 1 and 2. Player 1 knows whether the outcome of the die is low or high, and player 2 knows whether the outcome is odd or even.

The corresponding information collecting game is given by $v(\emptyset) = v(\{0\}) = v(\{1\}) = v(\{2\}) = v(\{1,2\}) = 0$, $v(\{0,1\}) = v(\{0,2\}) = 8$ and $v(\{0,1,2\}) = 34$. Note that $\mathcal{I}_{\{1,2\}} = \{\{1,3\},\{4,6\},\{2\},\{5\}\}$. Thus the latter value is obtained as follow.

$$\begin{aligned} v(\{0,1,2\}) &= \max_{a\in A}\frac{1}{6}(r_a(1)+r_a(3)) + \max_{a\in A}\frac{1}{6}(r_a(4)+r_a(6)) + \\ &\quad + \max_{a\in A}\frac{1}{6}r_a(2) + \max_{a\in A}\frac{1}{6}r_a(5) \\ &= \tfrac{1}{6}(r_1(1)+r_1(3)+r_4(4)+r_4(6)+r_2(2)+r_5(5)) = 34. \end{aligned}$$

Example 3. (The right amount of ice-cream)

Consider the information collecting situation with $N = \{0,1\}$, $\Omega = \{n,h\}$, $\mu(n) = 0.8$, $\mu(h) = 0.2$, $\mathcal{I}_1 = \{\{n\},\{h\}\}$, $A = \{s,\ell\}$, and $r_s(n) = 50$, $r_s(h) = 60$, $r_\ell(n) = 30$, $r_\ell(h) = 100$.

One can think of an ice-cream seller who always sells ice-cream at a sales outlet far away from his house. He distinguishes two posible situations: the normal situation (n) (prior probability 0.8) and the situation where there is a higher activity (h). He can make a phone call to player 1 before leaving home and ask him about the state. Depending on this he takes either a small (s) or a large (ℓ) amount of ice-cream with him to the sales outlet.

To calculate the corresponding IC–game $<\{0,1\},v>$, note that

(i) $w(0) = \max\{0.8r_s(n)+0.2r_s(h), 0.8r_\ell(n)+0.2r_\ell(h)\} = \max\{52,44\} = 52$. (Without extra information it is optimal to take a small amount of ice-cream to the selling place.)

(ii) $w(0,1) = 0.8\max\{r_s(n), r_\ell(n)\} + 0.2\max\{r_s(h), r_\ell(h)\} = 40+20 = 60$. (With information, player 0 takes a small amount if $\omega = n$ and a large amount otherwise.)

So, $v(\{0,1\}) = 8$ and $v(\{0\}) = v(\{1\}) = 0$.

3 IC–games and monotonic games with one veto player

A game $<N, v>$ with player set $N = \{0, 1, 2, ..., n\}$ is called a *game with* 0 *as veto player* if

(V_0) $v(S) = 0$ for each $S \subset N \setminus \{0\}$,

it is *monotonic* if

(M) $v(S) \leq v(T)$ for all S, T with $S \subset T \subseteq N$,

and it is called *zero-normalized* if $v(i) = 0$ for all $i \in N$. From now on we denote the cone of zero-normalized games satisfying (V_0) and (M) by $MV_0(N)$ and $ICG(N)$ is the family of IC–games with player set N.

The following two propositions tell that the IC–games with player set N form a subcone of $MV_0(N)$. The main result of this section, theorem 3.5, shows that in fact these cones coincide.

Proposition 3.1. *Let* $v \in ICG(N)$ *where* $N = \{0, 1, 2, ..., n\}$. *Then* $v \in MV_0(N)$.

Proof. (i) By definition, $v(S) = 0$ if $0 \notin S$ and $v(\{0\}) = 0$, so v satisfies (V_0). This implies that v is a zero-normalized game.

(ii) To prove (M), take $S, T \in 2^N$ with $S \subset T$. Note that in case $0 \notin S$ we have $v(S) = 0 \leq v(T)$. In case $0 \in S \subset T$ the inequality $v(S) \leq v(T)$ follows from the fact that the partition $\mathcal{I}_{T\setminus\{0\}}$ is a refinement of the partition $\mathcal{I}_{S\setminus\{0\}}$:

$$\begin{aligned} v(S) + w(0) &= \sum_{I \in \mathcal{I}_{S\setminus\{0\}}} \sup_{a \in A} \int_I r_a(\omega) d\mu(\omega) \\ &= \sum_{I \in I_{S\setminus\{0\}}} \sup_{a \in A} \left(\sum_{J \in \mathcal{I}_{T\setminus\{0\}}, J \subset I} \int_J r_a(\omega) d\mu(\omega) \right) \\ &\leq \sum_{I \in \mathcal{I}_{S\setminus\{0\}}} \sum_{J \in \mathcal{I}_{T\setminus\{0\}}, J \subset I} \sup_{a \in A} \int_J r_a(\omega) d\mu(\omega) \\ &= \sum_{J \in \mathcal{I}_{T\setminus\{0\}}} \sup_{a \in A} \int_J r_a(\omega) d\mu(\omega) = v(T) + w(0), \end{aligned}$$

where $w(0) = \sup_{a \in A} \int_\Omega r_a(\omega) d\mu(\omega)$. ∎

Proposition 3.2. *$ICG(N)$ is a convex cone of games.*

Proof. Let $\alpha, \alpha' \in \mathrm{I\!R}_+$ and let $v, v' \in ICG(N)$. Let

$$< N, (\Omega, \mathcal{F}, \mu), (\mathcal{I}_i)_{i \in \underline{N}}, A, \{r_a : \Omega \longrightarrow \mathrm{I\!R} \mid a \in A\} >$$

be an IC–situation leading to the IC–game $< N, v >$ and let the IC–situation $< N, (\Omega', \mathcal{F}', \mu'), (\mathcal{I}'_i)_{i \in \underline{N}}, A', \{r_{a'} | a' \in A'\} >$ generate the game $< N, v' >$. Then

$$\begin{aligned} C = < N, & (\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mu \times \mu'), \\ & (\mathcal{I}_i \times \mathcal{I}'_i)_{i \in \underline{N}}, A \times A', \{r_{(a,a')} \mid (a, a') \in A \times A'\} > \end{aligned}$$

is an IC–situation where $\mu \times \mu'$ is the product probability measure on the product σ–algebra $\mathcal{F} \times \mathcal{F}'$, $\mathcal{I}_i \times \mathcal{I}'_i$ is the partition of $\Omega \times \Omega'$ with atoms $E \times E'$ where $E \in \mathcal{I}_i$, $E' \in \mathcal{I}'_i$ and $r_{(a,a')} : \Omega \times \Omega' \longrightarrow \mathrm{I\!R}$ is defined by $r_{(a,a')}(\omega, \omega') = \alpha r_a(\omega) + \alpha' r_{a'}(\omega')$ for all $(a, a') \in A \times A'$ and $(\omega, \omega') \in \Omega \times \Omega'$. Corresponding to this situation is the IC–game $< N, v_C > = < N, \alpha v + \alpha' v' >$, To prove that $v_C = \alpha v + \alpha' v'$, note first that $v_C(S) = 0 = \alpha v(S) + \alpha' v'(S)$ for all $S \subset \underline{N}$. Furthermore,

$$\begin{aligned} w_C(0) \quad &:= \sup_{(a,a') \in A \times A'} \int_{\Omega \times \Omega'} r_{(a,a')}(\omega, \omega') d\mu \times \mu'(\omega, \omega') \\ &= \alpha \sup_{a \in A} \int_\Omega r_a(\omega) d\mu(\omega) + \alpha' \sup_{a' \in A'} \int_{\Omega'} r_{a'}(\omega') d\mu'(\omega') \\ &= \alpha w(0) + \alpha' w'(0). \end{aligned}$$

Then $v_C(S) = \alpha v(S) + \alpha' v'(S)$ for $S \subset N$ with $0 \in S$ follows from

$$\begin{aligned} & v_C(S) + w_C(0) \\ &= \sum_{I \times I' \in \mathcal{I}_S \times \mathcal{I}'_S} \sup_{(a,a') \in A \times A'} \int_{I \times I'} (\alpha r_a(\omega) + \alpha' r_{a'}(\omega')) d\mu \times \mu'(\omega, \omega') \\ &= \alpha \sum_{I \in \mathcal{I}_S} \sup_{a \in A} \int_I r_a(\omega) d\mu(\omega) + \alpha' \sum_{I' \in \mathcal{I}'_S} \sup_{a' \in A'} \int_{I'} r_{a'}(\omega') d\mu'(\omega') \\ &= \alpha(v(S) + w(0)) + \alpha'(v'(S) + w'(0)). \end{aligned}$$

Hence, $v_C = \alpha v + \alpha' v'$. ∎

The next proposition shows that the cone $MV_0(N)$ can be generated by taking non-negative combinations of simple games in $MV_0(N)$, which are games $<N, v>$ where $v(S) \in \{0, 1\}$ for all S and $v(N) = 1$.

Proposition 3.3. *For each* $v \in MV_0(N)$, $v \neq 0$, *there are* $k \in \underline{N}$, $\alpha_1, \alpha_2, ..., \alpha_k > 0$, *and simple games* $w_1, ..., w_k \in MV_0(N)$ *such that*

$$v = \sum_{r=1}^{k} \alpha_r w_r.$$

Proof. Take $v \neq 0$ in $MV_0(N)$. Let $\beta_0, \beta_1, ..., \beta_k$ be the worths of the different coalitions of v such that $0 = \beta_0 < \beta_1 < \cdots < \beta_k = v(N)$. For each $r \in \{1, 2, ..., k\}$ define the simple game w_r by

$$w_r(S) = \begin{cases} 1 & \text{if } \ v(S) \geq \alpha_r \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } S \in 2^N.$$

Then $v \in MV_0(N)$ implies that $w_r \in MV_0(N)$. Define $\alpha_r := \beta_r - \beta_{r-1}$, then it follows from the definition of w_r that $v = \sum_{r=1}^{k} \alpha_r w_r$. ■

We continue by showing that each simple game in $MV_0(N)$ is an IC–game.

Proposition 3.4. *Let* $<N, u>$ *be a simple zero–normalized monotonic game with* 0 *as veto player. Then there is an* IC*–situation* C *with corresponding* IC*–game* $<N, v>$ *such that* $u = v$.

Proof. Let $\mathcal{W} := \{S \subset N \mid u(S) = 1, \ u(T) = 0 \text{ for all } T \subset S, \ T \neq S\}$ be the set of minimum winning coalitions of the simple game $<N, u>$. Let C be the IC–situation where N is the player set in $<N, u>$, $\Omega = \{0, 1\}^N$, $\mu(E) = \dfrac{1}{2^n}|E|$ for each $E \subset \Omega$ and $\mathcal{I}_i = \{\{\omega \in \Omega \mid \omega_i = 0\}, \{\omega \in \Omega \mid \omega_i = 1\}\}$. Let $A = \{a_0\} \cup \{a(S, I) \mid S \in \mathcal{W}, \ I \in \mathcal{I}_S\}$. The rewards are $r_{a_0}(\omega) = 0$ for all $\omega \in \Omega$ and

$$r_{a(S,I)}(\omega) = \begin{cases} 1 & \text{if } \ \omega \in I \\ -2^n & \text{otherwise} \end{cases} \quad \text{for all } S \in \mathcal{W}, \ I \in \mathcal{I}_S.$$

(Note that the action $a(S, I)$ is only interesting in comparison to a_0, if I is known.) Now we prove that $u = v$, where v is the IC–game corresponding to C.

Obviously, $u(S) = v(S) = 0$ for all $S \subset \underline{N}$ and $u(\{0\}) = v(\{0\}) = 0$. It is clear that

$$w(\{0\}) = \max_{a \in A} \int_\Omega r_a(\omega) d\mu(\omega) = \int_\Omega r_{a_0} d\mu(\omega) = 0,$$

since $\int_\Omega r_{a(S,I)}(\omega) d\mu(\omega) \leq 0$ for all $(S, I) \in \mathcal{W} \times \mathcal{I}_S$. So for all $S \subset \underline{N}$ with $S \neq \emptyset$ we have

$$v(S \cup \{0\}) = \sum_{I \in \mathcal{I}_S} \max_{a \in A} \int_I r_a(\omega) d\mu(\omega).$$

To prove that $u(S \cup \{0\}) = v(S \cup \{0\})$ for all non empty $S \subset \underline{N}$, we consider two cases.

Case 1. $S \cup \{0\}$ contains a minimum winning coalition $T \in \mathcal{W}$. In this case $u(S \cup \{0\}) = 1$ by the monotonicity of u. We also have $v(S \cup \{0\}) \leq 1$ because $r_a(\omega) \leq 1$ for all $a \in A$ and $\omega \in \Omega$. On the other hand

$$\begin{aligned} v(S \cup \{0\}) &= \sum_{K \in \mathcal{I}_S} \max_{a \in A} \int_K r_a(\omega) d\mu(\omega) \\ &\geq \sum_{I \in \mathcal{I}_T} \sum_{K \in \mathcal{I}_S, K \subset I} \int_K r_{a(T,I)}(\omega) d\mu(\omega) \\ &= \sum_{I \in \mathcal{I}_T} \sum_{K \in \mathcal{I}_S, K \subset I} \int_K 1\, d\mu(\omega) = \sum_{I \in \mathcal{I}_T} \sum_{K \in \mathcal{I}_S, K \subset I} \mu(K) \\ &= \sum_{I \in \mathcal{I}_T} \mu(I) = 1. \end{aligned}$$

So $v(S \cup \{0\}) = 1 = u(S \cup \{0\})$.

Case 2. $S \cup \{0\}$ does not contain any minimum winning coalition of $<N, u>$. Then $u(S \cup \{0\}) = 0$. Further $v(S \cup \{0\}) \geq \int_\Omega r_{a_0}(\omega) d\mu(\omega) = 0$. To prove that $v(S \cup \{0\}) = 0$ it is sufficient to show that for each $T \in \mathcal{W}$, $I \in \mathcal{I}_T$ and each $K \in \mathcal{I}_S$

$$\int_K r_{a(T,I)} d\mu(\omega) \leq 0.$$

Now

$$\begin{aligned} \int_K r_{a(T,I)}(\omega) d\mu(\omega) &= \int_{K \cap I} 1\, d\mu - \int_{K \setminus I} 2^n d\mu \\ &= \mu(K \cap I) - 2^n 2^{-n} |K \setminus I| \leq 1 - |K \setminus I|. \end{aligned}$$

So, $\int_K r_{a(T,I)} d\mu(\omega) \leq 0$ if $|K \setminus I| \geq 1$. We prove that $K \setminus I \neq \emptyset$ by observing that K is of the form

$$\{x \in \Omega \mid x_i = \alpha_i \in \{0,1\} \text{ for all } i \in S\},$$

and that I is of the form $\{x \in \Omega \mid x_i = \beta_i \in \{0,1\} \text{ for all } i \in T\}$. Since S does not contain any minimum winning coalition we have that $T \setminus S \neq \emptyset$. Let $\hat{x} \in \Omega$ be such that $\hat{x}_i = \alpha_i$ for all $i \in S$, and $\hat{x}_i = 1 - \beta_i$ for all $i \in T \setminus S$. Then $\hat{x} \in K \setminus I$, which finishes the proof. ■

The theorem below formulates the main result of this section.

Theorem 3.5. $MV_0(N) = ICG(N)$.

Proof. From Proposition 3.1 it follows that $ICG(N) \subset MV_0(N)$. From Proposition 3.4 it follows that all zero–normalized simple monotonic games with 0 as veto player are elements of $ICG(N)$. Since $ICG(N)$ is a cone by Proposition 3.2, and each element of $ICG(N)$ is a non–negative linear combination of zero–normalized simple monotonic games with 0 as veto player (Proposition 3.3), we conclude that also the reverse inclusion $MV_0(N) \subset ICG(N)$ holds. ■

4 Solutions for IC–games

This last section contains many unexplained notions of cooperative game theory. For details we refer to the mentioned papers and to the books of Curiel (1997) and Driessen (1988).

The fact that an IC–game $<N, v>$ is monotonic and that 0 is a veto player implies that the vector $(v(N), 0, 0, ..., 0)$ is a core element, which gives all the gains obtained by cooperation to the action taker. So the core $C(v)$ is nonempty. We note that this core element coincides with $n!$ marginal vectors namely with those for which the entry order is such that 0 enters last. So the IC–game $<N, v>$ is a permutationally convex game (cf. Granot and Huberman (1982)). Note that the given core element can also be extended to a population monotonic allocation scheme (cf. Sprumont (1990))

$$[a_{S,i}]_{S \in 2^N \setminus \{\emptyset\}, i \in S}, \quad \text{where } a_{S,i} = \begin{cases} v(S) & \text{if } i = 0 \in S \\ 0 & \text{otherwise.} \end{cases}$$

From the paper of Arin and Feltkamp (1997), which deals with the class of veto rich games containing the class $MV_0(N) = ICG(N)$, we can conclude that the bargaining set (cf. Aumann and Maschler (1964)) of an IC–game coincides with the core. Also, the kernel (cf. Davis and Maschler (1965)) contains only one element, namely the nucleolus (cf. Schmeidler (1969)) and there is an efficient algorithm to calculate this value. Because IC–games are zero–monotonic the prekernel and the kernel coincide and the nucleolus is equal to the prenucleolus (cf. Maschler, Peleg and Shapley (1972)).

More about solutions can be said for games in two interesting subclasses of $MV_0(N) = ICG(N)$ namely $UMV_0(N)$ and $CMV_0(N)$, which we introduce now.

$UMV_0(N)$ is the cone of those games $v \in MV_0(N)$ that satisfy the so-called *union property*:

(U) $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(v)$ for all $S \subset N$ with $0 \in S$

where $M_i(v) = v(N) - v(N \setminus \{i\})$. This class $UMV_0(N)$ is the class of monotonic big boss games and is well-studied (cf. Muto et al. (1988), Tijs (1990)). If $v \in UMV_0(N)$, then

(i) the core $C(v)$ is equal to the parallelepiped

$$\left\{ x \in \mathbb{R}^N \,\middle|\, 0 \leq x_i \leq M_i(v) \text{ for all } i \in \underline{N},\ \sum_{i=0}^{n} x_i = v(N) \right\},$$

(ii) the τ-value (cf. Tijs (1981)) and the nucleolus coincide and are equal to the center of the core

$$\left(v(N) - \frac{1}{2}\sum_{i=1}^{n} M_i(v), \frac{1}{2}M_1(v), \frac{1}{2}M_2(v), \ldots, \frac{1}{2}M_n(v) \right).$$

$CMV_0(N)$ is the cone of those games in $MV_0(N)$ that are convex (cf. Shapley (1971)), that is, for which the property (C) holds:

(C) $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S, T \subset N$ and $i \in N$ with $S \subset T \subset N \setminus \{i\}$.

Games with property (C) have many other nice properties. We mention only that for a game $v \in CMV_0(N)$, the Shapley value (cf. Shapley (1953)) $\Phi(v)$ is a core element and that the core is the convex hull of the $(n+1)!$ marginal vectors.

Let us again look at the three examples in section 2. Example 1 (catch the monster) resulted in the IC–game with $v(\{0,1\}) = v(\{0,2\}) = 4$, $v(\{0,1,2\}) = 6$, and $v(S) = 0$ for all other coalitions S. Since $M_1(v) + M_2(v) = 2 + 2 < 6 = v(N) - v(0)$, $v \in UMV_0(N)$. The core is

$$\left\{x \in \mathbb{R}^{\{0,1,2\}} \mid 0 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 2,\ x_0 = 6 - x_1 - x_2\right\}$$

and the τ–value and the nucleolus are equal to $(4,1,1)$. The Shapley value is the average of the marginal vectors: $(0,4,2)$, $(0,2,4)$, $(4,0,2)$, $(6,0,0)$, $(4,2,0)$, $(6,0,0)$ and equals $(11/3, 4/3, 4/3)$. Note that the last four marginal vectors are extreme points of the core. The marginal vectors $(0,4,2)$ and $(0,2,4)$, corresponding to entry orders where 0 enters first, are no core elements.

Example 2 (gamble or not) resulted in the IC–game with $v(\{0,1\}) = v(\{0,2\}) = 8$, $v(\{0,1,2\}) = 34$ and $v(S) = 0$ otherwise. This game v is convex, so $v \in CMV_0(N)$. The core equals

$$\mathrm{conv}\{(0,8,26),(0,26,8),(8,0,26),(34,0,0),(8,26,0),(34,0,0)\}$$

where conv stands for convex hull, and the Shapley value is $\Phi(v) = (14,10,10) \in C(v)$.

Example 3 (the right amount of ice cream) resulted in a 2–person game which has the properties (U) and (C). Note that for all 2–person IC–games we have $MV_0(\{0,1\}) = CMV_0(\{0,1\}) = UMV_0(\{0,1\})$.

The next proposition tells that each 3-person IC–game has at least one of the properties (U) and (C). This is not necessarily the case for n–person IC–games with $n \geq 4$, as the final example 4 shows.

Proposition 4.1. $MV_0(\{0,1,2\}) = CMV_0(\{0,1,2\}) \cup UMV_0(\{0,1,2\})$.

Proof. Clearly, the set on the right sight of the equality sign is included in $MV_0(\{0,1,2\})$. For the converse inclusion, take $v \in MV_0(\{0,1,2\})$. Then it is sufficient to show that the following two claims hold.

$$(C.1) \qquad v \in UMV_0(\{0,1,2\}) \quad \text{iff} \quad v(\{0,1,2\} - v(\{0\}) \geq M_1(v) + M_2(v)$$

$$(C.2) \qquad v \in CMV_0(\{0,1,2\}) \quad \text{iff} \quad v(\{0,1,2\} - v(\{0\}) \leq M_1(v) + M_2(v).$$

$(C.1)$ is obvious. To prove $(C.2)$ note first that for all $v \in MV_0(\{0,1,2\})$ we have by monotonicity that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ holds if $\{0\} \neq S \subset T \subset N \setminus \{i\}$. Then $v \in CMV_0(\{0,1,2\})$

$$\Longleftrightarrow \quad v(\{0,i\}) - v(\{0\}) \leq v(\{0,1,2\}) - v(\{0,j\}) \quad \text{for } i,j \in \{1,2\}, \quad i \neq j$$
$$\Longleftrightarrow v(\{0,1,2\}) - v(\{0\}) \leq M_1(v) + M_2(v). \ \blacksquare$$

Example 4. Let $<N, v>$ be given by $N = \{0,1,2,3\}$, $v(S) = 2$ if $|S| = 2$ and $0 \in S$, $v(S) = 3$ if $|S| = 3$ and $0 \in S$, $v(N) = 5$, and $v(S) = 0$ in all other cases. Then $v \in MV_0(N)$, v does not satisfy (U) because

$$v(N) - v(0) = 5 < \sum_{i=1}^{3} M_i(v) = 6$$

and v does not satisfy (C) because

$$v(\{0,1\}) - v(\{0\}) = 2 > 1 = v(\{0,1,2,3\}) - v(\{0,2\}).$$

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